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The Degree of Approximation of functions III

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Abstract: *Popoviciu (1935) proved his result for Bernstein Polynomials. We tested the degree of approximation of function by our newly defined Bernstein type Polynomials, and so the corresponding results of Popoviciu have been extended for Lebesgue integrable function in $L_1 - norm$ by our newly defined Bernstein type Polynomials*

$$U_{nr}^{\alpha}(f,x) = (n+r+1)\sum_{k=0}^{n+r}\left\{\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} f(t)\,dt\right\} q_{nr,k}(x;\alpha)$$

where

$$q_{nr,k}(x;\alpha) = \binom{n+r}{k}\frac{x(x+k\alpha)^{k-1}(1-x)(1-x+(n+r-k)\alpha)^{n+r-k-1}}{(1+(n+r)\alpha)^{n+r-1}}$$

Keywords: *Bernstein Polynomials, Degree of Approximation, Bernstein type Polynomial,$L_1 - norm$, Lebesgue Integrable function.*

I. Introduction And Results

If $f(x)$ is a function defined on [0, 1], the Bernstein polynomial $B_n^f(x)$ of f is given as

$$B_n^f(x) = \sum_{k=0}^{n} f(k/n)\, p_{n,k}(x) \qquad (1.1)$$

where

$$p_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k} \qquad (1.2)$$

Bernstein (1912-13) proved that if $f(x)$ is continuous in closed interval [0,1], then $B_n^f(x)$ tends to $f(x)$ uniformly as $n \to \infty$. This Yields a simple constructive proof of Weierstrass's approximation theorem. A more precise version of this result due to Popoviciu(1935) states that

$$\left|B_n^f(x) - f(x)\right| \le \frac{5}{2} w_f(n^{-1/2})$$

wherew_f is the uniform modulus of continuity of f defined by
$w_f(h) = \max\{|f(x) - f(h)|; x, y \in [0,1], |x-y| \le h\}$
A slight modification of Bernstein polynomials due to Kantorovitch[7] makes it possible to approximate Lebesgue integrable function in L_1-norm by the modified polynomials

$$P_n^f(x) = (n+1)\sum_{k=0}^{n}\left\{\int_{k/(n+1)}^{(k+1)/(n+1)} f(t)\,dt\right\} p_{n,k}(x) \qquad (1.3)$$

where $p_{n,k}(x)$ is defined by (1.2)
By Abel's formula ([5])

$$(x+y)(x+y+n\alpha)^{n-1} = \sum_{k=0}^{n}\binom{n}{k} x(x+k\alpha)^{k-1} y(y+(n-k)\alpha)^{n-k-1} \qquad (1.4)$$

If we put$y = 1 - x$, we obtain ([4])

$$1 = \sum_{k=0}^{n}\binom{n}{k}\frac{x(x+k\alpha)^{k-1}(1-x)(1-x+(n-k)\alpha)^{n-k-1}}{(1+n\alpha)^{n-1}} \qquad (1.5)$$

Thus defining

$$q_{n,k}(x;\alpha) = \binom{n}{k}\frac{x(x+k\alpha)^{k-1}(1-x)(1-x+(n-k)\alpha)^{n-k-1}}{(1+n\alpha)^{n-1}} \qquad (1.6)$$

we have

$$\sum_{k=0}^{n} q_{n,k}(x;\alpha) = 1 \qquad (1.7)$$

Schurer [6] introduced an operator

S_{nr}: c [0, 1+$\frac{r}{n}$] → c [0, 1]

defined by

$$S_{nr}(f, x) = \sum_{k=0}^{n+r} f\left(\frac{k}{n+r}\right) p_{nr,k}(x) \tag{1.8}$$

where

$$p_{nr,k}(x) = \binom{n+r}{k} x^k (1-x)^{n+r-k} \tag{1.9}$$

and r is a non-negative integer.

A slight modification in Abel's formula with certain substitution Anwar Habib [3] constructed a generalized form of a Bernstein type Polynomials on [0, $1+\frac{r}{n}$] for Lebesgue Integral in L_1 norm as

$$U_{nr}^{\alpha}(f, x) = (n+r+1)\sum_{k=0}^{n+r}\left\{\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} f(t)\,dt\right\} q_{nr,k}(x;\alpha) \tag{1.10}$$

where

$$q_{nr,k}(x;\alpha) = \binom{n+r}{k}\frac{x(x+k\alpha)^{k-1}(1-x)(1-x+(n+r-k)\alpha)^{n+r-k-1}}{(1+(n+r)\alpha)^{n+r-1}} \tag{1.11}$$

such that $\sum_{k=0}^{n+r} q_{nr,k}(x;\alpha) = 1$ (1.12)

when $r = 0$ and $\alpha = 0$ then (1.11) & (1.12) reduces to (1.3) & (1.2) respectively.

In this paper, we shall test the degree of approximation by our polynomial (1.10) for Lebesgue integrable function in L_1-norm.

In fact we state our results as follows

Theorem 1: If $f(x)$ is continuous Lebesgue Integrable function on [0,1+$\frac{r}{n}$] and w(δ) is the modulus of continuity of f(x) , then for $\alpha = \alpha_{nr} = o(1/(n+r))$ we have

$$\left|U_{nr}^{\alpha}(f, x) - f(x)\right| \le \frac{3}{2} w\left(\frac{1}{\sqrt{n+r}}\right)$$

Theorem 2: If $f(x)$ is continuous Lebesgue integrable function on [0,1+$\frac{r}{n}$] such that its first derivative is bounded and $w_1(\delta)$ is the modulus of continuity of $f'(x)$, for $\alpha = \alpha_{nr} = o(1/(n+r))$ we have

$$\left|U_{nr}^{\alpha}(f, x) - f(x)\right| \le \frac{3}{4}\frac{1}{\sqrt{n+r}} w\left(\frac{1}{\sqrt{n+r}}\right) + o\left(\frac{1}{n+r}\right) .$$

I. LEMMAS

We need the following lemmas to prove our results (Anwar Habib [3])

Lemma 2.1: For all values of x

$$\sum_{k=0}^{n+r} k q_{nr,k}(x;\alpha) \le \frac{1+(n+r)\alpha}{1+\alpha}(n+r)x - \frac{(n+r)(n+r-1)x\alpha}{1+2\alpha}$$

Lemma 2.2: For all values of x

$$\sum_{k=0}^{n+r} k(k-1) q_{nr,k}(x;\alpha) \le (n+r)(n+r-1)(x+2\alpha)\{\frac{1+(n+r)\alpha}{(1+2\alpha)^2} - \frac{(n+r-2)\alpha}{(1+3\alpha)^3}$$

$$+(n+r-2)\alpha^2\left(\frac{1+(n+r)\alpha}{(1+3\alpha)^3} - \frac{(n+r-3)\alpha}{(1+4\alpha)^4}\right)\}$$

Lemma 2.3: For all values of x of [0, 1+$\frac{r}{n}$] and for $\alpha = \alpha_{nr} = 0\left(\frac{1}{n+r}\right)$, we have

$$(n+r+1)\sum_{k=0}^{n+r}\left\{\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} ((t-x)^2)\,dt\right\} q_{nr,k}(x;\alpha) \le \frac{x(1-x)}{n+r}$$

II. Proof Of The Theorems

Proof of theorem 1:

$$\left|U_{nr}^{\alpha}(f,x)-f(x)\right| \le (n+r+1)\sum_{k=0}^{n+r}\left\{\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} |f(t)-f(x)|dt\right\} q_{nr,k}(x;\alpha)$$

Using the property of modulus of continuity
$|f(x_2)-f(x_1)| \le w(|x_2-x_1|)$
$w(\lambda\delta) \le ([\lambda]+1)w(\delta) \le (\lambda+1)w(\delta)$, $\lambda > 0$ (3.1)

we obtain
$$|f(x)-f(t)| \le w(|x-t|)$$
$$= w(\tfrac{1}{\delta}|x-t|\delta)$$
$$= (1+\tfrac{1}{\delta}|x-t|)w(\delta)$$

$$\left|U_{nr}^{\alpha}(f,x)-f(x)\right| \le (n+r+1)w(\delta)\sum_{k=0}^{n+r}\left\{\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} (1+\frac{1}{\delta}|x-t|)dt\right\} q_{nr,k}(x;\alpha)$$
$$= w(\delta)[(n+1)\delta^{-1}\textstyle\sum_{k=0}^{n+r}\left\{\int_{k/(n+r+1)}^{(k+1)/(n+r+1)}|x-t|dt\right\}q_{nr,k}(x;\alpha) \quad \text{------} \quad (3.2)$$

then by Cauchy 's inequality , we have
$$(n+r+1)\sum_{k=0}^{n+r}\left\{\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} |x-t|dt\right\} q_{nr,k}(x;\alpha)$$
$$\le [(n+r+1)\sum_{k=0}^{n+r}\left\{\int_{\frac{k}{n+r+1}}^{\frac{k+1}{n+r+1}} (x-t)^2dt\right\} q_{nr,k}(x;\alpha)]^{1/2}$$
by lemma2. 3 and the fact $x(1-x) \le \frac{1}{4}$ on $[0,1+\frac{r}{n}]$ for large n
$$\le \left(\frac{1}{4(n+r)}\right)^{1/2} \quad (3.3)$$

and hence from (3.2) and (3.3) we have
$$\left|U_{nr}^{\alpha}(f,x)-f(x)\right| \le [1+\delta^{-1}\left(\frac{1}{2\sqrt{n+r}}\right)]w(\delta)$$
But for $\delta = (n+r)^{-1/2}$
$$\left|U_{nr}^{\alpha}(f,x) - f(x)\right| \le \frac{3}{2}w\left(\frac{1}{\sqrt{n+r}}\right)$$
which completes the proof of theorem 1.

Proof of theorem 2:
By applying the Mean Value Theorem of differential calculus, we can write
$$f(x)-f(t) = (x-t)\,f'(\xi)$$
$$= (x-t)\,f'(x) + (x-t)\{f'(\xi) - f'(x)\} \quad (3.4)$$

where ξ is an interior point of the interval determined by x and t .
If we multiply (3.4) by $(n+1)q_{nr,k}(x;\alpha)$ and integrate it from $\frac{k}{n+r+1}$ to $\frac{k+1}{n+r+1}$ and sum over k, there follows

$$f(x) - U_{nr}^{\alpha}(f, x)$$
$$= (n+r+1)\sum_{k=0}^{n+r}\left\{\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} |f(x) - f(t)|\,dt\right\} q_{nr,k}(x;\alpha)$$
$$= (n+r+1)\sum_{k=0}^{n+r}\left\{\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} (x-t)\, f'(x)\,dt\right\} q_{nr,k}(x;\alpha)$$
$$+ (n+r+1)\sum_{k=0}^{n+r}\left\{\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} (x-t)\{f'(\xi) - f'(x)\}dt\right\} q_{nr,k}(x;\alpha)$$
$$\left|(n+r+1)\sum_{k=0}^{n+r}\left\{\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} (x-t)\, f'(x)\,dt\right\} q_{nr,k}(x;\alpha)\right|$$
$$= \left|(n+r+1)\sum_{k=0}^{n+r}\left\{\frac{x}{n+r+1} - \frac{2k+1}{2(n+r+1)^2}\right\} f'(x)\; q_{nr,k}(x;\alpha)\right|$$
$$= \left|\sum_{k=0}^{n}\left\{x - \frac{k}{n+r+1} - \frac{1}{2(n+r+1)}\right\} f'(x)\; q_{nr,k}(x;\alpha)\right|$$

By Lemma 2.1 and (1.12) , we have

$$= \left|\left\{x - \frac{1}{n+r+1}\left[\frac{1+(n+r)\alpha}{1+\alpha} nx - \frac{(n+r)(n+r-1)x\alpha}{1+2\alpha}\right] - \frac{1}{2(n+r+1)}\right\} f'(x)\right|$$

$\leq \frac{M}{n+r}$ whereas $|f'(x)| \leq M$ and $\alpha = \alpha_{nr} = o(1/(n+r))$ and for large n and by (3.1)

$$|f'(\xi) - f'(x)| \leq w_1(|\xi - x|) \leq \left(1 + \frac{1}{\delta}|\xi - x|\right) w_1(\delta)$$

where δ is a positive number does not defined on k .

Consequently we can

$$\left|f(x) - U_{nr}^{\alpha}(f, x)\right| \leq \frac{M}{n+r} + w_1(\delta)[(n+r+1)\sum_{k=0}^{n+r}\left\{\int_{\frac{k}{n+r+1}}^{\frac{k+1}{n+r+1}} |x-t|\,dt\right\} q_{nr,k}(x;\alpha)$$
$$+\frac{1}{\delta}(n+r+1)\sum_{k=0}^{n+r}\left\{\int_{k/(n+r+1)}^{(k+1)/(n+r+1)} (x-t)^2 dt\right\} q_{nr,k}(x;\alpha)]$$

Hence by (3.3) and the Lemma (2.3) and with the fact $x(1-x) \leq \frac{1}{4}$ on $[0,1+\frac{r}{n}]$ for large n we have

$$\left|f(x) - U_{nr}^{\alpha}(f, x)\right| \leq \frac{M}{n+r} + w_1(\delta)\{\frac{1}{2\sqrt{n+r}} + \frac{1}{\delta}\left(\frac{1}{4(n+r)}\right)$$

But for $\delta = (n+r)^{-1/2}$

$$\leq \frac{M}{n+r} + w_1\left(\frac{1}{\sqrt{n+r}}\right)\{\frac{1}{2\sqrt{n+r}} + \frac{1}{4\sqrt{n+r}}$$
$$\leq \frac{3}{4\sqrt{n+r}} w_1\left(\frac{1}{\sqrt{n+r}}\right) + o\left(\frac{1}{n+r}\right)$$

which completes the proof of theorem 2.

III. Conclusion

The results of Popoviciu have been extended for Lebesgue Integrable function in L_1-norm by our newly defined Bernstein type Polynomials.

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