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Numerical Strategies of the Lorenz's Nonlinear Problems Using Adomian Decomposition Method

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Abstract: *In this paper an interesting and famous realistic Lorenz's nonlinear problem is discussed using the Adomian Decomposition Method (ADM). The results (approximate solutions) obtained very accurate using classical Runge-Kutta (RK) method, single-term Haar Wavelet series [8] and ADM methods are compared with the ODE45 in Matlab. It is found that the solution obtained using ADM is closer to the ODE45 in Matlab. The high accuracy and the wide applicability of ADM approach will be demonstrated with numerical example. Solution graphs for discrete exact solutions are presented in a graphical form to show the efficiency of the ADM. The results obtained show that ADM is more useful for solving Lorenz's nonlinear problems and the solution can be obtained for any length of time.*

Keywords: *Lorenz's equation, ODE45 in Matlab, Runge-Kutta method, Single-term Haar wavelet series, Adomian Decomposition Method.*

I. Introduction

Mathematical modeling aims to describe the different aspects of the real world, their interaction, and their dynamics through mathematics. It constitutes the third pillar of science and engineering, achieving the fulfillment of the two more traditional disciplines, which are theoretical analysis and experimentation. Now days, mathematical modeling has a key role also in fields such as the environment and industry, while its potential contribution in many other areas is becoming more and more evident. One of the reasons for this growing success is definitely due to the impetuous progress of scientific computation; this discipline allows the translation of a mathematical model, which can be explicitly solved only occasionally, into algorithms that can be treated and solved by ever more powerful computers.[2,9]

In this paper we developed numerical methods for addressing Lorenz's equation by an application of the Adomian Decomposition Method which was studied by Sekar and team of his researchers [3-7]. Recently, Sekar et al. [8] discussed the Lorenz's equation using STHW. In this paper, the same Lorenz's nonlinear problem was considered (discussed by Sekar et al. [8]) but present a different approach using the Adomian Decomposition Method with more accuracy for Lorenz's nonlinear equation. We also establish a simulation of the famous Lorenz's nonlinear equation, which is arisen in circuits, biological models, robotics and nonlinear dynamics etc. In this paper we show the simulation results in graphical form to highlight the effectives of ADM compare to RK, STHW and ODE45.

II. Adomian Decomposition Method

Suppose k is a positive integer and $f_1, f_2, \ldots, f_k$ are k real continuous functions defined on some domain G. To obtain k differentiable functions $y_1, y_2, \ldots, y_k$ defined on the interval I such that $(t, y_1(t), y_2(t), \ldots, y_k(t)) \in G$ for $t \in I$.

Let us consider the problems in the following system of ordinary differential equations:

$$\frac{dy_i(t)}{dt} = f_i(t, y_1(t), y_2(t), \ldots, y_k(t)) \ , \quad y_i(t)\,|_{t=0} = \beta_i \tag{1}$$

where β_i is a specified constant vector, $y_i(t)$ is the solution vector for $i = 1, 2, \ldots, k$. In the decomposition method, (1) is approximated by the operators in the form: $Ly_i(t) = f_i(t, y_1(t), y_2(t), \ldots, y_k(t))$ where L is the first order operator defined by $L = d/dt$ and $i = 1, 2, \ldots, k$.

Assuming the inverse operator of L is L^{-1} which is invertible and denoted by $L^{-1}(.) = \int_{t_0}^{t} (.)\, dt$, then applying L^{-1} to $L\, y_i(t)$ yields

$$L^{-1}Ly_i(t) = L^{-1}f_i(t, y_1(t), y_2(t), \ldots, y_k(t))$$

where $i = 1, 2, \ldots, k$. Thus

$$y_i(t) = y_i(t_0) + L^{-1}f_i(t, y_1(t), y_2(t), \ldots, y_k(t)).$$

Hence the decomposition method consists of representing $y_i(t)$ in the decomposition series form given by

$$y_i(t) = \sum_{n=0}^{\infty} f_{i,n}(t, y_1(t), y_2(t), \ldots, y_k(t)) \quad (2)$$

where the components $y_{i,n}$, $n \geq 1$ and $i = 1, 2, \ldots, k$ can be computed readily in a recursive manner. Then the series solution is obtained as

$$y_i(t) = y_{i,0}(t) + \sum_{n=1}^{\infty} \{L^{-1}f_{i,n}(t, y_1(t), y_2(t), \ldots, y_k(t))\}. \quad (3)$$

For a detailed explanation of decomposition method and a general formula of Adomian polynomials, we refer reader to [Adomian 1].

III. Lorenz's Equation

The Lorenz equations were discovered by Ed Lorenz in 1963 as a very simplified model of convection rolls in the upper atmosphere. Later these same equations appeared in studies of lasers, batteries, and in a simple chaotic waterwheel that can be easily built. Lorenz found that the trajectories of this system, for certain settings, never settle down to a fixed point, never approach a stable limit cycle, yet never diverge to infinity. What Lorenz discovered was at the time unheard of in the mathematical community, and was largely ignored for many years. Now this beautiful attractor is the most well known strange attractor that chaos has to offer.

Lorenz's equations are actually three differential equations, a first order equation for each of the x, y, and z components of the trajectories position. They are represented in (4). Where σ, ρ and β are parameters that change the behavior of the system. There are a lot of resources available if you wish to study the Lorenz equations in detail. These equations are usually the first chaotic differential equations introduced in any book on chaos. Also Ed Lorenz's paper (1963) is a very good source for information. Solution of the Lorenz's equation using STHW is presented in Figure 1.

IV. Numerical Example for Lorenz's Nonlinear Problem

The Lorenz equations describe a complex, 3 dimensional dynamical systems with 3 parameters σ, ρ and β. This system has the form,

$$\begin{aligned} \dot{x} &= \sigma(y - x) \\ \dot{y} &= \rho x - y - xz \\ \dot{z} &= xy - \beta z \end{aligned} \quad (4)$$

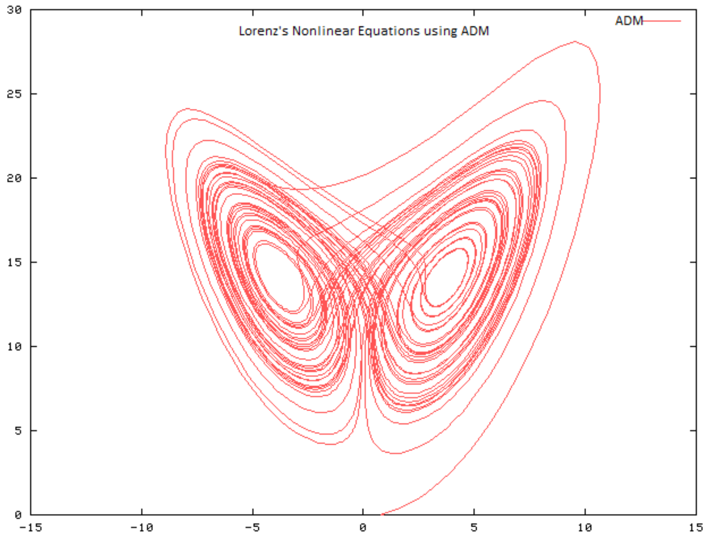

Fig. 1 Solution of Lorenz's equation using the ADM

and is nonlinear with two quadratic terms xy and xz. The properties of this deceptively simple system of equations turn out to be very complicated, and a good summary of results concerning the Lorenz equations is provided in Strogatz [32]. However, one property of Equation (4) which we will make use of is the chaotic behavior of solutions for certain parameter values.

In Strogatz [32], the term chaos is loosely associated with the idea that long-term behavior of solutions in a deterministic system is exhibiting sensitive dependence on initial conditions. That makes any long term integration difficult as nearby trajectories will separate exponentially fast. This behavior makes for a basis for comparing different numerical methods, especially if we want to check the accuracy of solutions from numerical integrators against one such as ODE45 which is highly accurate. For the same step size h, the time it takes in the integration period for the solutions from a particular method to diverge from that of ODE45's can be the measure of how accurate that method is. That is, the longer the solutions from a particular integrator are able to follow the solutions from ODE45, then the more accurate that method is. It should be easy to identify this length of time, since solutions in the Lorenz equations are sensitive to initial conditions, and slight changes to the values of the solution in any one of the 3 components can cause extreme differences in the solutions at a later time.

Also, it turns out that the Lorenz equations are an example in a Matlab toolbox known as DiffMan [2]. Developed by EngǾ, Marthinsen and Munthe-Kaas in Norway, DiffMan utilizes the calculation and graphical powers of Matlab to solve ordinary differential equations on manifolds, using a variety of Lie group techniques.

Now, let us follow the approach of Lorenz used when he numerically solved the Lorenz equations to study the behavior of the trajectories for a given initial condition $Y(t) = (x(t), y(t), z(t)) \in \mathbf{R}^3$. He studied the particular case when $\sigma = 10, \rho = 28$ and $\beta = 8/3$ [2]. We set the initial condition Y(t) as Y(0) = (17, -21, 54), and integrate over the interval t = [0, 10] with a constant step size of h = 1/20. The integrators we used for solving this example include the classical 4th order RK method, the STHW method, the ADM method, and finally, Matlab's ODE45 for comparison.

Plotting the results in three dimensions in Figure 2, we see the appearance of the famous **Lorenz butterfly** from all four of the integrators used to solve Equation (4). This is known as the chaotic attractor, which is loosely defined as a set A to which all neighboring trajectories converge. A is a closed set, such that A is an invariant set, and attracts an open set of initial conditions within the basin of attraction of A [2].

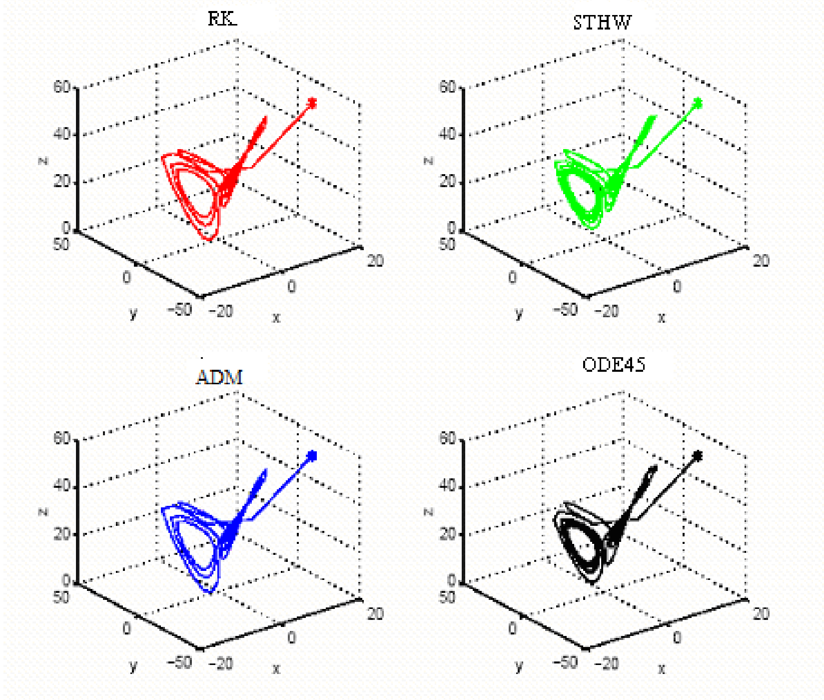

Fig. 2 Three dimensional graphs showing the Lorenz attractors from 4 different integrators, when solving Equation (4). y_0 = [17, 21, 54], t_0 = 0, t_f = 10, and h = 1/20.

Although all four attractors have the same form in three dimensions, the behavior of trajectories from different methods is not the same. This can be seen from the plots of the three components of solutions Y(t) from each of the four integrators tested, against the integration time interval t = [0, 10] in Figure 3. Note that the three component solutions from both RK and STHW are exactly the same.

Assuming that ODE45 method produced the most accurate solutions, then we can see that the behavior of solutions from other integrators do not follow that of ODE45's for the entire integration period. This is due to the sensitivity of solutions to the initial conditions, and in this case, the value of Y_i at the beginning of the i^{th} integration step. For all three components x(t), y(t), z(t), the solutions from RK and STHW, and ADM methods are pretty close to the solutions from ODE45 from the initial value t = 0 to roughly t = 3.

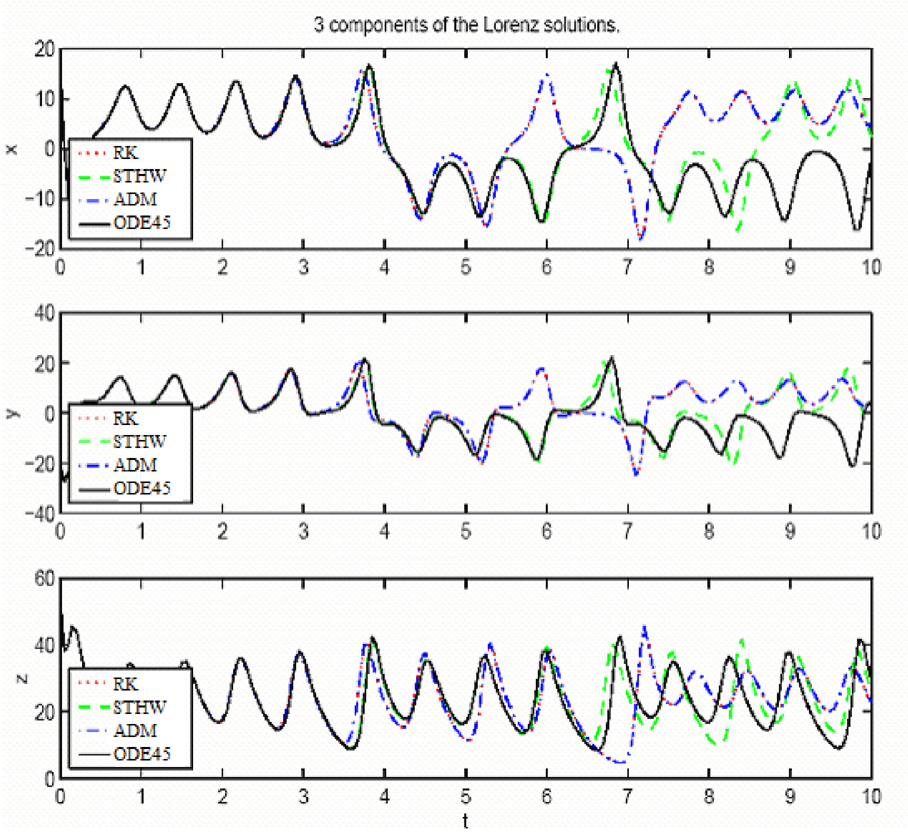

Fig. 3 Three component solutions Y (t) = (x(t), y(t), z(t)) against time t, from solving the Lorenz Equations in Equation (4). y_0 = [17, 21, 54], t_0 = 0, t_f = 10, and h = 1/20.

However, after that, solutions from RK and STHW methods slowly drifted away from the ODE45's trajectories until their behaviors are completely different from roughly t = 5:5. Similarly, the solutions from the ADM method are slightly better than those from RK and STHW methods, in the sense that the three components solutions from the ADM method followed ODE45's trajectories for a longer length of time, until they drifted apart at roughly t = 6. In this case, of the three fixed step size integrators the ADM method performs better than the RK and STHW methods.

V. Conclusion

The accuracy achieved from the ADM method is higher than that of the RK and the STHW methods. This can be observed if we compare results from the 3 integrators and that ODE45. The differences in this comparison are also plotted in the bottom 3 graphs of Figure 2 and Figure 3. The difference between ODE45 and ADM are several degrees smaller in magnitude than the differences between smaller in magnitude than the differences between numerical solutions from RK and STHW methods against ODE45's. With a relatively low computational cost, and a relatively good accuracy for fixed step size h, these brief experiments suggest the suitability of using the ADM to integrate systems of Lorenz's equation. Hence the ADM method is more suitable for studying the Lorenz's equation.

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