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Co – Isolated Locating Domination Number For Unicyclic Graphs

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Abstract : *Let G (V, E) be a simple, finite, undirected connected graph. A non – empty set $S \subseteq V$ of a graph G is a dominating set, if every vertex in V – S is adjacent to atleast one vertex in S. A dominating set $S \subseteq V$ is called a locating dominating set, if for any two vertices v, w $\in$ V – S, $N(v) \cap S \neq N(w) \cap S$. A locating dominating set $S \subseteq V$ is called a co – isolated locating dominating set, if there exists atleast one isolated vertex in <V – S >. The co – isolated locating domination number γ_{cild} is the minimum cardinality of a co – isolated locating dominating set. In this paper, the number γ_{cild} is obtained for unicyclic graphs.*
Keywords: *Dominating set, locating dominating set, co – isolated locating dominating set, co – isolated locating domination number.*

I. Introduction

Let G = (V, E) be a simple graph of order p. For v $\in$V(G), the neighborhood $N_G(v)$ (or simply N(v)) of v is the set of all vertices adjacent to v in G. For a connected graph G, the eccentricity $e_G(v)$ of a vertex v in G is the distance to a vertex farthest from v. Thus, $e_G(v) = \{d_G(u, v) : u \in V(G)\}$, where $d_G(u, v)$ is the distance between u and v in G. The minimum and maximum eccentricities are the radius and diameter of G, denoted r(G) and diam(G) respectively. A pendant vertex in a graph G is a degree of vertex one and a vertex is called a support if it is adjacent to a pendant vertex. A unicyclic graph G is a graph with exactly one cycle. The concept of domination in graphs was introduced by Ore [1]. A non – empty set S ⊆ V(G) of a graph G is a dominating set, if every vertex in V(G) – S is adjacent to some vertex in S. A special case of dominating set S is called a locating dominating set. It was defined by D. F. Rall and P. J. Slater in [2]. A dominating set S in a graph G is called a locating dominating set in G, if for any two vertices v, w ∈ V(G) – S, $N_G(v) \cap S$, $N_G(w) \cap S$ are distinct. The locating dominating number of G is defined as the minimum number of vertices in a locating dominating set in G. A locating dominating set S ⊆ V(G) is called a co – isolated locating dominating set , if <V – S> contains atleast one isolated vertex. The minimum cardinality of a co – isolated locating dominating set is called the co – isolated locating domination number $\gamma_{cild}(G)$. In this paper, the unicyclic graphs having co – isolated locating domination number $\gamma_{cild}(G)$ = 3, 4, and 5 are characterized.

II. Prior Results

The following results are obtained in [3], [4], [5] & [6]

Theorem 2.1[3]:

For every non – trivial simple connected graph G with p vertices, $1 \leq \gamma_{cild}(G) \leq p - 1$.

Theorem 2.2[3]:

$\gamma_{cild}(G) = 1$ if and only if $G \cong K_2$.

Observation 2.3 [3]:

If S is a co – isolated locating dominating set of G(V, E) with | S | = k, then V(G) – S contains atmost $pC_1 + pC_2 + \ldots + pC_k$ vertices.

Theorem 2.4 [3]:

$\gamma_{cild}(G) = p - 1 (p \geq 4)$ if and only if V(G) can be partitioned into two sets X and Y such that one of the sets X and Y say, Y is independent and each vertex in Y and the subgraph < X > of G induced by X is one of the following

(a) < X > is a complete graph
(b) < X > is totally disconnected
(c) Any two non – adjacent vertices in V(< X >) have common neighbours in < X >.

Theorem 2.5 [4]:

$\gamma_{cild}(G) = 2$ if and only if G is one of the following graphs

(a) P_p (p = 3, 4, 5), where P_p is a path on p vertices.
(b) C_p (p = 3, 5), where C_p is a path on p vertices.
(c) C_5 with a chord.
(d) G is the graph obtained by attaching a pendant edge at a vertex of C_3 (or) at a vertex of degree 2 in K_4 – e.
(e) G is the graph obtained by attaching a path of length 2 at a vertex of C_3.
(f) G is the Bull graph.

Theorem 2.6 [5]:
For a path P_p on p vertices,
$\gamma_{cild}(P_p) = \left\lfloor \frac{2p+4}{5} \right\rfloor$, p ≥ 3.

Theorem 2.7 [6]:
If C_p (p ≥ 3) is a cycle on p vertices, then $\gamma_{cild}(C_p) \leq \left\lceil \frac{2p}{5} \right\rceil$.

III. Main Results

In the following, the unicyclic graphs having co–isolated locating domination number $\gamma_{cild}(G)$ = 3, 4 and 5 are characterized.

Notations 3.1:

1. $C_p @ P_k$ is a graph obtained by attaching a path of length k at exactly one vertex of C_p.

Example 3.1.1: The graph G ≅ $C_4 @ P_3$ is given in Fig. 3.1.

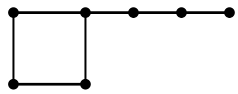
G ≅


Fig. 3.1.

2. $C_p @ P_{k_1} @_r P_{k_2}$ is a graph obtained by attaching paths of length k_1 and k_2 respectively at vertices u and v of C_p such that d(u, v) = r.

Example 3.1.2: The graph G ≅ $C_5 @ P_3 @_2 P_2$ is given in Fig. 3.2.

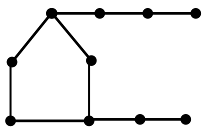
G ≅


Fig. 3.2.

3. $C_p @ P_{k_1} @_r P_{k_2} @_s P_{k_3}$ is a graph obtained by attaching paths of length k_1, k_2 and k_3 respectively at vertices u, v and of C_p such that d(u, v) = r; d(v, w) = s.

Example 3.1.3: The graph G ≅ $C_8 @ P_2 @_2 P_4 @_3 P_3$ is given in Fig. 3.3.

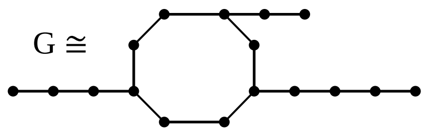
G ≅


Fig. 3.3.

4. $C_p @ P_{k_1} @_q P_{k_2} @_r P_{k_3} @_s P_{k_4}$(n ≥ 4) is a graph obtained by attaching paths of length k_1, k_2, k_3 and k_4 respectively at vertices u, v, w and x on C_p such that d(u, v) = q; d(v, w) = r; d(w, x) = s.

Example3.1.4: The graph G ≅ $C_8 @ P_2 @_2 P_4 @_3 P_3 @_1 P_1$ is given in Fig. 3.4.

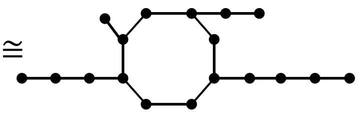
G ≅


Fig. 3.4.

5. $C_p @_s P_k$ is a graph obtained by attaching a support of a path of length k at a vertex of C_p.

Example 3.1.5: The graph G ≅ $C_4 @_s P_5$ is given in Fig. 3.5.

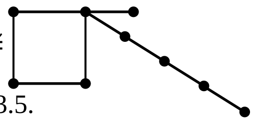
G ≅ 

Fig. 3.5.

6. $C_p\ @_c\ P_k$ is a graph obtained by attaching the central vertex of a path of length k (k is even) at a vertex of C_p.

Example 3.1.6: The graph $G \cong C_8\ @_c\ P_4$ is given in Fig. 3.6.

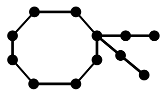
G ≅ 

Fig. 3.6.

7. $C_p\ @\ P_1\ @_{es}\ P_k$ is a graph obtained by attaching a path of length one at a vertex of C_p and then attaching a support of a path of length k to the pendant vertex of P_1.

Example 3.1.7: The graph $G \cong C_5\ @\ P_1\ @_{es}\ P_3$ is given in Fig. 3.7.

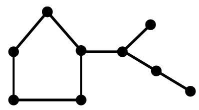
G ≅ 

Fig. 3.7.

8. $C_p\ @\left(\begin{matrix} P_1 \\ P_{k_1} @_s P_k \end{matrix}\right)$ is a graph obtained by attaching a path of length 1 and also a path of length k_1 at a vertex of C_p and then attaching a support of path of length k at a pendant vertex of the path P_{k_1}.

Example 3.1.8: The graph $G \cong C_3\ @\left(\begin{matrix} P_1 \\ P_1\ @_{es} P_3 \end{matrix}\right)$ is given in Fig. 3.8.

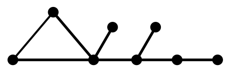
G ≅ 

Fig. 3.8.

Example 3.1.9: $G \cong C_5\ @\left(\begin{matrix} P_1 \\ P_2\ @_{es} P_4 \end{matrix}\right)$ is given in Fig. 3.9.

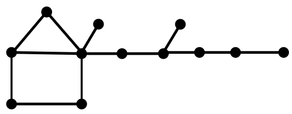
G ≅ 

Fig. 3.9.

9. A graph can also be obtained by performing the combinations of the above operations.

Example3.1.10: The graphs $G \cong C_6\ @\ P_1\ @_2\ P_1\ @_{2s}\ P_3$ and

$G \cong C_4\ @\left(\begin{matrix} P_1 \\ P_2\ @_{ec} P_6 \end{matrix}\right) @_2\ P_2$. are given in Fig. 3.10.

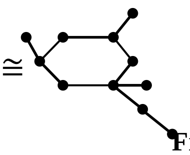
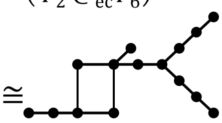
G ≅  G ≅ 

Fig. 3.10

Theorem 3.2:

For a connected unicyclic graph G, $\gamma_{cild}(G) = 2$ if and only if G is one of the graphs in the family $\mathcal{A}$, where $\mathcal{A} = \{ C_3, C_5, C_3\ @\ P_1, C_3\ @\ P_2\}$

Proof:

If G is one of the graphs of $\mathcal{A}$, then $\gamma_{cild}(G) = 2$.

Conversely, assume that $\gamma_{cild}(G) = 2$. Let S = {a, b} be a γ_{cild} – set of G with |S| = 2. Then $|V – S| \leq 2^2 – 1 = 3$ and <V – S> contains atleast one isolated vertex.

Case (1): |V – S| = 1
If <S> ≅ $2K_1$, then G is not unicyclic.
If <S> ≅ K_2, then G ≅ C_3.

Case (2): |V – S| = 2
Let V – S = {x_1, x_2}. Then <V – S> ≅ $2K_1$.
If $N(x_1) \cap S$ = {a, b} $N(x_2) \cap S$ = {a} (or) {b},then G ≅ C_3 @ P_1.
In all the other cases, G is not unicyclic.

Case (3): |V – S| = 3
Let V – S = {x_1, x_2, x_3} and $N(x_1) \cap S$ = {a}; $N(x_2) \cap S$ = {b} and $N(x_2) \cap S$ = {a, b}.

Subcase(3.a): $x_2x_3 \in$ <V – S> and x_1 is isolated in <V – S>
If ab ∉ E(G), then G ≅ C_3 @ P_2 and if ab ∈ E(G), then G is not unicyclic.

Subcase(3.b): x_1, x_2 and x_3 are all isolated in <V – S>.
If ab ∉ E(G), then G ≅ C_5 and if ab ∈ E(G), then G is not unicyclic.
Hence the theorem follows.

Notation 3.3:
The family of graphs $\mathcal{B}$ = {$\mathcal{B}_1$, $\mathcal{B}_2$, ..., $\mathcal{B}_8$, $\mathcal{B}_9$} are defined as follows, where

$\mathcal{B}_1$ = {$B_{1,1}$, $B_{1,2}$, ..., $B_{1,5}$, $B_{1,6}$ } = { C_6, C_6 @ P_1, C_7, C_6 @ P_2, C_7 @ P_1, C_6 @ P_3 }
$\mathcal{B}_2$ = {$B_{2,1}$, $B_{2,2}$, $B_{2,3}$, $B_{2,4}$} = { C_5 @ P_1 ,C_5 @ P_1 @ P_1, C_5 @ P_1 $@_2$ P_1, C_5 @ P_1 @ P_1 $@_2$ P_1}
$\mathcal{B}_3$ = {$B_{3,1}$, $B_{3,2}$, ..., $B_{3,6}$} = { C_4, C_4 @ P_1, C_4 @ P_1 @ P_1, C_4 @ P_1 $@_2$ P_1, C_3 @ P_1 $@_s$ P_2, C_4 @ P_1 @P_1 @ P_1}
$\mathcal{B}_4$ = {$B_{4,1}$} = { C_3 @ P_1 @ P_1 @ P_1 }
$\mathcal{B}_5$ = {$B_{5,1}$, $B_{5,2}$, ..., $B_{5,7}$} = { C_3 $@_s$ P_3, C_3 @ P_1 $@_{es}$ P_3, C_3 @ P_2 @ P_2, C_3 @ P_4, C_3 @ P_1 $@_{es}$ P_4,C_3 @ P_1 @ P_2 @ P_2,C_3 @ P_3 @ P_2}
$\mathcal{B}_6$ = { $B_{6,1}$, $B_{6,2}$ } = { C_3 @ P_1 $@_{es}$ P_2, C_3 @ P_1 $@_{es}$ P_3 }
$\mathcal{B}_7$ = { $B_{7,1}$, $B_{7,2}$, ..., $B_{7,7}$ } = { C_4 @ P_2, C_4 @ P_1 @ P_2, C_5 @ P_2, C_5 @ P_1 @ P_1, C_4 @ P_1 @ P_2 @ P_1, C_5 @ P_1 @ P_2, C_5 @ P_3 }
$\mathcal{B}_8$ = { $B_{8,1}$, $B_{8,2}$, $B_{8,3}$ } = { C_3 @ P_1 @ P_2, C_3 @ P_1 @ P_3, C_3 @ P_1 $@_s$ P_3 }and
$\mathcal{B}_9$ = { $B_{9,1}$, $B_{9,2}$, $B_{9,3}$} = { C_3 $@_s$ P_2, C_3 $@_s$ P_3, C_3 @ P_1 $@_s$ P_3 }

Theorem 3.4:
For a connected unicyclic graph G, $\gamma_{cild}(G) = 3$ if and only if G is one of the graphs in the family $\mathcal{B}$.
Proof:
If G is one of the graphs in the family $\mathcal{B}$, then $\gamma_{cild}(G) = 3$.
Conversely, let S be a γ_{cild} – set of a unicyclic graph G with |S| = 3 and therefore
$|V – S| \leq 2^3 – 1 = 7$.

Case(1): All the vertices of S lie on the cycle.
Then <S> ≅ $3K_1$, $K_1 \cup K_2$, P_3 (or) C_3.

Subcase(1.a.): <S> ≅ $3K_1$
Since all the vertices of S lie on the cycle and <S> ≅ $3K_1$, the cycle in this case is C_6 (or) C_7. Hence, $3 \leq |V – S| \leq 6$.

(i) $|V - S| = 3$
If $<V - S> \cong 3K_1$, then $G \cong B_{1,1}$
If $<V - S> \cong K_1 \cup K_2$, then G is not unicyclic.
(ii) $|V - S| = 4$
If $<V - S> \cong 4K_1$, then $G \cong B_{1,2}$
If $<V - S> \cong 2K_1 \cup K_2$, then $G \cong B_{1,3}$.
(iii) $|V - S| = 5$
If $<V - S> \cong 5K_1$, then $G \cong B_{1,4}$.
If $<V - S> \cong 3K_1 \cup K_2$, then $G \cong B_{1,5}$.
(iv) $|V - S| = 6$
If $<V - S> \cong 6K_1$, then $G \cong B_{1,6}$.
If $<V - S> \cong 4K_1 \cup K_2$, then $G \cong C_7 @ P_2$ and for this graph $\gamma_{cild}(G) = 4$.
(v) $|V - S| = 7$, then either G is not unicyclic or S will not be a γ_{cild} – set of G.

Subcase(1.b)*:* $<S> \cong K_1 \cup K_2$
The cycle in this case is C_5 (or) C_6.
(i) $|V - S| = 3$
If $<V - S> \cong 3K_1$, then $G \cong B_{2,1.}$
If $<V - S> \cong K_1 \cup K_2$, then $G \cong B_{1,1}$.
(ii) $|V - S| = 4$
If $<V - S> \cong 4K_1$, then $G \cong B_{2,2}$ (or) $B_{2,3}$.
If $<V - S> \cong 2K_1 \cup K_2$, then $G \cong B_{1,2}$.
(iii) $|V - S| = 5$
If $<V - S> \cong 5K_1$, then $G \cong B_{2,4}$.
If $<V - S> \cong 3K_1 \cup K_2$, then S will not be a γ_{cild} – set of G.
(iv) $|V - S| = 6$ (or) 7, then G is not unicyclic.

Subcase(1.c): $<S> \cong P_3$
The cycle in this case is C_3 (or) C_4.
(i) $|V - S| = 1$
If $<V - S> \cong K_1$, then $G \cong B_{3,1.}$
(ii) $|V - S| = 2$
If $<V - S> \cong 2K_1$, then $G \cong B_{3,2}$.
(iii) $|V - S| = 3$
If $<V - S> \cong 3K_1$, then $G \cong B_{3,3,}$ $B_{3,4}$ (or) $B_{3,5}$.
If $<V - S> \cong K_1 \cup K_2$, then $G \cong B_{2,1}$.
(v) $|V - S| = 4$
If $<V - S> \cong 4K_1$, then $G \cong B_{3,6}$.
If $<V - S> \cong 2K_1 \cup K_2$, then G is not unicyclic.
(vi) $|V - S| = 5$ (or) 6 (or) 7, then G is not unicyclic.

Subcase(1.d): $<S> \cong C_3$
If $|V - S| = 1$ (or) 2, then S will not be a γ_{cild} – set of G. If $|V - S| > 3$, then G is not unicyclic. Hence $|V - S| = 3$.
If $<V - S> \cong 3K_1$, then $G \cong B_{4,1.}$

Case(2): One vertex of S lie on the cycle and the other two vertices does not lie on the cycle.
The only cycle with this property is C_3. Also, $<S> \cong 3K_1$ (or) $K_1 \cup K_2$.
Subcase(2.a): $<S> \cong 3K_1$
Then $<V - S>$ must contain K_2 to form C_3. Also, $<V - S>$ must have atleast one isolated vertex. Therefore $|V - S| \geq 3$.
(i) $|V - S| = 3$

If $<V - S> \cong K_1 \cup K_2$, then $G \cong B_{5,1}$.
(ii) $|V - S| = 4$
If $<V - S> \cong 2K_1 \cup K_2$, then $G \cong B_{5,2}$ (or) $B_{5,3}$ (or) $\mathcal{B}_{5,4}$.
(iii) $|V - S| = 5$
If $<V - S> \cong 3K_1 \cup K_2$, then $G \cong B_{5,5}$ (or) $B_{5,6}$ (or) $B_{5,7}$.
(iv) $|V - S| = 6$ (or) 7, then G is not unicyclic.

Subcase(2.b): $<S> \cong K_1 \cup K_2$
By a similar argument as in Subcase(2.a), $|V - S| \geq 3$.
(i) $|V - S| = 3$
If $<V - S> \cong K_1 \cup K_2$, then $G \cong B_{6,1}$.
(ii) $|V - S| = 4$
If $<V - S> \cong 2K_1 \cup K_2$, then $G \cong B_{6,2}$.
(iii) $|V - S| = 5$ (or) 6 (or) 7, then G is not unicyclic.

Case(3): Two vertices of S lie on the cycle and the other vertex does not lie on the cycle.
In this case, $<S> \cong 3K_1$ (or) $K_1 \cup K_2$ (or) P_3.

Subcase(3.a): $<S> \cong 3K_1$
(i) $|V - S| = 3$
If $<V - S> \cong 3K_1$, then $G \cong B_{7,1}$.
If $<V - S> \cong K_1 \cup K_2$, then is not unicyclic.
(ii) $|V - S| = 4$
If $<V - S> \cong 4K_1$, then $G \cong B_{7,2}$.
If $<V - S> \cong 2K_1 \cup K_2$, then $G \cong B_{7,3}$ (or) $B_{7,4}$.
(iii) $|V - S| = 5$
If $<V - S> \cong 5K_1$, then $G \cong B_{7,5}$.
If $<V - S> \cong 3K_1 \cup K_2$, then $G \cong B_{7,6}$ (or) $B_{7,7}$.
(iv) $|V - S| = 6$ (or) 7, then G is not unicyclic.

Subcase(3.b): $<S> \cong K_1 \cup K_2$
If $<V - S>$ contains K_2, then G is not unicyclic. The only cycle in this case is C_3. If $|V - S| = 1$ (or) 2, then S will not be a γ_{cild} – set of G.
(i) $|V - S| = 3$
If $<V - S> \cong 3K_1$, then $G \cong B_{8,1}$.
(ii) $|V - S| = 4$
If $<V - S> \cong 4K_1$, then $G \cong B_{8,2}$ (or) $B_{8,3}$.
(iii) $|V - S| = 5$ (or) 6 (or) 7 then G is not unicyclic.

Subcase(3.c): $<S> \cong P_3$
The only cycle in this case is C_3.
(i) $|V - S| = 1$
If $<V - S> \cong K_1$, then G is not unicyclic.
(ii) $|V - S| = 2$
If $<V - S> \cong 2K_1$, then $G \cong B_{9,1}$
(iii) $|V - S| = 3$
If $<V - S> \cong 3K_1$, then $G \cong B_{9,1}$ (or) $B_{9,2}$
(iv) $|V - S| = 4$
If $<V - S> \cong 4K_1$, then $G \cong B_{9,3}$
(v) $|V - S| = 5$ (or) 6 (or) 7, then G is not unicyclic.
Hence the theorem follows.

Notation 3.5:

The family of graphs $\mathcal{C} = \{ \mathcal{C}_1, \mathcal{C}_2 \}$ are defined as follows, where

$\mathcal{C}_1 = \{ C_{1,1}\ C_{1,2}, ..., C_{1,43}, C_{1,44}\}$; and $\mathcal{C}_2 = \{ C_{2,1}, C_{2,2}, ..., C_{2,11}, C_{2,12} \}$

$C_{1,1} = C_3 @ P_2 @_s P_2$	$C_{1,20} = C_3 @ P_1 @ P_1 @ P_1 @_{es} P_3$	$C_{1,39} = C_3 @ P_1 @_{es} P_3 @ P_3$
$C_{1,2} = C_3 @ P_1 @ P_1 @_s P_3$	$C_{1,21} = C_3 @ P_1 @ P_2 @_s P_3$	$C_{1,40} = C_3 @ P_2 @ P_2 @_s P_4$
$C_{1,3} = C_3 @ P_1 @_c P_4$	$C_{1,22} = C_3 @ P_1 @ P_2 @ P_3$	$C_{1,41} = C_3 @ P_1 @ P_2 @ P_1 @_{es} P_4$
$C_{1,4} = C_3 @ P_1 @ P_1 @_{es} P_3$	$C_{1,23} = C_3 @ P_1 @_{es} P_4$	$C_{1,42} = C_3 @ P_2 @_{es} P_4 @ P_2$
$C_{1,5} = C_3 @ P_1 @ P_1 @ P_3$	$C_{1,24} = C_3 @ P_1 @ P_2 @_{es} P_3$	$C_{1,43} = C_3 @ P_1 @_{es} P_4 @ P_3$
$C_{1,6} = C_3 @ P_2 @_s P_3$	$C_{1,25} = C_3 @ P_2 @ P_1 @_{ec} P_4$	$C_{1,44} = C_3 @ P_2 @_c P_6$
$C_{1,7} = C_4 @ P_1 @ P_1 @ P_1 @ P_1$	$C_{1,26} = C_4 @ P_1 @ P_1 @_{es} P_4$	$C_{2,1} = C_3 @ P_1 @ P_1 @_s P_2$
$C_{1,8} = C_4 @ P_1 @_s P_3$	$C_{1,27} = C_4 @ P_1 @ P_2 @ P_1 @ P_2$	$C_{2,2} = C_3 @ P_1 @ P_1 @_{es} P_2$
$C_{1,9} = C_5 @ P_1 @ P_1 @ P_1$	$C_{1,28} = C_3 @ P_1 @ P_2 @_{es} P_4$	$C_{2,3} = C_3 @ P_1 @ P_2 @_s P_2$
$C_{1,10} = C_5 @ P_1 @ P_1 @ P_2$	$C_{1,29} = C_3 @ P_2 @_{es} P_3 @ P_2$	$C_{2,4} = C_3 @ P_1 @_{es} P_2 @ P_2$
$C_{1,11} = C_3 @ P_1 @_{ec} P_4$	$C_{1,30} = C_3 @ P_2 @ P_2 @ P_3$	$C_{2,5} = C_3 @ P_1 @ P_2 @_{es} P_2$
$C_{1,12} = C_4 @ P_1 @ P_1 @ P_1 @ P_2$	$C_{1,31} = C_3 @ P_2 @ P_2 @_s P_3$	$C_{2,6} = C_3 @ P_1 @ P_1 @ P_1 @_{es} P_2$
$C_{1,13} = C_4 @ P_1 @_c P_4$	$C_{1,32} = C_3 @ P_1 @ P_2 @ P_1 @_{es} P_3$	$C_{2,7} = C_4 @ P_1 @ P_1 @_{es} P_2$
$C_{1,14} = C_4 @ P_1 @ P_1 @_{es} P_3$	$C_{1,33} = C_3 @ P_1 @ P_2 @_s P_4$	$C_{2,8} = C_3 @ P_2 @_{es} P_2 @ P_2$
$C_{1,15} = C_3 @ P_1 @ P_1 @_s P_4$	$C_{1,34} = C_3 @ P_1 @ P_2 @ P_4$	$C_{2,9} = C_3 @_s P_2 @ P_2 @ P_2$
$C_{1,16} = C_3 @ P_2 @_c P_4$	$C_{1,35} = C_3 @ P_1 @_{es} P_4 @ P_2$	$C_{2,10} = C_3 @ P_1 @_{es} P_2 @ P_2 @ P_1$
$C_{1,17} = C_3 @ P_1 @ P_1 @ P_4$	$C_{1,36} = C_3 @ P_1 @ P_1 @ P_1 @_{es} P_4$	$C_{2,11} = C_3 @_s P_4 @ P_2$
$C_{1,18} = C_3 @ P_1 @_{es} P_3 @ P_2$	$C_{1,37} = C_3 @ P_1 @_c P_6$	$C_{2,12} = C_3 @ P_1 @_{es} P_2 @ P_3$
$C_{1,19} = C_3 @ P_1 @_c P_5$	$C_{1,38} = C_3 @ P_2 @_c P_5$	

Theorem 3.6:

Let G be a connected unicyclic graph in which one vertex of a γ_{cild} – set lies on the cycle. Then $\gamma_{cild}(G) = 4$ if and only if G is one of the graphs in the family $\mathcal{C}$.

Proof:

If G is one of the graphs in the family $\mathcal{C}$, then $\gamma_{cild}(G) = 4$.

Conversely, let S be a γ_{cild} – set of the unicyclic graph G with $|S| = 4$ and therefore $|V – S| \leq 2^4 – 1 = 15$ and <V – S> contains atleast one isolated vertex. Let a vertex of S lie on the cycle. Then the cycle in G is one of C_3, C_4 and C_5. Also it is observed that, $|N(u) \cap S| = 1$ (or) 2, for any $u \in V – S$. Hence $|V – S| \leq 7$.

Therefore, $\langle S \rangle \cong 4K_1, 2K_1 \cup K_2, K_1 \cup P_3$ (or) $K_{1,3}$

Case (1): $\langle S \rangle \cong 4K_1$

Then <V – S> must contain K_2. Since <V – S> contains atleast one isolated vertex, $|V – S| \geq 3$.

Subcase(1.a): $|V – S| = 3$

If $\langle V – S \rangle \cong K_1 \cup K_2$, then $G \cong C_{1,1}$

Subcase(1.b): $|V – S| = 4$

If $\langle V – S \rangle \cong 2K_1 \cup K_2$, then G is one of the graphs from $C_{1,2}$ to $C_{1,5}$

If $\langle V – S \rangle \cong K_1 \cup P_3$, then G is one of the graphs from $C_{1,6}$ to $C_{1,9}$

Subcase(1.c): $|V – S| = 5$

If $\langle V – S \rangle \cong K_1 \cup P_4$, then $G \cong C_{1,10}$

If $\langle V – S \rangle \cong 2K_1 \cup P_3$, then G is one of the graphs from $C_{1,11}$ to $C_{1,14}$

If $\langle V – S \rangle \cong 3K_1 \cup K_2$, then G is one of the graphs from $C_{1,15}$ to $C_{1,24}$

Subcase(1.d): $|V – S| = 6$

If $\langle V – S \rangle \cong 3K_1 \cup P_3$, then G is one of the graphs from $C_{1,25}$ to $C_{1,27}$

If $\langle V – S \rangle \cong 4K_1 \cup K_2$, then G is one of the graphs from $C_{1,28}$ to $C_{1,39}$

Subcase(1.e): $|V – S| = 7$

If $\langle V – S \rangle \cong 5K_1 \cup K_2$, then G is of the graphs from $C_{1,40}$ to $C_{1,44}$

If $\langle V – S \rangle \cong 3K_1 \cup 2K_2$ (or) $K_1 \cup 3K_2$, then S will not be a γ_{cild} – set of G.

Case (2): $\langle S\rangle \cong 2K_1 \cup K_2$

By a similar argument as in Case(1), $|V – S| \geq 3$.

Subcase(2.a): $|V – S| = 3$

If $\langle V – S\rangle \cong K_1 \cup K_2$, then G $\cong C_{2,1}$ and $C_{2,2}$

Subcase(2.b): $|V – S| = 4$

If $\langle V – S\rangle \cong 2K_1 \cup K_2$, then G is one of the graphs from $C_{2,3}$ to $C_{2,7}$ and $C_{1,\,2}$

Subcase(2.c): $|V – S| = 5$

If $\langle V – S\rangle \cong 3K_1 \cup K_2$, then G is one of the graphs from $C_{2,8}$ to $C_{2,12}$ and $C_{1,19}$

If $\langle V – S\rangle \cong 2K_1 \cup P_3$, then G $\cong C_{1,14}$

Subcase(2.d): $|V – S| = 6$

If $\langle V – S\rangle \cong 4K_1 \cup K_2$, then G $\cong C_{1,31}$

Subcase(2.e): $|V – S| = 7$

Then either G is not unicyclic or S will not be a γ_{cild} – set of G.

Case (3): $\langle S\rangle \cong K_{1,3}$

In this case, it is observed that all vertices in $\langle V – S\rangle$ are isolated vertices.

Therefore,$|V – S| = 4$.

Subcase(3.a): $|V – S| = 4$

If $\langle V – S\rangle \cong 4K_1$, then G $\cong C_{1,3}$

Case (4): $\langle S\rangle \cong K_1 \cup P_3$

By a similar argument as in Case(1), $|V – S| \geq 3$.

Subcase(4.a): $|V – S| = 3$

If $\langle V – S\rangle \cong K_1 \cup K_2$, then G $\cong C_{1,1}$

Subcase(4.b): $|V – S| = 4$

If $\langle V – S\rangle \cong 2K_1 \cup K_2$, then G $\cong C_{1,6}$ and $C_{1,3}$

Subcase(4.c): $|V – S| = 5$

If $\langle V – S\rangle \cong 3K_1 \cup K_2$, then G $\cong C_{1,16}$

Subcase(4.d): $|V – S| = 6$ (or) 7

If $\langle V – S\rangle \cong 3K_1 \cup K_2$, then G $\cong C_{1,16}$

Then either G is not unicyclic or S will not be a γ_{cild} – set of G.

This completes the proof of the theorem.

Notation 3.7:

The family of graphs $\mathcal{D} = \{ \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3, \mathcal{D}_4 \}$ are defined as follows, where

$\mathcal{D}_1 = \{ D_{1,1}\ D_{1,2}, ..., D_{1,49}, D_{1,50}\}$; $\mathcal{D}_2 = \{ D_{2,1}, D_{2,2}, ..., D_{2,37}, D_{2,38}\}$;

$\mathcal{D}_3 = \{ D_{3,1}, D_{3,2}, ..., D_{3,6}, D_{3,7}\}$; and $\mathcal{D}_4 = \{ D_{4,1}, D_{4,2}, D_{4,3}, D_{4,4}, D_{4,5} \}$

$D_{1,1} = C_4 @ P_1 @_s P_2$

$D_{1,2} = C_4 @ P_1 @ P_3$

$D_{1,3} = C_4 @_s P_3$

$D_{1,4} = C_5 @ P_3$

$D_{1,5} = C_6 @ P_1 @ P_1$

$D_{1,6} = C_4 @_s P_3 @ P_1 @ P_1$

$D_{1,7} = C_4 @ P_2 @ P_3$

$D_{1,8} = C_4 @ P_1 @_s P_4$

$D_{1,9} = C_4 @ P_1 @_c P_4$

$D_{1,10} = C_4 @ P_1 @ P_1 @ P_3$

$D_{1,11} = C_4 @ P_1 @ P_1 @_{es} P_2$

$D_{1,12} = C_4 @ P_1 @_{es} P_3$

$D_{1,13} = C_4 @ P_2 @_s P_3$

$D_{1,14} = C_5 @ P_1 @_s P_3$

$D_{1,15} = C_5 @ P_2 @_3 P_2$

$D_{1,16} = C_5 @ P_1 @ P_1 @ P_1 @ P_1$

$D_{1,17} = C_5 @ P_4$

$D_{1,18} = C_5 @ P_1 @ P_1 @ P_1$

$D_{1,19} = C_5 @ P_1 @_{es} P_3$

$D_{1,20} = C_5 @ P_2 @ P_2$

$D_{1,21} = C_5 @ P_2 @_2 P_1 @ P_1$

$D_{1,22} = C_6 @ P_1 @_3 P_2$

$D_{1,23} = C_6 @ P_2 @ P_1$

$D_{1,24} = C_6 @ P_1 @ P_1 @ P_1$

$D_{1,25} = C_4 @ P_2 @_s P_3 @ P_1$

$D_{1,26} = C_4 @ P_1 @ P_1 @_s P_4$

$D_{1,27} = C_4 @ P_1 @ P_4 @ P_1$

$D_{1,28} = C_4 @ P_2 @_s P_4$

$D_{1,29} = C_4 @ P_1 @ P_2 @ P_3$

$D_{1,30} = C_4 @ P_1 @ P_1 @_{es} P_4$

$D_{1,31} = C_4 @ P_1 @ P_1 @_{es} P_3 @ P_1$

$D_{1,32} = C_5 @ P_2 @ P_1 @ P_2$

$D_{1,33} = C_5 @ P_1 @_{es} P_4$

$D_{1,34} = C_5 @ P_2 @_s P_3$

$D_{1,35} = C_5 @_s P_4 @ P_1$

$D_{1,36} = C_5 @ P_2 @_{es} P_3$

$D_{1,37} = C_5 @ P_1 @ P_2 @ P_2$

$D_{1,38} = C_5 @ P_1 @ P_1 @_{es} P_3$

$D_{1,39} = C_5 @ P_1 @ P_2 @ P_1 @ P_1$

$D_{1,40} = C_5 @ P_1 @ P_1 @_2 P_3$

$D_{1,41} = C_5 @ P_2 @ P_1 @_2 P_2$

$D_{1,42} = C_6 @ P_2 @ P_2$

$D_{1,43} = C_6 @ P_1 @ P_2 @ P_1$

$D_{1,44} = C_4 @ P_1 @ P_2 @_s P_4$

$D_{1,45} = C_5 @ P_1 @ P_2 @ P_2 @ P_1$

$D_{1,46} = C_5 @_s P_4 @ P_2$

$D_{1,47} = C_5 @ P_2 @_{es} P_4$

$D_{1,48} = C_4 @ P_1 @ P_1 @_{es} P_4 @ P_1$

$D_{1,49} = C_5 @ P_1 @ P_1 @_{es} P_4$
$D_{1,50} = C_5 @ P_2 @ P_1 @_2 P_3$
$D_{2,1} = C_4 @ P_1 @_{es} P_2$
$D_{2,2} = C_3 @ P_1 @_{es} P_2 @ P_1$
$D_{2,3} = C_3 @ P_1 @ P_1 @ P_3$
$D_{2,4} = C_3 @ P_1 @ P_1 @_{es} P_3$
$D_{2,5} = C_3 @ P_1 @_{es} P_4$
$D_{2,6} = C_4 @ P_2 @_s P_2$
$D_{2,7} = C_4 @ P_1 @ P_1 @_s P_2$
$D_{2,8} = C_4 @ P_1 @ P_1 @ P_2$
$D_{2,9} = C_4 @ P_1 @ P_1 @ P_1 @ P_1$
$D_{2,10} = C_5 @ P_1 @_s P_2$
$D_{2,11} = C_3 @ P_1 @ P_1 @ P_1 @_{es} P_3$
$D_{2,12} = C_4 @ P_1 @ P_1 @_{es} P_3$
$D_{2,13} = C_3 @ P_2 @_{es} P_3 @ P_1$
$D_{2,14} = C_3 @ \begin{pmatrix} P_1 \\ P_2 @_{es} P_3 \end{pmatrix}$
$D_{2,15} = C_3 @ P_1 @ P_1 @_{es} P_4$
$D_{2,16} = C_3 @ P_1 @ P_1 @ P_4$
$D_{2,17} = C_3 @ P_1 @ P_2 @ P_3$
$D_{2,18} = C_3 @ P_2 @ P_1 @_s P_3$
$D_{2,19} = C_3 @ P_2 @_s P_4$
$D_{2,20} = C_3 @ P_2 @_{es} P_4$
$D_{2,21} = C_3 @ P_1 @ P_1 @_s P_4$
$D_{2,22} = C_4 @ P_1 @ P_1 @_{es} P_2 @ P_1$
$D_{2,23} = C_4 @ P_1 @ P_2 @_s P_2$
$D_{2,24} = C_4 @ P_1 @ P_1 @ P_1 @ P_2$
$D_{2,25} = C_4 @ P_2 @_{es} P_3$
$D_{2,26} = C_4 @ P_1 @_{es} P_4$
$D_{2,27} = C_5 @ P_2 @_s P_2$
$D_{2,28} = C_5 @ P_2 @_{es} P_2$
$D_{2,29} = C_5 @ P_1 @ P_1 @_{es} P_2$
$D_{2,30} = C_3 @ P_1 @ P_1 @ P_1 @_{es} P_4$
$D_{2,31} = C_3 @ P_1 @ P_2 @_s P_4$
$D_{2,32} = C_3 @ \begin{pmatrix} P_1 \\ P_2 @_{es} P_3 \end{pmatrix} @ P_1$
$D_{2,33} = C_3 @ \begin{pmatrix} P_1 \\ P_2 @_{es} P_4 \end{pmatrix}$
$D_{2,34} = C_3 @ P_1 @ P_2 @_{es} P_4$
$D_{2,35} = C_3 @ P_1 @ P_2 @ P_1 @_{es} P_3$
$D_{2,36} = C_4 @ P_2 @_{es} P_4$
$D_{2,37} = C_4 @ P_1 @ P_1 @ P_2 @ P_2$
$D_{2,38} = C_3 @ \begin{pmatrix} P_1 \\ P_2 @_{es} P_4 \end{pmatrix} @ P_1$
$D_{3,1} = C_3 @ P_2 @_{es} P_2$
$D_{3,2} = C_3 @ P_1 @ P_1 @_{es} P_2$
$D_{3,3} = C_3 @ P_1 @ P_1 @ P_1 @_{es} P_2$
$D_{3,4} = C_3 @ \begin{pmatrix} P_1 \\ P_2 @_{es} P_2 \end{pmatrix}$
$D_{3,5} = C_3 @ P_2 @_{es} P_3$
$D_{3,6} = C_3 @ P_1 @ P_2 @_{es} P_2$
$D_{3,7} = C_4 @ P_2 @_{es} P_3$
$D_{4,1} = C_3 @ P_1 @ P_1 @_s P_2$
$D_{4,2} = C_3 @ P_2 @_s P_2$
$D_{4,3} = C_3 @ P_2 @_s P_3$
$D_{4,4} = C_3 @ P_1 @ P_1 @_s P_3$
$D_{4,5} = C_3 @ P_1 @ P_2 @_s P_2$

Theorem 3.8:

Let G be a connected unicyclic graph in which two vertices of γ_{cild} – set lie on the cycle. Then $\gamma_{cild}(G) = 4$ if and only if G is one of the graphs in the family $\mathcal{D}$.

Proof:

If G is one of the graphs in the family $\mathcal{D}$, then $\gamma_{cild}(G) = 4$.

Conversely, let S be a γ_{cild} – set of the unicyclic graph G with |S| = 4 and two vertices of S lie on the cycle of G. By Theorem 3.6, $3 \leq |V – S| \leq 7$. Since <V – S> contains atleast one isolated vertex,
$\langle S \rangle \cong 4K_1, 2K_1 \cup K_2, 2K_2$ (or) $K_1 \cup P_3$.

Case (1): $\langle S \rangle \cong 4K_1$

Subcase(1.a): |V – S| = 3

If $\langle V – S \rangle \cong 3K_1$, then $G \cong D_{1,1}$

Subcase(1.b): |V – S| = 4

If $\langle V – S \rangle \cong 4K_1$, then $G \cong D_{1,2}$ and $D_{1,3}$
If $\langle V – S \rangle \cong 2K_1 \cup K_2$, then $G \cong D_{1,4}$
If $\langle V – S \rangle \cong K_1 \cup P_3$, then $G \cong D_{1,5}$

Subcase(1.c): |V – S| = 5

If $\langle V – S \rangle \cong 5K_1$, then G is one of the graphs from $D_{1,6}$ to $D_{1,13}$
If $\langle V – S \rangle \cong 3K_1 \cup K_2$, then G is one of the graphs from $D_{1,14}$ to $D_{1,21}$
If $\langle V – S \rangle \cong K_1 \cup 2K_2$, then $G \cong D_{1,22}$ and $D_{1,23}$
If $\langle V – S \rangle \cong 2K_1 \cup P_3$, then $G \cong D_{1,24}$, $D_{1,22}$ and $D_{1,23}$

Subcase(1.d): |V – S| = 6

If $\langle V – S \rangle \cong 6K_1$, then G is one of the graphs from $D_{1,25}$ to $D_{1,31}$
If $\langle V – S \rangle \cong 4K_1 \cup K_2$, then G is one of the graphs from $D_{1,32}$ to $D_{1,41}$
If $\langle V – S \rangle \cong 2K_1 \cup K_2$, then $G \cong D_{1,42}$ and $D_{1,22}$
If $\langle V – S \rangle \cong 3K_1 \cup P_3$, then $G \cong D_{1,43}$

Subcase(1.e): |V – S| = 7

If $\langle V – S \rangle \cong 7K_1$, then G is one of the graphs from $D_{1,44}$ to $D_{1,48}$
If $\langle V – S \rangle \cong 5K_1 \cup K_2$, then $G \cong D_{1,49}$ and $D_{1,50}$
If $\langle V – S \rangle \cong 3K_1 \cup 2K_2$ (or) $K_1 \cup 3K_2$, then S will not a γ_{cild} – set of G.

Case (2): $\langle S \rangle \cong 2K_1 \cup K_2$

Subcase(2.a): $|V - S| = 3$

If $\langle V - S\rangle \cong 3K_1$, then $G \cong D_{2,1}$, $D_{2,2}$ and $D_{1,1}$

If $\langle V - S\rangle \cong 2K_1 \cup K_2$, then $G \cong D_{2,2}$

Subcase(2.b): $|V - S| = 4$

If $\langle V - S\rangle \cong 4K_1$, then G is one of the graphs from $D_{2,3}$ to $D_{2,7}$ and $D_{1,12}$

If $\langle V - S\rangle \cong 2K_1 \cup K_2$, then $G \cong D_{2,8}$ to $D_{2,10}$ $D_{1,8}$ and $D_{1,19}$

Subcase(2.c): $|V - S| = 5$

If $\langle V - S\rangle \cong 5K_1$, then G is one of the graphs from $D_{2,11}$ to $D_{2,23}$, $D_{1,30}$ and $D_{1,31}$

If $\langle V - S\rangle \cong 3K_1 \cup K_2$, then G is one of the graphs from $D_{2,24}$ to $D_{2,29}$, $D_{1,14}$, $D_{1,19}$ and $D_{2,15}$

Subcase(2.d): $|V - S| = 6$

If $\langle V - S\rangle \cong 6K_1$, then G is one of the graphs from $D_{2,30}$ to $D_{2,34}$ and $D_{1,11}$

If $\langle V - S\rangle \cong 4K_1 \cup K_2$, then G is one of the graphs from $D_{2,35}$ to $D_{2,37}$, $D_{1,30}$, $D_{1,34}$ and $D_{1,36}$

Subcase(2.e): $|V - S| = 7$

If $\langle V - S\rangle \cong 7K_1$, then $G \cong D_{2,38}$

If $\langle V - S\rangle \cong 5K_1 \cup K_2$, then G is not unicyclic.

Case (3): $\langle S\rangle \cong 2K_2$

Subcase(3.a): $|V - S| = 3$

If $\langle V - S\rangle \cong 3K_1$, then $G \cong D_{3,1}$ and $D_{3,2}$

If $\langle V - S\rangle \cong K_1 \cup K_2$, then $G \cong D_{1,11}$

Subcase(3.b): $|V - S| = 4$

If $\langle V - S\rangle \cong 4K_1$, then G is one of the graphs from $D_{3,3}$ to $D_{3,6}$ and $D_{2,13}$

If $\langle V - S\rangle \cong 2K_1 \cup K_2$, then $G \cong D_{3,7}$

Subcase(3.c): $|V - S| = 5$

If $\langle V - S\rangle \cong 5K_1$, then G is one of the graphs $D_{3,5}$, $D_{3,6}$, $D_{2,13}$ and $D_{2,16}$

If $\langle V - S\rangle \cong 3K_1 \cup K_2$, then $G \cong D_{3,7}$

Subcase(3.d): $|V - S| = 6$

If $\langle V - S\rangle \cong 6K_1$, then $G \cong D_{2,35}$

Subcase(3.d): $|V - S| = 7$

Then either G is not unicyclic or S will not be a γ_{cild} – set of G.

Case (4): $\langle S\rangle \cong K_1 \cup P_3$

Subcase(4.a): $|V - S| = 3$

If $\langle V - S\rangle \cong 3K_1$, then $G \cong D_{4,1}$ and $D_{4,2}$

Subcase(4.b): $|V - S| = 4$

If $\langle V - S\rangle \cong 4K_1$, then $G \cong D_{4,3}$, $D_{4,4}$ and $D_{4,5}$

Subcase(4.c): $|V - S| = 5$

If $\langle V - S\rangle \cong 5K_1$, then $G \cong D_{4,5}$

Subcase(4.d): $|V - S| = 6$ (or) 7

Then either G is not unicyclic or S will not be a γ_{cild} – set of G.

This completes the proof of the theorem.

Notation 3.9:

The family of graphs $\mathcal{E} = \{ \mathcal{E}_1, \mathcal{E}_2, ..., \mathcal{E}_5, \mathcal{E}_6 \}$ are defined as follows, where $\mathcal{E}_1 = \{ E_{1,1}, E_{1,2}, ..., E_{1,31}, E_{1,32} \}$; $\mathcal{E}_2 = \{ E_{2,1}, E_{2,2}, ..., E_{2,27}, E_{2,28} \}$; $\mathcal{E}_3 = \{ E_{3,1}, E_{3,2}, E_{3,3}, E_{3,4} \}$; $\mathcal{E}_4 = \{ E_{4,1}, E_{4,2}, ..., E_{4,14}, E_{4,14} \}$; $\mathcal{E}_5 = \{ E_{5,1}, E_{5,2}, E_{5,3}, E_{5,4} \}$; and $\mathcal{E}_6 = \{ E_{6,1} \}$.

$E_{1,1} = C_6 @ P_2$	$E_{1,7} = C_6 @ P_1 @_3 P_2$	$E_{1,13} = C_6 @_s P_4$
$E_{1,2} = C_6 @ P_3$	$E_{1,8} = C_7 @ P_1 @ P_1$	$E_{1,14} = C_6 @ P_1 @_2 P_3$
$E_{1,3} = C_6 @_s P_3$	$E_{1,9} = C_7 @ P_1 @_3 P_1$	$E_{1,15} = C_6 @ P_1 @ P_2 @ P_1$
$E_{1,4} = C_6 @ P_1 @ P_2$	$E_{1,10} = C_7 @ P_2$	$E_{1,16} = C_6 @ P_1 @_2 P_1 @ P_2$
$E_{1,5} = C_6 @ P_1 @_2 P_2$	$E_{1,11} = C_8 @ P_1$	$E_{1,17} = C_6 @ P_1 @_2 P_1 @_2 P_2$
$E_{1,6} = C_6 @ P_1 @ P_1 @ P_1$	$E_{1,12} = C_5 @_s P_4 @ P_1$	$E_{1,18} = C_6 @ P_1 @ P_1 @ P_1 @_2 P_1$

$E_{1,19} = C_7 @ P_3$

$E_{1,20} = C_7 @_s P_3$

$E_{1,21} = C_7 @ P_1 @ P_2$

$E_{1,22} = C_7 @ P_1 @_2 P_2$

$E_{1,23} = C_7 @ P_1 @_3 P_2$

$E_{1,24} = C_7 @ P_1 @_2 P_1 @ P_2$

$E_{1,25} = C_8 @ P_2$

$E_{1,26} = C_6 @ P_1 @_{2s} P_4$

$E_{1,27} = C_6 @ P_1 @_2 P_1 @_2 P_3$

$E_{1,28} = C_6 @ P_1 @_2 P_1 @_{2s} P_3$

$E_{1,29} = C_6 @ P_1 @_2 P_1 @ P_2 @ P_1$

$E_{1,30} = C_7 @_s P_4$

$E_{1,31} = C_7 @ P_1 @_2 P_3$

$E_{1,32} = C_6 @ P_1 @_{2s} P_4 @_2 P_1$

$E_{2,1} = C_5 @ P_1 @ P_1 @ P_1$

$E_{2,2} = C_5 @ P_2 @_2 P_1$

$E_{2,3} = C_5 @_s P_3$

$E_{2,4} = C_6 @_s P_2$

$E_{2,5} = C_6 @ P_1 @ P_1$

$E_{2,6} = C_5 @ P_1 @ P_1 @ P_1 @ P_1$

$E_{2,7} = C_5 @ P_2 @ P_1 @ P_1$

$E_{2,8} = C_5 @ P_1 @_2 P_1 @ P_2$

$E_{2,9} = C_5 @ P_1 @ P_2 @ P_1$

$E_{2,10} = C_5 @ P_1 @_2 P_3$

$E_{2,11} = C_5 @_s P_4$

$E_{2,12} = C_5 @_s P_3 @_2 P_1$

$E_{2,13} = C_5 @ P_1 @ P_3$

$E_{2,14} = C_6 @_s P_2 @ P_1$

$E_{2,15} = C_7 @_s P_2$

$E_{2,16} = C_6 @ P_1 @ P_1 @_2 P_1$

$E_{2,17} = C_5 @ P_1 @ P_2 @ P_1 @ P_1$

$E_{2,18} = C_5 @ P_1 @ P_1 @_2 P_3$

$E_{2,19} = C_5 @ P_1 @_2 P_1 @ P_3$

$E_{2,20} = C_5 @_s P_4 @_2 P_1$

$E_{2,21} = C_5 @_s P_3 @_2 P_1 @ P_1$

$E_{2,22} = C_5 @ P_1 @ P_3 @_2 P_1$

$E_{2,23} = C_6 @_s P_2 @_2 P_1 @_2 P_1$

$E_{2,24} = C_6 @_s P_3 @_2 P_1$

$E_{2,25} = C_6 @ P_3 @_2 P_1 @_2 P_2$

$E_{2,26} = C_6 @ P_1 @_3 P_2 @ P_1$

$E_{2,27} = C_5 @_s P_4 @_2 P_1 @ P_1$

$E_{2,28} = C_5 @ P_1 @_s P_4 @_2 P_1$

$E_{3,1} = C_5 @ P_1 @_{2s} P_2$

$E_{3,2} = C_5 @ P_1 @_{2s} P_3$

$E_{3,3} = C_5 @ P_1 @ P_1 @_{2s} P_2$

$E_{3,4} = C_5 @ P_1 @ P_1 @_{2s} P_3$

$E_{4,1} = C_5 @_s P_2$

$E_{4,2} = C_4 @ P_1 @ P_1 @ P_1 @ P_1$

$E_{4,3} = C_4 @ P_2 @ P_1 @ P_1$

$E_{4,4} = C_5 @ P_1 @_s P_2$

$E_{4,5} = C_5 @ P_1 @_{2s} P_2$

$E_{4,6} = C_4 @ P_1 @ P_1 @ P_1 @ P_2$

$E_{4,7} = C_4 @ P_1 @ P_1 @_s P_3$

$E_{4,8} = C_4 @ P_1 @ P_1 @ P_3$

$E_{4,9} = C_5 @ P_1 @_s P_2 @_2 P_1$

$E_{4,10} = C_5 @ P_1 @_s P_3$

$E_{4,11} = C_5 @ P_1 @ P_1 @_2 P_2$

$E_{4,12} = C_5 @ P_1 @_2 P_1 @_s P_3$

$E_{4,13} = C_4 @ P_1 @ P_1 @_s P_4$

$E_{5,1} = C_3 @ P_1 @ P_1 @ P_2$

$E_{5,2} = C_3 @ P_1 @ P_1 @ P_3$

$E_{5,3} = C_3 @ P_1 @ P_1 @_s P_3$

$E_{5,4} = C_3 @ P_1 @ P_1 @_s P_4$

$E_{6,1} = C_4 @ P_1 @ P_1 @_s P_2$

Theorem 3.10:

Let G be connected unicyclic graph in which three vertices of γ_{cild} – set lie on the cycle. Then $\gamma_{cild}(G) = 4$ if and only if G is one of the graphs in the family $\mathcal{E}$.

Proof:

If G is one of the graphs in the family $\mathcal{E}$, then $\gamma_{cild}(G) = 4$.

Conversely, let S be a γ_{cild} – set of the unicyclic graph G with $|S| = 4$. Since three vertices of S lie on the cycle and G is unicyclic, $3 \leq |V - S| \leq 8$.

Since $\langle V - S \rangle$ contains atleast one isolated vertex,

$\langle S \rangle \cong 4K_1, 2K_1 \cup K_2, 2K_2, P_3, C_3$ (or) $K_{1,3}$.

Case (1): $\langle S \rangle \cong 4K_1$

Subcase(1.a): $|V - S| = 3$

Then S is not a γ_{cild} –set of G.

Subcase(1.b): $|V - S| = 4$

If $\langle V - S \rangle \cong 4K_1$, then $G \cong E_{1,1}$

If $\langle V - S \rangle \cong 2K_1 \cup K_2$, then G is not unicyclic.

Subcase(1.c): $|V - S| = 5$

If $\langle V - S \rangle \cong 5K_1$, then G is one of the graphs from $E_{1,2}$ to $E_{1,7}$

If $\langle V - S \rangle \cong 3K_1 \cup K_2$, then G is one of the graphs from $E_{1,8}$ to $E_{1,10}$

If $\langle V - S \rangle \cong K_1 \cup 2K_2$, then $G \cong E_{1,11}$

Subcase(1.d): $|V - S| = 6$

If $\langle V - S \rangle \cong 6K_1$, then G is one of the graphs from $E_{1,12}$ to $E_{1,18}$

If $\langle V - S \rangle \cong 4K_1 \cup K_2$, then G is one of the graphs from $E_{1,19}$ to $E_{1,24}$

If $\langle V - S \rangle \cong 2K_1 \cup K_2$, then $G \cong E_{1,25}$

Subcase(1.e): $|V - S| = 7$

If $\langle V - S \rangle \cong 7K_1$, then G is one of the graphs from $E_{1,26}$ to $E_{1,29}$ and $E_{1,24}$

If $\langle V - S \rangle \cong 5K_1 \cup K_2$, then $G \cong E_{1,30,}$ $E_{1,31}$ and $E_{1,24}$

If $\langle V - S \rangle \cong 3K_1 \cup 2K_2$ (or) $K_1 \cup 3K_2$, then S will not be a γ_{cild} – set of G.

Subcase(1.f): $|V - S| = 8$

If $\langle V - S\rangle \cong 8K_1$, then $G \cong E_{1,32}$

If $\langle V - S\rangle$ contains K_2 as one of its components, then S will not be a γ_{cild} – set of G.

Case (2): $\langle S\rangle \cong 2K_1 \cup K_2$

Subcase(2.a): $|V - S| = 3$

Then S is not a γ_{cild} –set of G.

Subcase(2.b): $|V - S| = 4$

If $\langle V - S\rangle \cong 4K_1$, then G is one of the graphs from $E_{2,1}$ to $E_{2,4}$ and $E_{1,1}$

If $\langle V - S\rangle \cong 2K_1 \cup K_2$, then $G \cong E_{2,5}$

Subcase(2.c): $|V - S| = 5$

If $\langle V - S\rangle \cong 5K_1$, then G is one of the graphs from $E_{2,6}$ to $E_{2,14}$ and $E_{1,3}$

If $\langle V - S\rangle \cong 3K_1 \cup K_2$, then G is one of the graphs $E_{2,15}$, $E_{2,16}$, $E_{1,10}$, $E_{2,3}$ and $E_{2,10}$

Subcase(2.d): $|V - S| = 6$

If $\langle V - S\rangle \cong 6K_1$, then G is one of the graphs from $E_{2,17}$ to $E_{2,25}$

If $\langle V - S\rangle \cong 4K_1 \cup K_2$, then $G \cong E_{2,26}$, $E_{1,16}$ and $E_{1,19}$

Subcase(2.e): $|V - S| = 7$

If $\langle V - S\rangle \cong 7K_1$, then G is one of the graphs $E_{2,27}$, $E_{2,28}$, $E_{1,26}$ and $E_{1,27}$

If $\langle V - S\rangle \cong 5K_1 \cup K_2$, then G is not unicyclic.

Subcase(2.f): $|V - S| = 8$

Then either G is not unicyclic or S will not be a γ_{cild} – set of G.

Case (3): $\langle S\rangle \cong 2K_2$

Subcase(3.a): $|V - S| = 3$

Then S is not a γ_{cild} –set of G.

Subcase(3.b): $|V - S| = 4$

If $\langle V - S\rangle \cong 4K_1$, then $G \cong E_{3,1}$

If $\langle V - S\rangle \cong 2K_1 \cup K_2$, then S will not be a γ_{cild} – set of G.

Subcase(3.c): $|V - S| = 5$

If $\langle V - S\rangle \cong 5K_1$, then $G \cong E_{3,2}$ and $E_{3,3}$

If $\langle V - S\rangle \cong K_1 \cup K_2$ (or) $3K_1 \cup K_2$, then S will not be a γ_{cild} – set of G.

Subcase(3.d): $|V - S| = 6$.

If $\langle V - S\rangle \cong 6K_1$, then $G \cong E_{3,4}$

If $\langle V - S\rangle \cong 4K_1 \cup K_2$ (or) $2K_1 \cup 2K_2$, then G is not unicyclic.

Subcase(3.e): $|V - S| = 7$ (or) 8

Then either G is not unicyclic or S will not be a γ_{cild} – set of G.

Case (4): $\langle S\rangle \cong K_1 \cup P_3$

Subcase(4.a): $|V - S| = 3$

If $\langle V - S\rangle \cong 3K_1$, then $G \cong E_{4,1}$

Subcase(4.b): $|V - S| = 4$

If $\langle V - S\rangle \cong 4K_1$, then G is one of the graphs from $E_{4,2}$ to $E_{4,5}$ and $E_{2,3}$

If $\langle V - S\rangle \cong 2K_1 \cup K_2$, then S will not be a γ_{cild} – set of G.

Subcase(4.c): $|V - S| = 5$

If $\langle V - S\rangle \cong 5K_1$, then G is one of the graphs from $E_{4,6}$ to $E_{4,11}$ and $E_{1,3}$

If $\langle V - S\rangle \cong K_1 \cup K_2$ (or) $3K_1 \cup K_2$, then S will not be a γ_{cild} – set of G.

Subcase(4.d): $|V - S| = 6$

If $\langle V - S\rangle \cong 6K_1$, then $G \cong E_{4,12}$ and $E_{4,13}$

If $\langle V - S\rangle \cong 4K_1 \cup K_2$ (or) $2K_1 \cup 2K_2$, then G is not unicyclic.

Subcase(4.e): $|V - S| = 7$ (or) 8

Then either G is not unicyclic or S will not be a γ_{cild} – set of G.

Case (5): $\langle S\rangle \cong K_1 \cup C_3$

Subcase(5.a): $|V - S| = 3$

If $\langle V - S\rangle \cong 3K_1$, then $G \cong E_{5,1}$

Subcase(5.b): $|V - S| = 4$

If $<V - S> \cong 4K_1$, then $G \cong E_{5,2}$ and $E_{5,3}$

If $<V - S> \cong 2K_1 \cup K_2$, then S will not be a γ_{cild} – set of G.

Subcase(5.c): $|V - S| = 5$

If $<V - S> \cong 5K_1$, then $G \cong E_{5,4}$

If $<V - S> \cong K_1 \cup K_2$ (or) $3K_1 \cup K_2$, then S will not be a γ_{cild} – set of G.

Subcase(5.d): $|V - S| = 6, 7$ (or) 8

Then either G is not unicyclic or S will not be a γ_{cild} – set of G.

Case (6): $<S> \cong P_4$

Subcase(6.a): $|V - S| = 3$

Then S is not a γ_{cild} –set of G.

Subcase(6.b): $|V - S| = 4$

If $<V - S> \cong 4K_1$, then $G \cong E_{6,1}$

Subcase(6.c): $|V - S| = 5$

If $<V - S> \cong 5K_1$, then $G \cong E_{4,8}$

If $<V - S> \cong 2K_1 \cup K_2$, then S will not be a γ_{cild} – set of G.

Subcase(6.d): $|V - S| = 5, 6, 7$ (or) 8

Then either G is not unicyclic or S will not be a γ_{cild} – set of G.

This completes the proof of the theorem.

Notation 3.11:

The family of graphs $\mathcal{F} = \{ \mathcal{F}_1, \mathcal{F}_2, ..., \mathcal{F}_5\}$ are defined as follows, where

$F_{1,1} = C_8$	$F_{1,9} = C_8@P_1@_2P_1@_2P_1@_2 P_1$	$F_{3,1} = C_6 @ P_1 @ P_1$
$F_{1,2} = C_8 @ P_1$	$F_{2,1} = C_7 @ P_1 @ P_1$	$F_{3,2} = C_6 @ P_1 @ P_1 @_3 P_1$
$F_{1,3} = C_9$	$F_{2,2} = C_7 @ P_1 @_2 P_1$	$F_{3,3} = C_6 @P_1@P_1@_2 P_1@ P_1$
$F_{1,4} = C_8 @ P_1 @_2 P_1$	$F_{2,3} = C_7 @ P_1 @_3 P_1$	$F_{4,1} = C_6 @ P_1 @ P_1 @ P_1$
$F_{1,5} = C_9 @ P_1$	$F_{2,4} = C_7 @ P_1 @_2 P_1 @_2 P_1$	$F_{4,2} = C_6@P_1@ P_1@ P_1 @_2 P_1$
$F_{1,6} = C_{10}$	$F_{2,5} = C_7 @ P_1 @ P_1 @_2 P_1$	$F_{5,1} = C_5 @ P_1 @ P_1 @ P_1$
$F_{1,7} = C_8 @ P_1@_2 P_1 @_2 P_1$	$F_{2,6} = C_8 @ P_1 @ P_1$	$F_{5,2} = C_5@P_1@ P_1@ P_1 @ P_1$
$F_{1,8} = C_9 @ P_1 @_2 P_1$	$F_{2,7} = C_7@P_1@_2P_1@_2P_1@ P_1$	$F_{5,3} = C_4@P_1@P_1@P_1@P_1$

Theorem 3.12:

Let G be a connected unicyclic graph G in which four vertices of γ_{cild} – set lie on the cycle. Then $\gamma_{cild}(G) = 4$ if and only if G is one of the graphs in the family $\mathcal{F}$.

Proof:

If G is one of the graphs in the family $\mathcal{F}$, then $\gamma_{cild}(G) = 4$.

Conversely, let S be a γ_{cild} – set of the unicyclic graph G. Since four vertices of S lie on the cycle, $4 \leq |V - S| \leq 8$.

Since $<V - S>$ contains atleast one isolated vertex, $<S>$ is one of the graphs $4K_1$, $2K_1 \cup K_2$, $K_1 \cup P_3$, P_4 and C_4.

Case (1): $<S> \cong 4K_1$

Subcase(1.a): $|V - S| = 4$

If $<V - S> \cong 4K_1$, then $G \cong F_{1,1}$

If $<V - S> \cong 2K_1 \cup K_2$, then G is not unicyclic.

Subcase(1.b): $|V - S| = 5$

If $<V - S> \cong 5K_1$, then $G \cong F_{1,2}$

If $<V - S> \cong 3K_1 \cup K_2$, then $G \cong F_{1,3}$

Subcase(1.c): $|V - S| = 6$

If $<V - S> \cong 6K_1$, then $G \cong F_{1,4}$

If $<V - S> \cong 4K_1 \cup K_2$, then $G \cong F_{1,5}$

If $<V - S> \cong 2K_1 \cup K_2$, then $G \cong F_{1,6}$

Subcase(1.d): $|V - S| = 7$

If $<V - S> \cong 7K_1$, then $G \cong F_{1,7}$

If $<V - S> \cong 5K_1 \cup K_2$, then $G \cong F_{1,8}$

If $<V - S> \cong 3K_1 \cup 2K_2$ (or) $K_1 \cup 3K_2$, then S will not be a γ_{cild} – set of G.

Subcase(1.e): $|V - S| = 8$

If $<V - S> \cong 8K_1$, then $G \cong F_{1,9}$

If $<V - S>$ contains K_2 as one of its components, then S will not be a γ_{cild} – set of G.

Case (2): $<S> \cong 2K_1 \cup K_2$

Subcase(2.a): $|V - S| = 4$

If $<V - S> \cong 4K_1$, then $G \cong F_{1,2}$

If $<V - S> \cong 2K_1 \cup K_2$, then $G \cong F_{1,1}$

Subcase(2.b): $|V - S| = 5$

If $<V - S> \cong 5K_1$, then $G \cong F_{2,1}$, $F_{2,2}$ and $F_{2,3}$

If $<V - S> \cong 3K_1 \cup K_2$, then $G \cong F_{1,3}$

Subcase(2.c): $|V - S| = 6$

If $<V - S> \cong 6K_1$, then $G \cong F_{2,4}$ and $F_{2,5}$

If $<V - S> \cong 4K_1 \cup K_2$, then $G \cong F_{2,6}$

Subcase(2.d): $|V - S| = 7$

If $<V - S> \cong 7K_1$, then $G \cong F_{2,7}$

If $<V - S> \cong 5K_1 \cup K_2$, then G is not unicyclic.

Subcase(2.e): $|V - S| = 8$

Then either G is not unicyclic or S will not be a γ_{cild} – set of G.

Case (3): $<S> \cong 2K_2$

Subcase(3.a): $|V - S| = 4$

If $<V - S> \cong 4K_1$, then $G \cong F_{3,1}$

If $<V - S> \cong 2K_1 \cup K_2$, then S will not a γ_{cild} – set of G.

Subcase(3.b): $|V - S| = 5$

If $<V - S> \cong 5K_1$, then $G \cong F_{3,2}$

If $<V - S> \cong 3K_1 \cup K_2$, then $G \cong F_{2,2}$

If $<V - S> \cong K_1 \cup K_2$, then S will not a γ_{cild} – set of G.

Subcase(3.c): $|V - S| = 6$

If $<V - S> \cong 6K_1$, then $G \cong F_{3,3}$

If $<V - S> \cong 4K_1 \cup K_2$ (or) $2K_1 \cup 2K_2$, then G is not unicyclic.

Subcase(3.d): $|V - S| = 7$ (or) 8

Then either G is not unicyclic or S will not be a γ_{cild} – set of G.

Case (4): $<S> \cong K_1 \cup P_3$

Subcase(4.a): $|V - S| = 4$

If $<V - S> \cong 4K_1$, then $G \cong F_{3,1}$

If $<V - S> \cong 2K_1 \cup K_2$, then S will not be a γ_{cild} – set of G.

Subcase(4.b): $|V - S| = 5$

If $<V - S> \cong 5K_1$, then $G \cong F_{4,1}$ and $F_{2,2}$

If $<V - S> \cong 3K_1 \cup K_2$, then $G \cong F_{2,1}$

If $<V - S> \cong K_1 \cup K_2$, then S will not be a γ_{cild} – set of G.

Subcase(4.c): $|V - S| = 6$

If $<V - S> \cong 6K_1$, then $G \cong F_{4,2}$

If $<V - S> \cong 4K_1 \cup K_2$ (or) $2K_1 \cup 2K_2$, then G is not unicyclic.

Subcase(4.d): $|V - S| = 7$ (or) 8

Then either G is not unicyclic or S will not be a γ_{cild} – set of G.

Case (5): $<S> \cong P_4$

Subcase(5.a): $|V - S| = 4$

If <V – S> ≅ $4K_1$, then G ≅ $F_{5,1}$

If <V – S> ≅ $2K_1 \cup K_2$, then G ≅ $F_{3,1}$

Subcase(5.b): |V – S| = 5

If <V – S> ≅ $5K_1$, then G ≅ $F_{5,2}$

If <V – S> ≅ $K_1 \cup K_2$, then G is not unicyclic.

Subcase(5.c): |V – S| = 6, 7 (or) 8

Then either G is not unicyclic or S will not be a γ_{cild} – set of G.

Case (6): <S> ≅ C_4

Subcase(6.a): |V – S| = 4

If <V – S> ≅ $4K_1$, then G ≅ $F_{5,3}$

Subcase(6.b): |V – S| = 5, 6, 7 (or) 8

Then either G is not unicyclic or S will not be a γ_{cild} – set of G.

This completes the proof of the theorem.

Remark 3.13:

Let G be a connected unicyclic graph. Then $\gamma_{cild}(G) = 4$ if and only if G is isomorphic to one of the graphs in the family of graphs $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and $\mathcal{D}$.

IV. Conclusion

This paper results on finding the co – isolated locating domination number for unicyclic graphs. Determining the co – isolated locating domination number remain open. In particular the co – isolated locating domination number equal to 3 (or) 4 (or) 5 are of interest. For large values of n ≥ 6 proof similar to those presented in this paper get too complicated. So a new approach seems necessary.

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