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Decomposition of Line Graph into Paths and Cycles

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Abstract: *Let P_{k+1} denote a path of length k and let C_k denote a cycle of length k. As usual K_n denotes the complete graph on n vertices. In this paper we investigate decompositions of line graph of K_n into p copies of P_5 and q copies of C_4 for all possible values of $p \geq 0$ and $q \geq 0$.*
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I. Introduction

Unless stated otherwise all graphs considered here are finite, simple, and undirected. For the standard graph-theoretic terminology the readers are referred to [7].Let P_{k+1} denote a path of length k and let C_k denote a cycle of length k. Let S_k denotes a star on k vertices, i.e. $S_k = K_{1,k-1}$. As usual K_n denotes the complete graph on n vertices. Let $K_{m,n}$ denote the complete bipartite graph with m and n vertices in the parts. If G = (V,E) is a simple graph then the line graph of G is the graph L(G) = (E,L), where L = $\{\{e,f\} | \{e,f\} \subseteq E, |e \cap f| = 1\}$. For the sake of convenience, we use uv to denote an edge $\{u,v\}$ and u,v are called the ends of the edge $\{u,v\}$. In general, if one allows more than one edge (but a finite number) between same pair of vertices, the resulting graph is called a multigraph. In particular, if G is a simple graph then for any $\lambda \geq 1$, G (λ) and λG respectively denote the multigraph with edge-multiplicity λ and the disjoint union of λ copies of G.

By a decomposition of a graph G, we mean a list of edge-disjoint subgraphs $H_1,\ldots, H_k$, of G whose union is G (ignoring isolated vertices). When each subgraph in a decomposition is isomorphic to H, we say that G has an H-decomposition. It is easily seen that $\sum e(H_i) = e(G)$ is one of the obvious necessary condition for the existence of a $\{H_1,H_2,\ldots,H_k\}$ – decomposition of G. A $\{H_1,H_2\}$ – decomposition of G is a decomposition of G into copies of H_1 and H_2 using at least one of each. If G has a $\{H_1,H_2\}$-decomposition, we say that G is $\{H_1,H_2\}$-decomposable.

The problem of H-decomposition of $K_n(\lambda)$ is the well-known Alspach's conjecture [6] when H is any set of cycles of length at most n satisfying the necessary sum conditions and $2 \mid \lambda (n-1)$. For the case λ =1, Alspach conjecture is also stated for even values of n , where in this case the cycles should decompose K_n minus a one-factor. There are many related results, but only special cases of this conjecture are solved completely. When H is a set of paths, in this case the problem of H- decomposition has been investigated by Tarsi [19] who showed that if $(n-1)\lambda$ is even and H is any set of paths of length at most $n-3$ satisfying the necessary sum condition, then $K_n(\lambda)$ has an H-decomposition. The problem of H-decomposition of $K_{m,n}(\lambda)$ has been investigated by Truszczynski [20] when m and n are even and H is any set of the paths with some constraints on length satisfying the necessary sum condition. It is natural to consider the problem of H-decomposition of K_n, where H is a combination of paths, cycles, and some other subgraphs. We will restrict our attention to H which is any set of paths and cycles satisfying the necessary sum condition. There are several similarly known results as follows. A graph-pair of order t consists of two non-isomorphic graphs G and H on t non-isolated vertices for which G ∪ H is isomorphic to K_t.

Study on $\{H_1,H_2\}$-decomposition of graphs is not new. Abueida, Daven, and Rob- lee [1,3] completely determined the values of n for which $K_n(\lambda)$ admits the $\{H_1,H_2\}$-decomposition such that $H_1 \cup H_2 \cong K_t$, when $\lambda \geq 1$ and $|V(H_1)| = |V(H_2)| = t$, where $t \in \{4, 5\}$. Abueida and Daven [2] proved that there exists a $\{K_k, S_{k+1}\}$-decomposition of K_n for $k \geq 3$ and $n \equiv 0, 1 \pmod k$. Abueida and O'Neil [4] proved that for $k \in \{3, 4, 5\}$, the $\{C_k, S_k\}$-decomposition of $K_n(\lambda)$ exists, whenever $n \geq k + 1$ except for the ordered triples $(k,n, \lambda) \in \{(3, 4, 1), (4, 5, 1), (5, 6, 1), (5, 6, 2), (5, 6, 4), (5, 7, 1), (5, 8, 1)\}$. Abueida and Daven [2] obtained necessary and sufficient conditions for the $\{C_4, (2K_2)\}$- decomposition of the Cartesian product and tensor product of paths, cycles, and complete graphs. Shyu [14] obtained a necessary and sufficient condition for the existence of a $\{P_5, C_4\}$-decomposition of K_n. Shyu [15] proved that K_n has a $\{P_4, S_4\}$-decomposition if and only if $n \geq 6$ and $3(p+q)=\binom{n}{2}$. Also he proved that K_n has a $\{P_k, S_k\}$-decomposition with a restriction $p \geq k/2$, when k even (resp., $p \geq k$, when k odd). Shyu [16] obtained a necessary and sufficient condition for the existence of a $\{P_4,K_3\}$-decomposition of K_n. Shyu [17] proved that K_n has a $\{C_4, S_5\}$ - decomposition if and only if $4(p+q) = \binom{n}{2}$,

$q \neq 1$, when n is odd and $q \geq \max\{3, \frac{n}{4}\}$, when n is even. Shyu [18] proved that $K_{m,n}$ has a $\{P_k, S_k\}$-decomposition for some m and n and also obtained some necessary and sufficient condition for the existence of a $\{P_4, S_4\}$-decomposition of $K_{m,n}$. Sarvate and Zhang [13] obtained necessary and sufficient conditions for the existence of a $\{pP_3, qK_3\}$-decomposition of $K_n(\lambda)$, when $p = q$.

Chou et al. [8] proved that for a given triple (p,q,r) of nonnegative integers, G decompose into p copies of C_4, q copies of C_6, and r copies of C_8 such that $4p+6q+8r = |E(G)|$ in the following two cases: (a) $G = Km,n$ with m and n both even and greater than four (b) $G = K_{n,n} - I$, where n is odd. Chou and Fu [9] proved that the existence of a $\{C_4, C_{2t}\}$-decomposition of $K_{2u,2v}$, where $t/2 \leq u,v < t$, when t even (resp., $(t + 1)/2 \leq u,v \leq (3t - 1)/2$, when t odd) implies such decomposition in $K_{2m,2n}$, where $m,n \geq t$ (resp., $m,n \geq (3t + 1)/2$). Lee and Chu [10,11] obtained a necessary and sufficient condition for the existence of a $\{P_k, S_k\}$-decomposition of $K_{n,n}$ and $K_{m,n}$. Lee and Lin [12] obtained a necessary and sufficient condition for the existence of a $\{C_k, S_{k+1}\}$-decomposition of $K_{n,n} - I$. Abueida and Lian [5] obtained necessary and sufficient conditions for the existence of a $\{C_k, S_{k+1}\}$-decomposition of K_n for some n.

In this paper we investigate decompositions of line graph of K_n into p copies of P_5 and q copies of C_4 for all possible values of $p \geq 0$ and $q \geq 0$.

II. {P5,C4}-decomposition of L(Kn)

In this section, we investigate the existence of $\{P_5,C_4\}$-decomposition of $L(K_n)$. Note that, the line graph of C_4 is C_4.

III. Construction:

Let $\mathbb{C}_4^a$ and $\mathbb{C}_4^b$ be two cycles of length 4, where $\mathbb{C}_4^a = (abcda)$ and $\mathbb{C}_4^b = (wxyzw)$. If v is a only common vertex of $\mathbb{C}_4^a$ and $\mathbb{C}_4^b$, say $c = y = v$, then we have two paths of length 4 as follows: $abvzw, wxvda$.

Lemma 2.1. There exists a $\{P_5, C_4\}$-decomposition of $K_{4,4}$.

Proof . Let $V(K_{4,4}) = \{x_1, x_2, x_3, x_4\} \cup \{y_1, y_2, y_3, y_4\}$. We exhibit the $\{P_5,C_4\}$-decomposition of $K_{4,4}$ as follows:

(1) $p = 0$ and $q = 4$. The required cycles are
$(x_1y_1x_2y_2x_1)$, $(x_1y_3x_2y_4x_1)$, $(x_3y_1x_4y_2x_3)$, $(x_3y_3x_4y_4x_3)$.
(2) $p = 2$ and $q = 2$. The required paths and cycles are
$y_1x_1y_2x_2y_4$, $y_4x_1y_3x_2y_1$ and $(x_3y_1x_4y_2x_3)$, $(x_3y_3x_4y_4x_3)$.
(3) $p = 3$ and $q = 1$. The required paths and cycles are
$x_2y_2x_3y_1x_1$, $x_1y_3x_2y_1x_4$, $x_4y_2x_1y_4x_2$ and $(x_3y_3x_4y_4x_3)$.
(4) p = 4 and q = 0. The required paths are
$y_1x_1y_2x_2y_4$, $y_4x_1y_3x_2y_1$, $y_3x_3y_4x_4y_2$, $y_2x_3y_1x_4y_3$.

Lemma 2.2. There exists a $\{P_5,C_4\}$-decomposition of the complete bipartite graph $K_{8,8m}$ for all $m \geq 1$.

Proof. Let $V(K_{8,8m}) = (X, Y)$ with $X = \{x_1, \ldots, x_8\}$ $Y = \{y_1, \ldots, y_{8m}\}$. Partition (X, Y) into 4-subsets (A_i^x, A_j^y) such that $\bigcup_{i=1}^{2} A_i^x = X$, $\bigcup_{j=1}^{2m} A_j^y = Y$. Then $G[A_i^x, A_j^y] \cong K_{4,4}$. Thus $K_{8,8m} = 4m(K_{4,4})$. By Lemma 2.1, the graph $K_{4,4}$ has a $\{P_5,C_4\}$-decomposition. Hence the graph $K_{8,8m}$ has the desired decomposition.

Lemma 2.3. There exists a $\{P_5,C_4\}$-decomposition of the graph $K_{8m} - F$ for all $m \geq 1$, where F is a 1-factor of K_{8m}.

Proof. Let $8m = 4k$, where k is a positive integer and $V(K_{4k} - I) = \{x_0, \ldots, x_{4k-1}\} = \{\bigcup_{i=0}^{2k-1}(A_i)\}$, where $I = \{x_{2i}x_{2i+1} : 0 \leq i \leq 2k-1\}$ and $Ai = \{x_{2i}, x_{2i+1}\}$, $0 \leq i \leq 2k-1$. We obtain a new graph G from $K_{4k} - I$, by identifying each set Ai with a single vertex ai, join two vertices a_i and a_j by an edge if the corresponding sets A_i and A_j form a $K|A_i|,|A_j|$ in K_{4k}. Then the new graph $G = K_{2k}$. The graph K_{2k} has a hamilton path decomposition $\{G_0, \ldots, G_k\}$, where each G_i, $0 \leq i \leq k$ is a hamilton path. Each G_i decomposes some copies of P_3 or P_4. When we go back to $K_{4k} - I$, each P_3 (resp., P_4) of G_i, $0 \leq i \leq k$ will gives rise to $2C_4$ or $2P_5$ (resp., $\{2P_5, 1C_4\}$ or $3P_5$) in $K_{4k} - I$. Hence the graph $K_m - I$ has the desired decomposition.

Lemma 2.4. There exit a $\{P_5, C_4\}$ – decomposition of the graph $L(K_9)$.

Proof. For $0 \leq i \leq 8$, by Lemma 2.3 there exit a $\{P_5,C_4\}$-decomposition $Gi \cong K8 - Fi$, where Gi defined on the vertex set $\{\{i,j\}|0\leq j \leq 8, j \neq i\}$ and where $Fi = \{\{\{ i , 1+i\},\{ i,2+i\}, \{i ,3+i\},\{ i,7+i\},\},\{\{i, 4+i\},\{i, 8+i\}, \{i, 5+i\},\{i,6+i\}\}\}$, reducing all sums modulo 9.Then a $\{P_5,C_4\}$-decomposition of $L(K_9)$ on the vertex set $\{\{i,j\}|\{i,j\} \subseteq \{0,1,....,8\}\}$ follows from the partition of the edges of $\bigcup_{i=0}^{8}$ into P_5 or C_4, $\{(\{j,j+1\},\{1+j,5+ j \}, \{5+ j,2+ j \},\{2+ j , j \}) \mid 0 \leq j \leq 8\}$, again reducing all sums modulo 9.

Theorem 2.1. If $n \equiv 1$ (mod 8) then there exists a $\{P_5,C_4\}$ – decomposition of $L(K_n)$.
Proof. The result is true for $n = 9$, so we proceed by induction on n. Let $n = 8m + 1$, $m \geq 2$, and let the vertex set of k_n be $\{\infty\} \cup \{1,...,8m\}$. $L(K_n)$ can be partitioned into the following edge-disjoint subgraphs:

1. $L(K_9)$, K_9 being defined on the vertex set $\{\infty\} \cup \{1,\ldots,8\}$;
2. $L(K_{8m-7})$, K_{8m-7} being defined on the vertex set $\{\infty\} \cup \{9,\ldots,8m\}$;
3. For $1 \leq i \leq 8$, $G_i \cong K_{8m-8} - F_i$ on the vertex set $\{\{ i , j \}| \ 9 \leq j \leq 8m\}$, where $F_i =\{\{ i , 2k-1\}, \{ i , 2k\} \mid 5 \leq k \leq 4m$;
4. For $9 \leq j \leq 8m$, $G_i \cong K_8 - F_j$ on the vertex set $\{\{ i,j \}| \ 1 \leq i \leq 8\}$, where $F_j = \{\{2k-1,j\},\{2k,j\}| \ 1 \leq k \leq 4$;
5. $(\bigcup_{i=1}^{8}(F_i)) \cup (\bigcup_{j=9}^{8m}(F_i))$;
6. $K_{8,8m-8}$ with b-partition $\{\{\infty , i\}|1 \leq i \leq 8\}$ and $\{\{\infty, j\}| \ 9 \leq j \leq 8m$;
7. For $1 \leq i \leq 8$, $H_i \cong K_{8,8m-8}$ with bipartition $\{\{i,k\}| \ k \in\{\infty,1,2,\ldots,8\} \setminus\{i\}\}$ and $\{\{i,j\}| \ 9 \leq j \leq 8m\}$; and
8. For $9 \leq j \leq 8m$, $H_j \cong K_{8,8m-8}$ with bi-partition $\{\{i,j\}|1 \leq i \leq 8\}$ and $\{\{k ,j\}| \ k \in \{\infty,9,10,\ldots,8m\}\setminus \{j\}\}$.

The result now follows, since there exist $\{P_5,C_4\}$-decomposition of graphs defined in (1) and (2) by induction and Lemma 2.4, (3) and (4) by Lemma 2.3, (5) since these edges form vertex-disjoint C_4, and (6),(7) and (8) by Lemma 2.2. Hence the graph $L(K_n)$ has the desired decomposition.

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