IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 11, Issue 5 Ver. IV (Sep. - Oct. 2015), PP 53-58
www.iosrjournals.org

Bifurcation Analysis for the Delay Logistic Equation with Two Delays

Mohammed O.Mahgoub[1], Mohammed A.Gubara[2]
[1]*(Mathematics Department, College of Science / Majmaa University, Saudi Arabia)*
[2]*(Mathematics Department, College of technology of mathematical sciences/ Alnelain University, Sudan)*

Abstract : *Time-delay dynamical systems are utilized to describe many phenomena of physical interest. Special effort of this paper is given to Hopf bifurcation of time-delay dynamical systems. Linear analysis is carried out to determine the critical conditions under which Hopf bifurcation can occur. Delays being inherent in any biological system, we seek to analyze the effect of delays on the growth of populations governed by the logistic equation. The local stability analysis and local bifurcation analysis of the logistic equation with two delays is carried out considering one of the delays as a bifurcation parameter.*
Keywords: *Logistic equation, delay, stability, Hopf bifurcation.*

I. Introduction

Time delays have been incorporated into biological models to represent resource regeneration times, maturation periods, feeding times, reaction times, etc. by many researchers[1],[2],[3] for discussions of general delayed biological systems. In general, delay differential equations exhibit much more complicated dynamics than ordinary differential equations since a time delay could cause a stable equilibrium to become unstable and cause the populations to fluctuate.

The logistic equation is a simple model of population growth in conditions where there are limited resources. It was formulated by Verhulst in 1838 to describe the self limiting growth of a biological population.

According to the logistic equation, the growth rate of a population is proportional to the current population and the availability of resources in the ecosystem. The logistic equation model is then reads

$$\frac{dx}{dt} = r\,x\,(1 - \frac{x}{K}) \qquad (1)$$

Where x is the population at any instant, $r\,(>0)$ is the intrinsic growth rate and K(> 0) is the carrying capacity of the population.

In practice, the process of reproduction is not instantaneous. For example, in a Daphnia a large clutch presumably is determined not by the concentration of unconsumed food available when the eggs hatch, but by the amount of food available when the eggs were forming, some time before they pass into the broad pouch.

Between this time of determination and the time of hatching many newly hatched animals may have been liberated from the brood pouches of other Daphnia in the culture, so increasing the population. Hutchinson (1948) incorporated the effect of delay into the logistic equation and assumed egg formation to occur τ units of time before hatching and proposed the following more realistic logistic equation

$$\frac{dx}{dt} = r\,x(t)[1 - \frac{x(t-\tau)}{K}] \qquad (2)$$

Many topological changes in the population size like damped oscillations, limit cycles and even chaos may occur as a consequence of delay [4]. The bifurcation analysis of a system with a single delay has been performed in [5]. The importance of two delays in the logistic equation can be found in [6]. In this paper, the effect of such delays will be analyzed, we will investigate local stability, and local bifurcation phenomena taking into account one of the two delays as a bifurcation parameter.

The paper is organized as follows; in section II we discuss model formulation and the accompanying hypothesis. Section III is devoted to discuss the stability analysis and section IV for the Hopf bifurcation analysis. A brief conclusion is given in section V.

II. Model Formulation

Milind and Preetish [7], in their paper describe the model employed with the accompanying assumptions for a single delay system and a two delays system.

A. Logistic equation with single delay

The model which has been employed to characterize population dynamics is as follows:

- The equation is normalized.

- The growth rate is r which is finite, positive and time independent.
- The delay τ_1 is also finite and positive.

With these assumptions, the single delay logistic equation simplifies to

$$\frac{dx}{dt} = r\,x(t)[1 - b\,x(t-\tau_1)] \qquad (3)$$

B. Logistic equation with multiple delays

The same assumptions as made in the single delay logistic equation may be applied here as well. In the case of a system with n delays, the equation takes the form

$$\frac{dx}{dt} = r\,x(t)[1 - \sum_{i=1}^{n} b_i\,x(t-\tau_i)] \qquad (4)$$

The case of two delays is investigated in this paper taking into consideration one of the two delays as a bifurcation parameter.

III. Stability Analysis

As it is well known the delay logistic equation is non-linear. We first linearize it then proceed to determine conditions for the stability of both the single delay logistic equation as well as the logistic equation with two delays.

A. Stability analysis of logistic equation with single delay

Let the equilibrium point (where the growth rate is zero) be denoted by x^*. Then $x(t) = x(t-\tau_1) = x^*$

So, there are two equilibrium points $x^* = 0$ (describing the initial interval) and $x^* = \frac{1}{b}$ (describing the saturating intervals).

Now after the linearization of the delay logistic equation about the first equilibrium point we reveal that $x^* = 0$ is unstable and this equilibrium point is not of much interest.

The second equilibrium point $x^* = \frac{1}{b}$ is of greater interest and after linearizing about it we get

$$\frac{dy}{dt} = -r\,y(t-\tau_1)$$

Its characteristic equation is therefore,

$$\lambda + r\,e^{-\lambda\tau_1} = 0 \qquad (5)$$

The system will be stable if $(\lambda < 0)$.Bifurcation point is the value of the parameters for which the values of λ lie on the imaginary axis.

Substituting $\lambda = i\theta$ into (5) we find,

$$i\theta + r[\cos(\theta\tau_1) - i\sin(\theta\tau_1)] = 0$$

After equating real parts in both sides, we get

$$\theta\tau_1 = (2n+1)\frac{\pi}{2}, \qquad n = 0,1,2,... \qquad (6)$$

And after equating imaginary parts in both sides, we get

$$\theta - r\sin(\theta\tau_1) = 0$$

To find the first solution we replace $n = 0$ in (6), so

$$\frac{\pi}{2} - r\,\tau_1 = 0$$

For $\tau_1 = 0$ the characteristic equation is $\lambda = -r$, which implies that $\lambda < 0$ because r is assumed positive and the system is stable. i.e., for $r\,\tau_1 = 0$ the roots of the characteristic equation are still negative (< 0) .Therefore, necessary and sufficient condition for stability is,

$$r\,\tau_1 < \frac{\pi}{2} \qquad (7)$$

It is very clear that if the system has no delays or $\tau_1 = 0$, the system is always stable.

B. Stability analysis of logistic equation with two delays

At the fixed points, $x(t) = x(t-\tau_1) = x(t-\tau_2) = x^*$, so we have

$$x^* = 0 \quad \text{or} \quad x^* = \frac{1}{b_1 + b_2}$$

We have seen in the previous subsection that the first equilibrium is unstable and is not of much interest. We investigate now the second value, and after linearizing the delay logistic equation about it we get

$$\frac{dy}{dt} = -r\,x^*\,[b_1\,y\,(t-\tau_1) + b_2\,y\,(t-\tau_2)] \qquad (8)$$

If $b_1 \gg b_2$, then the second delay term in (8) can be neglected and the analysis reduces to the case of a single delay system.

Consider the case when $b_1 = b_2$, the characteristic equation therefore is

$$\lambda + \frac{r}{2}e^{-\lambda\tau_1} + \frac{r}{2}e^{-\lambda\tau_2} = 0 \qquad (9)$$

The system will be stable if $(\lambda < 0)$ .Bifurcation point is the value of the parameters for which the values of λ lie on the imaginary axis.

Substituting $\lambda = i\theta$ into (9), we find

$$i\theta + \frac{r}{2}[\cos(\theta\tau_1) - i\sin(\theta\tau_1)] + \frac{r}{2}[\cos(\theta\tau_2) - i\sin(\theta\tau_2)] = 0 \qquad (10)$$

After equating real part to zero, we get

$$\theta = \frac{2(\pm\frac{\pi}{2} + 2\pi\,C_1)}{\tau_1 + \tau_2} \qquad , C_1 \in Z \qquad (11)$$

And after equating imaginary part to zero and using the value of θ obtained in (11), we get the bifurcation point at

$$r = \frac{\pi}{(\tau_1+\tau_2)\cos[\frac{\pi(\tau_1-\tau_2)}{2(\tau_1+\tau_2)}]} \qquad (12)$$

Where $\quad (\tau_1+\tau_2)\cos[\frac{\pi(\tau_1-\tau_2)}{2(\tau_1+\tau_2)}] \neq 0$

And

$$\tau_1 = \frac{\pi}{r\cos[\frac{\pi(\tau_1-\tau_2)}{2(\tau_1+\tau_2)}]} - \tau_2 \qquad (13)$$

On simplifying (13) we get the bifurcation point at

$$\tau_1 = \frac{\pi}{r\sin[\frac{\pi\,\tau_2}{2(\tau_1+\tau_2)}]} - \tau_2 \qquad (14)$$

Therefore, necessary and sufficient condition for stability is,

$$r(\tau_1+\tau_2)\cos[\frac{\pi(\tau_1-\tau_2)}{2(\tau_1+\tau_2)}] < \pi \qquad r, \tau_1, \tau_2 > 0 \qquad (15)$$

Substituting $\tau_1 = \tau_2 = \tau$, we get the familiar condition (7)

IV. Bifurcation Analysis

In this section, it is shown that the delay logistic equation undergoes a Hopf bifurcation at a critical value of the parameters growth rate (r) or delay (τ_1) for the single delay case [7], and at a critical value of the parameters growth rate (r) or one of the two delays (τ_1) or (τ_2) for the two delays case.

A. Hopf bifurcation analysis for the single delay case

From (7), at the critical point for the single delay case, $r\tau_1 = \frac{\pi}{2}$. Differentiating characteristic equation (5) with respect to the growth rate, we get:

$$\frac{d\lambda}{dr} = \frac{e^{-\lambda\tau_1}}{r\tau_1 e^{-\lambda\tau_1} - 1}$$

Evaluating at critical point $r^* = \frac{\pi}{2\tau_1}$,

$$\mathrm{Re}\{\frac{d\lambda}{dr}\}_{r=r^*} = \frac{\frac{\pi}{2}}{\frac{\pi^2}{4}+1} > 0 \qquad (16)$$

Similarly differentiating (5) with respect to the delay, we obtain:

$$\frac{d\lambda}{d\tau_1} = \frac{\lambda r e^{-\lambda\tau_1}}{1 - r\tau_1 e^{-\lambda\tau_1}}$$

Evaluating at critical point $\tau_1^* = \frac{\pi}{2r}$,

$$\mathrm{Re}\{\frac{d\lambda}{d\tau_1}\}_{\tau_1=\tau_1^*} = \frac{r^2}{\frac{\pi^2}{4}+1} > 0 \qquad (17).$$

The transversality condition of the Hopf bifurcation with respect to the growth rate and time delay is satisfied in (16) and (17) respectively. Thus, logistic equation with single delay undergoes a Hopf bifurcation at the critical point given by $r\tau_1 = \frac{\pi}{2}$

B. Hopf bifurcation analysis for the two delays case

B.1 the bifurcation parameter is the growth rate (r):

Consider the logistic equation with two delays given by (9). Differentiating (9) with respect to the growth rate we get,

$$\frac{d\lambda}{dr} = \frac{e^{-\lambda\tau_1} + e^{-\lambda\tau_2}}{r\tau_1 e^{-\lambda\tau_1} + r\tau_2 e^{-\lambda\tau_2} - 2}$$

Evaluating at critical point $r^* = \frac{\pi}{(\tau_1+\tau_2)\cos[\frac{\pi(\tau_1-\tau_2)}{2(\tau_1+\tau_2)}]}$, we get

$$\mathrm{Re}\{\frac{d\lambda}{dr}\}_{r=r^*} = \frac{r(\tau_1+\tau_2)+\pi}{r^2\tau_1^2 + r^2\tau_2^2 + \frac{2\pi r\tau_1\tau_2}{\tau_1+\tau_2}} > 0 \qquad (18)$$

Hence the transversality condition of the Hopf bifurcation is satisfied with respect to the growth rate.

It is important to realize that in this paper we assume that both of the delays are equally important, so we cannot scale time to obtain one of the delays equal to 1.

B.2 the bifurcation parameter is the first delay (τ_1):

Differentiating (9) with respect to the first delay τ_1 we get,

$$\frac{d\lambda}{d\tau_1} = \frac{r\lambda e^{-\lambda\tau_1}}{2 - r(\tau_1 e^{-\lambda\tau_1} + \tau_2 e^{-\lambda\tau_2})} \qquad (19)$$

Recall that from equality of real and imaginary parts to zero in (10) we found,

$$\cos\theta\tau_2 = -\cos\theta\tau_1 \qquad \sin\theta\tau_2 = \frac{2\theta}{r} - \sin\theta\tau_1$$

Using these together with $\lambda = i\theta$, (19) becomes,

$$\frac{d\lambda}{d\tau_1} = \frac{ir\theta(\cos\theta\tau_1 - i\sin\theta\tau_1)}{2 - r[\tau_2(-\cos\theta\tau_1 - i(\frac{2\theta}{r} - \sin\theta\tau_1)) + \tau_1(\cos\theta\tau_1 - i\sin\theta\tau_1)]} \qquad (20)$$

And on simplifying (20), we get

$$\frac{d\lambda}{d\tau_1} = -\frac{ir\theta}{r(\tau_1 - \tau_2) + (-2 - 2i\theta\tau_2)(\cos\theta\tau_1) + 2(-i + \theta\tau_2)(\sin\theta\tau_1)} \qquad (21)$$

So the real part of (21) is,

$$\mathrm{Re}\{\frac{d\lambda}{d\tau_1}\} = \frac{r\theta(2\theta\tau_2\cos\theta\tau_1 + 2\sin\theta\tau_1)}{[r(\tau_1 - \tau_2) - 2\cos\theta\tau_1 + 2\theta\tau_2\sin\theta\tau_1]^2 + [2\theta\tau_2\cos\theta\tau_1 + 2\sin\theta\tau_1]^2} \qquad (22)$$

According to Hopf classical theorem, the system undergoes a Hopf bifurcation at the critical value τ_1^* given by (14) if the value of $\mathrm{Re}\{\frac{d\lambda}{d\tau_1}\}$ evaluated at τ_1^* is not equal to zero.

It is very clear that the denominator of (22) is always positive whatever the value of τ_1, so we confine ourselves to the numerator only.

So replacing the value of θ from (11) and the critical value of the bifurcation parameter τ_1^* from (14), the numerator becomes

$$\frac{2r\pi\left[\pi\tau_2\cos\left[\frac{\pi^2\csc(\frac{\pi\tau_2}{\tau_1+\tau_2}) - r\pi\tau_2}{r(\tau_1+\tau_2)}\right] + (\tau_1+\tau_2)\sin\left[\frac{\pi^2\csc(\frac{\pi\tau_2}{\tau_1+\tau_2}) - r\pi\tau_2}{r(\tau_1+\tau_2)}\right]\right]}{(\tau_1+\tau_2)^2} \qquad (23)$$

Now it is clear that the denominator of (23) is always positive and in addition if,

$$0 < \frac{\pi^2\csc(\frac{\pi\tau_2}{\tau_1+\tau_2}) - r\pi\tau_2}{r(\tau_1+\tau_2)} < \frac{\pi}{2} \qquad (24)$$

The numerator of (23) will also be positive.

Recall from (14) that $\pi\csc(\frac{\pi\tau_2}{\tau_1+\tau_2}) = r(\tau_1+\tau_2)$, so inequality (24) becomes

$$0 < 2\tau_1 < (\tau_1+\tau_2) \qquad (25)$$

So (22) will be positive at the critical value of the bifurcation parameter τ_1^* if the condition (25) is satisfied.

Hence the transversality condition of the Hopf bifurcation is satisfied with respect to the first delay τ_1 if (25) is hold. Thus, the two delay logistic equation undergoes Hopf bifurcation with respect to the first delay τ_1 at the critical point given by (14).

B.3 the bifurcation parameter is the second delay (τ_2):

From (14) the critical value of τ_2 as a bifurcation parameter is

$$\tau_2^* = \frac{\pi}{r\sin[\frac{\pi\tau_2}{2(\tau_1+\tau_2)}]} - \tau_1 \qquad (26)$$

Following the same manner, (22) will be positive at the critical value of the bifurcation parameter τ_2^* given by (26) if the next condition is hold

$$0 < 2\tau_2 < (\tau_1+\tau_2) \qquad (27)$$

Hence the transversality condition of the Hopf bifurcation is satisfied with respect to the second delay τ_2 if (27) is hold. Thus, the two delay logistic equation undergoes Hopf bifurcation with respect to the second delay τ_2 at the critical point given by (26).

V. Conclusion

We have performed local stability analysis of the delay logistic equation with and without multiple time delays. Sufficient conditions to aid design were also extracted. While the ideal system (without delay) is always perfectly stable, the actual system that we have considered using delays undergoes a Hopf bifurcation for both parameters, the growth rate and each of the two delays.

The paper can be considered as a base for future work concerning delay logistic equation with more than two delays.

References

[1] E. H. Ait Dads and K. Ezzinbi (2002), Boundedness and almost periodicity for some state- dependent delay differential equations, Electron. J. Differential Equations 2002, No. 67, pp.1-13.

[2] W. C. Allee (1931), Animal Aggregations: A Study in General Sociology, Chicago University Press, Chicago.

[3] O. Arino and M. Kimmel (1986), Stability analysis of models of cell production systems, Math. Modelling 7, 1269-1300.

[4] S. H. Strogatz, Nonlinear Dynamics and Chaos. Perseus Books, 1994.

[5] C. Sun, M. Han and Y. Lin, "Analysis of stability and Hopf bifurcation for a delayed logistic equation," Chaos, Solitons and Fractals, vol. 31, pp. 672-682, 2007.

[6] R. L. Kitching , Time, resources and population dynamics in insects, Australian Journal of Ecology vol. 2, pp. 31-42, 1997.

[7] M. M. Rao and Preetish K. L., Stability and Hopf Bifurcation Analysis of the Delay Logistic Equation, Cornell university library, arXiv: 1211.7022v1 [q-bio.PE] 29 Nov 2012.