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An approach to Jordan canonical form of similarity

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Abstract: *Every linear transformation on a finite dimensional vector space gives different matrix representation w.r.t different basis. But as they are representation of same linear transformation. Therefore all the matrix representation of the transformation must have common properties of the transformation and must have a canonical representation. Here it is found out such canonical form when the field is algebraically closed. If eign vectors of the transformation can span the vector space, then canonical form will be diagonal matrix with eign values are diagonal elements. But if they not span the vector space, then using two famous well-known theorems (1)* ***Primary Decomposition*** *theorem and (2)* ***Cyclic Decomposition*** *theorem we get the famous* ***Jordan Canonical*** *Form which is the simplest representation of the linear transformation for algebraically closed field.*
Keywords– *Linear transformation, Primary Decomposition, Cyclic decomposition, Jordan Canonical form.*

I. Introduction

Every linear transformation over a finite dimensional vector space over a field represents by a similar class of matrices with respect to different bases. We want to find a canonical form of similarity. When eign vectors span the vector space then canonical form be diagonal matrix but we want to find when they can not span the vector space. Here we find a unique collection of subspaces, direct sum of which is the given vector space and the subspaces spanned by bases which can constructed form eign vectors.

II. General Discussion

With respect to different basis on V_n over F each linear transformation T represents different matrices. Thus the same similarity class must share in common those properties of T that are independent of the choice of basis and therefore are valid in any coordinate system. Thus we find a basis under which T is expressed in the simplest possible way. i.e. canonical matrix representation. Thus we find out such non zero vectors of V_n that are mapped by T in the simplest possible form. Here it is found out such canonical form when the field is algebraically closed. If eign vectors of the transformation can span the vector space, then canonical form will be diagonal matrix with eign values are diagonal elements. But if they not span the vector space, then using two famous wellknown theorem (1) Primary Decomposition theorem and (2) Cycilc Decomposition theorem we get the famous Jordan Canonical form. Now we are starting with to find out such vectors of V that are mapped by T into scalar multiple of itself. i.e. parallel to itself. i.e. $T\xi = \lambda\xi$ for some scalar λ. If B is any ordered basis of V_n and $A = [T]_B$ and X be the coordinate of the vector, then $AX = \lambda X$, $(A - \lambda I)X = 0$, As we want a basis , i.e. some non zero vectors, $\therefore |A - \lambda I| = 0$. The roots λ_i of this equation are called eign value and corresponding non zero vectors X_i are called eign vectors for the eign value λ_i .
If corresponding to all eign values we get n linearly independent vectors, then they form a basis of V_n . i.e. w.r.t the basis $\{X_1 X_2, \ldots\ldots, X_n\}$ the matrix representation of T will be the simplest form and which will be

$$\begin{pmatrix} \lambda_1 & 0 & . & . & 0 \\ 0 & \lambda_2 & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & \lambda_k \end{pmatrix} \qquad \text{where } k \leq n.$$

Again the subspace $w(\lambda_i)$ form by the eign vectors corresponding to λ_i be T invariant and in this case,

$$V_n = w(\lambda_1) \oplus w(\lambda_2) \oplus \dots\dots\dots \oplus w(k).$$

Thus we are now in a stage where we not consider the whole space, only to consider the subspaces, direct sum of gives the total space and the union of the basis of the subspaces gives the basis of V_n. Therefore action of T on the whole space is the sum of the action of T on the subspaces.

But there arises two problems

1. F is not algebraically closed. i. e. characteristic polynomial does not factors completely over F into a product of polynomials of degree one.
2. Even if (1)is possible , there may not be enough characteristic vectors for T to span V_n.

Now we state another direct sum decomposition of V_n, called **Primary Decomposition** theorem. Which states that , if p be minimal polynomial of T,

$$p = {p_1}^{r_1} {p_2}^{r_2} \dots\dots\dots\dots\dots {p_k}^{r_k}$$

where p_i are distinct irreducible monic polynomials over F and r_i are positive integers. If w_i be the null space of $p_i(T)^{r_i}$. Then

1. $V_n = W_1 \oplus W_2 \oplus \dots\dots \oplus W_k$
2. Each W_i is invariant under T ;
3. If T_i is the operator induced on W_i by T , then minimal polynomial for T_i is ${p_i}^{r_i}$.

But here it is difficult to find the basis of W_i. Now we state another decomposition on V_n, called **Cyclic decomposition** theorem and with the help of this two theorem we shall show that if F is algebraically closed then there exist a basis under which T is represented in the simplest possible way. Now Cyclic Decomposition theorem says that, If T be a linear operator on a finite dimensional vector space V and W_0 be a proper T admissible subspace of V. Then there exist nonzero vectors $\alpha_1, \alpha_2, \dots\dots, \alpha_r$ in V with respective T − annihilators $p_1, p_2, \dots\dots, p_r$ such that

1. $V = W_0 \oplus Z(\alpha_1, T) \oplus Z(\alpha_2, T) \oplus \dots\dots\dots Z(\alpha_r, T)$
2. p_k devides p_{k-1} for all $k = 2, \dots, r$.

Furthermore, the integer r and the annihilators $p_1, p_2, \dots\dots, p_r$ are uniquely determined and the fact that no α_k is zero. If we take $W_0 = \{0\}$, Then $V = Z(\alpha_1, T) \oplus Z(\alpha_2, T) \oplus \dots\dots\dots Z(\alpha_r, T)$.

Now with the help of this above two theorems we develop the canonical form and the corresponding basis for algebraically closed field .

Let the characteristic polynomial be ,

$f = (x - \lambda_1)^{d_1} (x - \lambda_2)^{d_2} \dots\dots\dots\dots (x - \lambda_k)^{d_k}$ where $\lambda_1, \lambda_2, \dots\dots \lambda_k$ are distinct elements of F . Then the minimal polynomial for T be,

$p = (x - \lambda_1)^{r_1} (x - \lambda_2)^{r_2} \dots\dots\dots\dots (x - \lambda_k)^{r_k}$ where $1 \le r_i \le d_i$

If W_i be the null space of $(T_i - \lambda_1 I)^{r_i}$, Then by primary decomposition theorem

$$V_n = W_1 \oplus W_2 \oplus \dots\dots \oplus W_k$$

and the operator T_i induced on W_i by T has minimal polynomial $(x - \lambda_i)^{r_i}$. Then $N_i = T_i - \lambda_i I$, is nilpotent on W_i and has minimal polynomial x^{r_i}. Now by cyclic decomposition there exist R_i non zero vectors $\alpha_{i1}, \alpha_{i2}, \dots\dots \alpha_{iR_i}$ of W_i such that $V = Z(\alpha_{i1}, N_i) \oplus Z(\alpha_{i2}, N_i) \oplus \dots\dots\dots Z(\alpha_{iR_i}, N_i)$ with unique N_i − annihilators $p_{i1}, p_{i2}, \dots\dots\dots, p_{iR_i}$ and each $p_{ij} = x^{k_{ij}}$ ($j = 1,2, \dots\dots, R_i$) s.t. $k_{i1} = r_i \ge k_{i2} \ge \dots \ge k_{iR_i}$

Again it can be shown that the R_i vectors $\{{N_i}^{k_{i1}-1}\alpha_{i1}, {N_i}^{k_{i2}-1}\alpha_{i2}, {N_i}^{k_{i1}-1}\alpha_{i1}, \dots\dots\dots\dots\dots\dots, {N_i}^{k_{iR_i}-1}\alpha_{iR_i}\}$ form the basis of the null space of $N_i = T_i - \lambda_i I$. i. e. eign vectors corresponding to the eign value λ_i of T_i.

Taken $\{\xi_{i1}, \xi_{i2}, \dots\dots, \xi_{iR_i}\}$ as eign vectors corresponding to the eign value λ_i.

Then we take ${N_i}^{k_{i1}-1}\alpha_{i1} = \xi_{i1} \Rightarrow {N_i}^{k_{i1}}\alpha_{i1} = N_i\xi_{i1} = (T_i - \lambda_i I)\xi_{i1} = 0$.

$\therefore {N_i}^{k_{i1}}\alpha_{i1} = 0$. thus the set $\{\alpha_{i1}, N_i\alpha_{i1}, \dots \dots, {N_i}^{k_{i1}-1}\alpha_{i1}\}$ of k_{i1} independent vectors span $Z(\alpha_{i1}, N_i)$. As $k_{i1} + k_{i2} + \dots + k_{iR_i} = d_i$, Thus union of such R_i basis form a basis of W_i and union of such basis of W_i form a basis of V_n andunder such basis T is represented in the simplest way.

Now ,

$N_i^{k_{i1}-1}\alpha_{i1} = \xi_{i1} \Rightarrow (T - \lambda_i I)^{k_{i1}-1}\alpha_{i1} = \xi_{i1}$

$N_i^{k_{i1}}\alpha_{i1} = (T - \lambda_i I)\xi_{i1} = 0 \Rightarrow T\xi_{i1} = \lambda_i\xi_{i1}$

Taken, $\alpha_{i1} = \alpha_{i1}^{(1)}, N_i\alpha_{i1} = \alpha_{i1}^{(2)}, N_i^2\alpha_{i1} = \alpha_{i1}^{(3)}, \dots\dots, N_i^{k_{i1}-1}\alpha_{i1} = \alpha_{i1}^{(k_{i1})} = \xi_{i1}$

$\therefore T\alpha_{i1}^{(k_{i1})} = \lambda_i\alpha_{i1}^{(k_{i1})}$,

$\therefore N_i\alpha_{i1}^{(k_{i1}-1)} = \alpha_{i1}^{(k_{i1})} \Rightarrow (T - \lambda_i I)\alpha_{i1}^{(k_{i1}-1)} = \alpha_{i1}^{(k_{i1})} \Rightarrow T\alpha_{i1}^{(k_{i1}-1)} = \lambda_i\alpha_{i1}^{(k_{i1}-1)} + \alpha_{i1}^{(k_{i1})}$,

$\therefore\ N_i\,\alpha_{i1}^{(k_{i1}-2)} = \alpha_{i1}^{(k_{i1}-1)} \Rightarrow (T-\lambda_i I)\alpha_{i1}^{(k_{i1}-2)} = \alpha_{i1}^{(k_{i1}-1)} \Rightarrow T\,\alpha_{i1}^{(k_{i1}-2)} = \lambda_i\,\alpha_{i1}^{(k_{i1}-2)} + \alpha_{i1}^{(k_{i1}-1)}$,

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$\therefore\ N_i\,\alpha_{i1}^{(2)} = \alpha_{i1}^{(3)} \Rightarrow (T-\lambda_i I)\alpha_{i1}^{(2)} = \alpha_{i1}^{(3)} \Rightarrow T\,\alpha_{i1}^{(2)} = \lambda_i\,\alpha_{i1}^{(2)} + \alpha_{i1}^{(3)}$,

$\therefore\ N_i\,\alpha_{i1}^{(1)} = \alpha_{i1}^{(2)} \Rightarrow (T-\lambda_i I)\alpha_{i1}^{(1)} = \alpha_{i1}^{(2)} \Rightarrow T\,\alpha_{i1}^{(1)} = \lambda_i\,\alpha_{i1}^{(1)} + \alpha_{i1}^{(2)}$,

Thus w.r.t the basis $B_{i1} = \{\alpha_{i1}{}^{(1)}, \alpha_{i1}{}^{(2)}, \dots\dots, \alpha_{i1}{}^{(k_{i1})}\}$matrix representation of $Z(\alpha_{i1}, N_i)$ will be

$$J_1^i = \begin{pmatrix} \lambda_i & 0 & . & . & . & 0 \\ 1 & \lambda_i & 0 & . & . & 0 \\ 0 & 1 & \lambda_i & . & . & . \\ . & 0 & 1 & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & \lambda_i \end{pmatrix}$$

which is elementary Jordan matrix with charateristic value λ_i.
Thus w.r.t the basis $B_i = \{B_{i1}, B_{i2}, \dots\dots, B_{iR_i}\}$ of W_i whose dimension is d_i, matrix representation of T_i will be

$$A_i = \begin{pmatrix} J_1^i & 0 & . & . & 0 \\ 0 & J_2^i & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & J_{R_i}^i \end{pmatrix}$$

Finanally matrix representation of w.r.t. the basis $B = \{B_1, B_2, \dots\dots, B_k\}$ of V_n will be

$$A = \begin{pmatrix} A_1 & 0 & . & . & 0 \\ 0 & A_2 & . & . & 0 \\ . & . & . & . & . \\ . & . & . & . & . \\ 0 & 0 & . & . & A_k \end{pmatrix}$$

which is the **Jordan canonical form** that is similar to every matrix which represents T on V_n .

References

[1]. kenneth hoffman & ray kunze, linear algebra, chapter 6 &7