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Convection Driven by Surface Tension and Buoyancy in a Relatively Hotter or Cooler Layer of Liquid with Insulating Boundaries

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Abstract: *Rayleigh-Bénard-Marangoni convection in a relatively hotter or cooler layer of liquid is studied theoretically by means of modified linear stability theory. The upper surface of the layer is considered to be non-deformable free where surface tension gradients arise on account of variation of temperature and the lower boundary surface is rigid, each subject to constant heat flux condition. The Galerkin technique is used to obtain the eigenvalue equation analytically. This analysis predicts that the onset of convection in a relatively hotter layer of liquid is more stable than a cooler one under identical conditions, irrespective of whether the two mechanisms causing instability act individually or simultaneously, and that the coupling between the two agencies causing instability remains perfect.*

Keywords: *Buoyancy, Convection, Coupling, Linear stability, Surface tension, Insulating.*

I. Introduction

The mechanism of the onset of surface tension induced convection in a thin horizontal liquid layer heated from below with free upper surface was reported experimentally by Block [1] and explained mathematically by Pearson [2]. They established that the patterned hexagonal cells observed by Bénard [3, 4] and explained by Rayleigh [5] in terms of buoyancy, were in fact due to temperature dependent surface tension. Convection induced by surface tension gradients is now commonly known as Bénard-Marangoni convection in contrast to the buoyancy induced Rayleigh- Bénard convection. Quantitative disagreement between experiment and theory has indicated that gravity was present in Bénard's experiments as well as in other experiments involving convection in a liquid layer with free surface in a laboratory on the earth, therefore, Nield [6] considered the combined effects of both the surface tension and buoyancy on the onset of convection in a liquid layer heated from below with free upper surface, called Rayleigh-Bénard-Marangoni convection, and found that the two effects causing instability are tightly coupled. For a detail study of convection one may be referred to the work of Chandrasekhar [7], Normand et al. [8], Koschmieder [9] and Schatz & Neitzel [10].

Since the process of controlling convection in a fluid has become important in material processing and because of its applications extending from producing large crystals of uniform properties to manufacturing new materials with unique properties. Recently, Gupta et al. [11] studied the Rayleigh-Bénard-Marangoni convection in a relatively hotter or cooler layer of liquid with thermally conducting rigid lower boundary surface and the upper free surface subject to general mixed thermal condition using the Fourier series method, and established that irrespective of the nature of the driving mechanism (surface tension or buoyancy or both) the hotter layer with its heat diffusivity apparently increased as a consequent of actual decrease in its specific heat at constant volume, must exhibit convection at a higher temperature difference, hence more stable, than a cooler layer of the same liquid under identical conditions. In this paper, we study the Rayleigh-Bénard-Marangoni convection in a relatively hotter or cooler layer of liquid whose lower boundary is rigid and upper boundary is free, each subject to constant heat-flux (thermally insulating). The Galerkin method is used to find the eigenvalue equation analytically. We find that the Galerkin method turns out to be simple and gives quite accurate results with minimum of mathematical computations compared to the cumbersome Fourier series method which leads to considerably more algebra. This analysis predicts that the onset of convection in a relatively hotter layer of liquid is more stable than a cooler one under identical conditions, irrespective of whether the two mechanisms causing instability act individually or simultaneously and that the coupling between the two agencies causing instability remains perfect whether the layer of liquid is relatively hotter or cooler.

II. Mathematical formulation of the problem

The physical configuration of the problem consists of an infinite horizontal layer of viscous fluid of uniform thickness d heated from below. The lower rigid boundary of the layer is maintained at a constant temperature $T_0(>0)$ whose upper boundary surface is open to the atmosphere at temperature $T_1 < (T_0)$, each subject to constant heat flux condition. We choose a Cartesian coordinate system of axes with the x and y axes in the plane of the lower surface and the z axis along the vertically upward direction so that the fluid is confined between the planes at $z = 0$ and $z = d$ as shown in Fig 1.

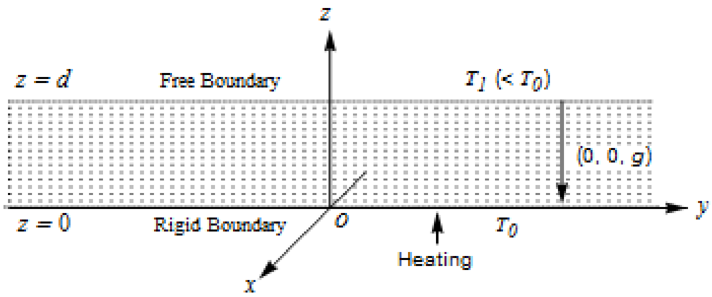

Fig 1: Schematic representation of the physical configuration of the problem.

The surface tension on the upper free surface of the fluid is regarded as a function of temperature only which is given by the simple linear law $\tau = \tau_1 - \sigma(T - T_1)$ where the constant τ_1 is the unperturbed value of τ at the unperturbed surface temperature $T = T_1$ and $-\sigma = (\partial\tau/\partial T)_{T=T_1}$ represents the rate of change of surface tension with temperature, evaluated at temperature T_1, and surface tension being a monotonically decreasing function of temperature, σ is positive.

Following Banerjee et al [12], the modified linearized perturbation equations governing the system under consideration are given as

$$\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\nabla^2 w = g\alpha\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\theta \qquad (1)$$

$$(1 - \alpha_2 T_0)\left(\frac{\partial\theta}{\partial t} - \beta w\right) = \kappa\nabla^2\theta \qquad (2)$$

where the dependent variables w and θ represent respectively the z-component of perturbation velocity and the temperature perturbation. The uniform temperature gradient $\beta\left[= (T_0 - T_1)/d\right]$, the gravitational acceleration g, the coefficient of volume expansion α, the kinematic viscosity ν, the thermal diffusivity κ are each assumed constant,

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

and t represents time. Further, the coefficient α_2 (due to variation in specific heat at constant volume on account of variation in the temperature) lies in the range from 0 to 10^{-4} and that range of the dimensionless parameter $\alpha_2 T_0$ covering the usual laboratory conditions is $0 \le \alpha_2 T_0 < 1$ for liquids with which we are mostly concerned. In this range, any given value of $\alpha_2 T_0 (\ne 0)$ corresponds to the layer of liquid which is relatively hotter compared to that associated with its value less than (including $\alpha_2 T_0 = 0$) the given one (Banerjee et al. [12]). Equations (1)-(2), must be solved subject to appropriate boundary conditions. We confine our attention to boundaries on which the heat flux is kept constant.

Thus, the boundary conditions at z = 0, are

$$w = 0, \quad \frac{\partial w}{\partial z} = 0, \quad \frac{\partial\theta}{\partial z} = 0, \qquad (3a, b, c)$$

and the boundary conditions at z = d, are

$$w = 0, \quad \rho\nu\frac{\partial^2 w}{\partial z^2} - \sigma\nabla_1^2\theta = 0, \quad \frac{\partial\theta}{\partial z} = 0, \qquad (4a, b, c)$$

where $\nabla_1 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

We now analyze an arbitrary disturbance in terms of normal modes assuming that the perturbations w and θ are of the form

$$[w(x, y, z), \theta(x, y, z)] = [w(z), \theta(z)]\exp\left\{i\left(a_x x + a_y y\right) + st\right\}$$

where a_x and a_y are components of the horizontal wave number $a = \sqrt{\left(a_x^2 + a_y^2\right)}$ of the disturbance, and s is the time growth rate (a complex number in general). Using the above expressions for w and θ in equations (1)-(2) and then making the resulting equations dimensionless by choosing d, $d^2/\nu, \nu/d$ and $\beta d(\nu/\kappa)$ as units of length, time, velocity and temperature scales respectively; on putting

$$a_* = ad, \quad W = \frac{wd}{\nu}, \quad \Theta_* = \frac{\theta\kappa R a_*^2}{\beta d\nu}, \quad p_* = \frac{sd^2}{\nu}$$

and omitting asterisks (*) for convenience, we obtain

$$\left(D^2 - a^2\right)\left(D^2 - a^2 - p\right)W = \Theta$$

(5)

$$\left(D^2 - a^2 - pP_r\left(1-\alpha_2 T_0\right)\right)\Theta = -Ra^2\left(1-\alpha_2 T_0\right)W \tag{6}$$

Here, $R = \dfrac{g\alpha\beta d^4}{\kappa\nu}$, $P_r = \dfrac{\nu}{\kappa}$ are respectively the Rayleigh number and Prandtl number.

We restrict our analysis to the case when the principle of exchange of stability is valid for the present problem so that instability first sets in as stationary convection. In this case, the marginal state is characterized by $p = 0$. Then the equation (5) and (6) relevant to marginal stability reduces to

$$\left(D^2 - a^2\right)^2 W = \Theta \tag{7}$$

$$\left(D^2 - a^2\right)\Theta = -Ra^2\left(1-\alpha_2 T_0\right)W \tag{8}$$

In terms of new variables, the non-dimensional form of boundary conditions (3a, b, c) and (4a, b, c) can be written as

$$\left.\begin{aligned} &W(0) = 0,\ DW(0) = 0,\ D\Theta(0) = 0, && \text{at } z = 0, \\ &W(1) = 0,\ D^2W(1) = -\gamma\Theta(1) = 0,\ D\Theta(1) = 0, && \text{at } z = 1. \end{aligned}\right\} \tag{9}$$

where $\gamma = \dfrac{M}{R} = \dfrac{\sigma}{\rho g\alpha d^2}$ with $M = \dfrac{\sigma\beta d^2}{\rho\kappa\nu}$ as the Marangoni number.

The parameter where $\gamma(\geq 0)$ characterizes the strength of surface-tension relative to buoyancy; which depends only on the fluid parameters and on the liquid depth, but not on the external heating Garcia [13] and Zeren [14]

In absence of surface tension ($M = 0$ or $\gamma = 0$) buoyancy is the sole agency causing instability determined by R while in absence of buoyancy ($R = 0$ or $\gamma \to \infty$), surface tension is the sole agency causing instability determined by M. In fact, each instability mechanism causing instability has a non-dimensional number (R or M), but when surface tension and buoyancy act simultaneously they are related by means of the relation $M = \gamma R$, however, for the sake of clarity results will be discussed in terms of usual (R, M) plane.

III. Solution Of The Problem

The Equations (7)-(8) together with boundary conditions (9) constitute an eigenvalue problem of order six. The single term Galerkin method (Finlayson [15]) is convenient for solving the present problem. Accordingly, the unknown variables W and Θ are written as

$$W = AW_1 \text{ and } \Theta = B\Theta_1 \tag{10}$$

in which A and B are constants and W_1 and Θ_1 are the trial functions, which are chosen suitably satisfying the boundary conditions (9). Multiplying equation (8) by W and equation (9) by Θ, integrating the resulting equations with respect to z from 0 to 1 using the boundary conditions (9). Substituting for W and Θ from (10) and eliminating A and B from resulting system of equations, we obtain yield the following eigenvalue equation

$$\begin{vmatrix} \left\langle \left(D^2W\right)^2 + 2a^2\left(DW\right)^2 + a^4\left(W\right)^2 \right\rangle & \left[\gamma DW(1)\Theta(1) - \left\langle W\Theta \right\rangle\right] \\ -Ra^2\left(1-\alpha_2 T_0\right)\left\langle W\Theta \right\rangle & \left\langle \left(D\Theta\right)^2 + a^2\left(\Theta\right)^2 \right\rangle \end{vmatrix} = 0, \tag{11}$$

where $\langle\ \rangle$ denotes integration with respect to z from z = 0 to z = 1, and suffixes have been dropped for simplicity while writing equation (11). The eigenvalue equation (11) may be put in the following form

$$R\left\langle W\Theta\right\rangle\left[\left\langle W\Theta\right\rangle-\gamma DW(1)\Theta(1)\right]=\frac{\left\langle\left(D^2W\right)^2+2a^2\left(DW\right)^2+a^4\left(W\right)^2\right\rangle\left\langle\left(D\Theta\right)^2+a^2\left(\Theta\right)^2\right\rangle}{a^2\left(1-\alpha_2T_0\right)} \quad (12)$$

IV. Result and discussion

We select the trial function

$$W=z^2(1-z)\left[\frac{\gamma}{4}+\frac{1}{24}\left(\frac{3}{2}-z\right)\right] \quad \text{and} \quad \theta=1, \quad (13)$$

Such that they satisfy all the boundary conditions in (9).

It is important to remark here that above choice of the velocity trial function given by (13) is found to be useful for cases in which the two mechanisms (buoyancy and surface tension) causing instability act individually or simultaneously. Substitution of trial functions given by (13) into the eigenvalue equation (12), we obtain

$$R\left(\frac{1}{320}+\frac{\gamma}{48}\right)=\frac{1}{(1-\alpha_2T_0)}\left\{1+\frac{a^2}{15}\left[\frac{\left(\gamma+\frac{1}{8}\right)^2+\frac{1}{448}}{\left(\gamma+\frac{1}{12}\right)^2+\frac{1}{180}}\right]+\frac{a^4}{420}\left[\frac{\left(\gamma+\frac{7}{48}\right)^2+\frac{5}{6912}}{\left(\gamma+\frac{1}{12}\right)^2+\frac{1}{180}}\right]\right\} \quad (14)$$

Case 1. When buoyancy is the sole agency causing instability

By setting $\gamma = 0$ ($M = 0$) in the relation (14), we obtain the case in which buoyancy is the sole agency causing instability. The eigenvalue equation (14) then yields

$$R=\frac{320}{(1-\alpha_2T_0)}\left\{1+\frac{2a^2}{21}+\frac{19a^4}{4536}\right\} \quad (15)$$

and R attain its minimum when a = 0, given by

$$R_c=\frac{320}{(1-\alpha_2T_0)} \quad (16)$$

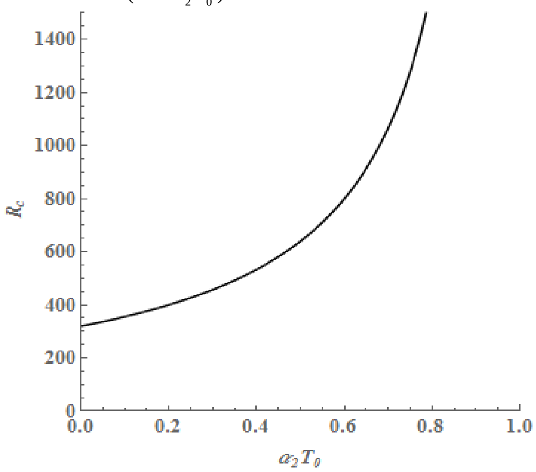

Figure 2: Variation of R_c as function of α_2T_0, when $M = 0$.

When $\alpha_2T_0=0$, then $R_c = 320$ which is exactly same as that obtained by Nield[16]. The effect of increasing values of α_2T_0 on the onset of buoyancy driven convection problem is plotted in Fig 2, using the expression (16). Increasing value of the critical Rayleigh number R_c with α_2T_0 as shown in Fig 2, illustrates that a relatively hotter layer of liquid is more stable than a cooler one under identical conditions.

Case 2. When surface tension is the sole agency causing instability

The relation (4) may be written as follows

$$\frac{R}{320}+\frac{M}{48}=\frac{1}{(1-\alpha_2 T_0)}\left\{1+\frac{a^2}{15}\left[\frac{\left(\gamma+\frac{1}{8}\right)^2+\frac{1}{448}}{\left(\gamma+\frac{1}{12}\right)^2+\frac{1}{180}}\right]+\frac{a^4}{420}\left[\frac{\left(\gamma+\frac{7}{48}\right)^2+\frac{5}{6912}}{\left(\gamma+\frac{1}{12}\right)^2+\frac{1}{180}}\right]\right\} \quad (17)$$

By setting $\gamma \to \infty$ (R = 0) in the relation (17), we obtain the case in which surface tension is the sole agency causing instability. The eigenvalue equation (17) then yields

$$M = \frac{48}{(1-\alpha_2 T_0)}\left\{1+\frac{a^2}{15}+\frac{a^4}{420}\right\} \quad (18)$$

and M attain its minimum when a = 0, given by

$$M_c = \frac{48}{(1-\alpha_2 T_0)} \quad (19)$$

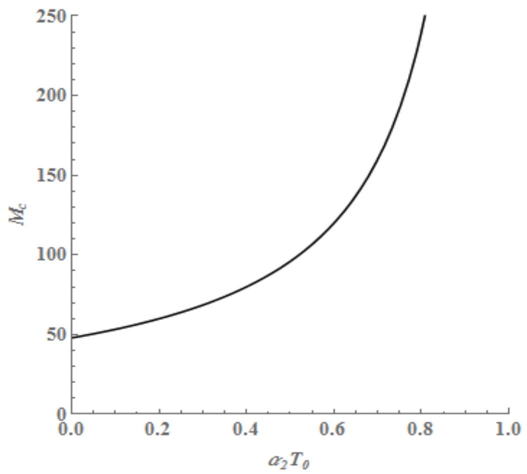

Figure 3: Variation of M_c as function of $\alpha_2 T_0$, when R = 0.

When $\alpha_2 T_0 = 0$, then M_c = 48 which is exactly same as that obtained by Pearson [2] and confirmed by Nield [16]. The effect of increasing values of $\alpha_2 T_0$ on the onset of surface tension driven convection problem is plotted in Fig 3, using the expression (19). Increasing value of the critical Marangoni number M_c with $\alpha_2 T_0$ as shown in Fig 3 illustrates that a relatively hotter layer of liquid is more stable than a cooler one under identical conditions.

Case 3. When both surface tension and buoyancy mechanisms cause instability

For the case when instability is caused by both surface tension and buoyancy mechanisms, from the relation (17) we find that coefficients of both a^2 and a^4 are positive for prescribed values of γ and $\alpha_2 T_0$, hence true minimum of R with respect to a exists at a = 0. In this case, the relation (17) reduces to the following neutral stability condition

$$\frac{R}{320}+\frac{M}{48}=\frac{1}{1-\alpha_2 T_0}. \quad (20)$$

When $\alpha_2 T_0$ = 0, then we have the relationship between R/320 and M/48 previously referred to by Nield [17] and corresponds to maximum reinforcement of the two agencies causing instability. Further, the relation (20) shows that the surface tension and buoyancy effects remain perfectly coupled for a prescribed value of $\alpha_2 T_0$.

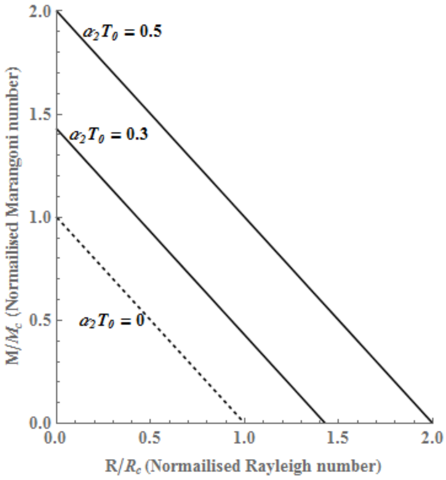

Figure 4: Variation of Marangoni and Rayleigh numbers on the onset of convection for various values of $\alpha_2 T_0$.

The (R, M)-loci corresponding to neutral stability for the combined surface tension and buoyancy driven instability are plotted in Fig 4, using the relation (20) for various values of $\alpha_2 T_0$. Fig 4 clearly demonstrates that a relatively hotter layer of liquid is more stable than a cooler one under identical conditions, and that the coupling between the two mechanisms remains perfect.

V. Conclusion

We conclude that the onset of Rayleigh-Bénard-Marangoni convection in a relatively hotter layer of liquid is more stable than a cooler one under identical conditions, irrespective of whether the two mechanisms causing instability act individually or simultaneously and that the coupling between two mechanisms causing instability remains perfect whether the layer of liquid is relatively hotter or cooler. This analysis will hopefully stimulate further theoretical studies as well as experimental work on this problem.

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