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Numerical Solution of the Linear and Nonlinear Stiff Problems Using Adomian Decomposition Method

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Abstract : *In this article, a new method of analysis for linear and nonlinear stiff problems using the Adomian Decomposition Method (ADM) is presented. To illustrate the effectiveness of the ADM an example from linear and nonlinear stiff problems have been considered and compared with the single-term Haar Wavelet series (STHW)[9] and with exact solutions of the problems, and are found to be very accurate. Error graphs for the linear and nonlinear stiff problems are presented in a graphical form to show the efficiency of this ADM. This ADM can be easily implemented in a digital computer and the solution can be obtained for any length of time.*

Keywords: *Linear stiff differential equations, nonlinear stiff differential equations, Single-term Haar wavelet series, Adomian Decomposition Method.*

I. Introduction

Among the models using differential equations (DE), ordinary differential equations are frequently used to describe various physical problems, for example, motions of the planet in a gravity field like the Kepler problem, the simple pendulum, electrical circuits and chemical kinetics problems.

An ordinary differential equation (ODE) has the form

$$y'(x) = f(x, y(x)) \tag{1}$$

where x is the independent variable which often to time in a physical problem and the dependent variable, *y(x)*, is the solution. Moreover, since *y(x)* could be an *N* dimensional vector valued function, the domain and range of the differential equation, *f* and the solution, *y* are given by

$$f : R \times R^N \to R^N,$$

$$y : R \to R^N.$$

The above equation (1) where *f* is a function of both *x* and *y* are called 'non-autonomous'. However, by simply introducing an extra variable which is always exactly equal to *x*, it can be easily rewritten in an equivalent 'autonomous' form below where *f* is a function of *y* only.

$$y'(x) = f(y(x)) \tag{2}$$

Even though many problems are naturally expressed in the non-autonomous form, the autonomous form of differential equation (2) is preferred for most of the theoretical investigations. Furthermore, the autonomous form has some advantages in numerical analysis since it gives a greater possibility that numerical methods can solve the differential equation exactly.

The differential equation by itself is not enough to find a unique solution. Hence, some other additional information is needed. However, if all components of y are given at a certain value of x i.e. 'initial conditions' then the differential equation is called as an 'initial value problem (IVP)' which is closely and naturally involved with physical modeling.

An initial value problem with the given initial condition $y(x_0) = y_0$ has the structure

$$y'(x) = f(x, y(x)), \; y(x_0) = y_0 \tag{3}$$

in (3) non-autonomous form and

$$y'(x) = f(y(x)), y(x_0) = y_0 \tag{4}$$

in (4) autonomous form.

In this paper we developed numerical methods for addressing linear and nonlinear stiff differential equations by an application of the Adomian Decomposition Method which was studied by Sekar and team of his researchers [4-7]. Recently, Sekar *et al.* [8] discussed the linear and nonlinear stiff differential equations using STHW. In this paper, the same linear and nonlinear stiff differential equations was considered (discussed by Sekar *et al.* [8]) but present a different approach using the Adomian Decomposition Method with more accuracy for linear and nonlinear stiff differential equations. In this paper we show the simulation results in graphical form to highlight the effectives of ADM compare to STHW.

II. Adomian Decomposition Method

Suppose k is a positive integer and $f_1, f_2, \dots, f_k$ are k real continuous functions defined on some domain G. To obtain k differentiable functions $y_1, y_2, \dots, y_k$ defined on the interval I such that $(t, y_1(t), y_2(t), \dots, y_k(t)) \in G$ for $t \in I$.

Let us consider the problems in the following system of ordinary differential equations:

$$\frac{dy_i(t)}{dt} = f_i(t, y_1(t), y_2(t), \dots, y_k(t)) \quad , \quad y_i(t)\,|_{t=0} = \beta_i \tag{5}$$

where β_i is a specified constant vector, $y_i(t)$ is the solution vector for $i = 1, 2, \dots, k$. In the decomposition method, (5) is approximated by the operators in the form: $Ly_i(t) = f_i(t, y_1(t), y_2(t), \dots, y_k(t))$ where L is the first order operator defined by $L = d/dt$ and $i = 1, 2, \dots, k$.

Assuming the inverse operator of L is L^{-1} which is invertible and denoted by $L^{-1}(.) = \int_{t_0}^{t} (.)\,dt$, then applying L^{-1} to $L\,y_i(t)$ yields

$$L^{-1}Ly_i(t) = L^{-1} f_i(t, y_1(t), y_2(t), \dots, y_k(t))$$

where $i = 1, 2, \dots, k$. Thus

$$y_i(t) = y_i(t_0) + L^{-1} f_i(t, y_1(t), y_2(t), \dots, y_k(t)) .$$

Hence the decomposition method consists of representing $y_i(t)$ in the decomposition series form given by

$$y_i(t) = \sum_{n=0}^{\infty} f_{i,n}(t, y_1(t), y_2(t), \dots, y_k(t))$$

where the components $y_{i,n}$, $n \ge 1$ and $i = 1, 2, \dots, k$ can be computed readily in a recursive manner. Then the series solution is obtained as

$$y_i(t) = y_{i,0}(t) + \sum_{n=1}^{\infty} \{L^{-1} f_{i,n}(t, y_1(t), y_2(t), \dots, y_k(t))\} .$$

For a detailed explanation of decomposition method and a general formula of Adomian polynomials, we refer reader to [Adomian 1].

III. Stiff Problems

Even if there exists the numerical solution to a differential equation, certain types of differential equations are difficult to solve, in fact, they need certain types of numerical methods. This phenomenon known as ‘stiffness’ was first recognized by Curtiss and Hirschfelder [3] in 1952. Stiffness occurs when some components of the solution decay much more rapidly than others. These problems have highly stable exact solutions but have highly unstable numerical solutions. There are several ways of characterizing ‘stiffness’ and one way of understanding is looking at the Lipschitz constant. Stiff problems typically have a large Lipschitz constant; however, many of them have a more moderate size one-sided Lipschitz constant.

Definition.

Butcher [2] *The function* $f : [a,b] \times R^N \to R^N$ *is said to satisfy a ‘one-sided Lipschitz condition’ if there exists a ‘one-sided Lipschitz constant’ l, such that for all* $x \in [a,b]$ *and all* $y, z \in R^N$,

$$\langle f(x,y) - f(x,z), y - z \rangle \le l \|y - z\|^2$$

where the norm is defined by $\|y\|^2 = \langle y, y \rangle$ *assuming that there exists an inner-product on* R^N.

Therefore, the Lipschitz constant could be large while the one-sided Lipschitz constant could be small, or even negative. This theorem leads me to deduce the following result.

Theorem.

Butcher [2] *If f satisfies a one-sided Lipschitz condition with one-sided Lipschitz constant l, and y and z are solutions of* $y'(x) = f(x, y(x))$, *then for all* $x \geq x_0$,

$$\|y(x) - z(x)\| \leq \exp(l(x - x_0))\|y(x_0) - z(x_0)\| .$$

Notice from this result that the distance between any two solutions will not increase rapidly or may even decrease if the equation has an adequate one-sided Lipschitz constant. Since stiffness is closely related to the behaviour of perturbations to a given solution, it is important to find out the effect of small perturbations with a one-sided Lipschitz condition.

Consider $$y'(x) = f(x, y(x)) \quad (6)$$

with *y(x)*, a solution, and $\varepsilon Y(x)$, a small perturbation to the given solution. Replace *y(x)* in the equation (6) by $y(x) + \varepsilon Y(x)$ and expand the solution in a series in powers of ε up to the second order, then get

$$y'(x) + \varepsilon Y'(x) = f(x, y(x)) + \varepsilon \frac{\partial f}{\partial y} Y(x). \quad (7)$$

Subtract the equation (6) from (7) and simplify it, then finally obtain the equation which controls the behaviour of the perturbation,

$$Y'(x) = \frac{\partial f}{\partial y} Y(x) = J(x) Y(x)$$

where *J(x)* is the Jacobian matrix of $f(x, y(x))$. I can use the spectrum of eigen values of *J(x)* to characterise stiffness. The eigen values of *J(x)* determine the growth rate of the perturbation with a moderate change in the value of the solution and a very small change in *J(x)* in a time interval Δx. The existence of one or more large and negative values of λ where $\lambda \in \sigma(J(x))$ where $x \in \Delta x$ indicates that stiffness is present.

IV. Numerical Example for Linear Stiff Problem

Stiffness can be understood by the practical difficulty found in numerical calculation as well. The stiff problems are impossible or very difficult to solve by explicit methods, mainly because the small bounded stability region of explicit methods forces the numerical method to take very small step sizes for the smooth solution. Two examples of stiff problems are given here to observe how explicit and implicit methods work for these problems.

Consider the stiff system of three linear ordinary differential equations with corresponding initial conditions is of the autonomous form.

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ -L & 1 & L \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix}, \begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \varepsilon \end{bmatrix}$$

where L= - 25 and ε =2.

The analytic solution is which is drawn in Figure 1. $\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix} = \begin{bmatrix} \sin(x) \\ \cos(x) \\ \sin(x) + \varepsilon \exp(Lx) \end{bmatrix}$

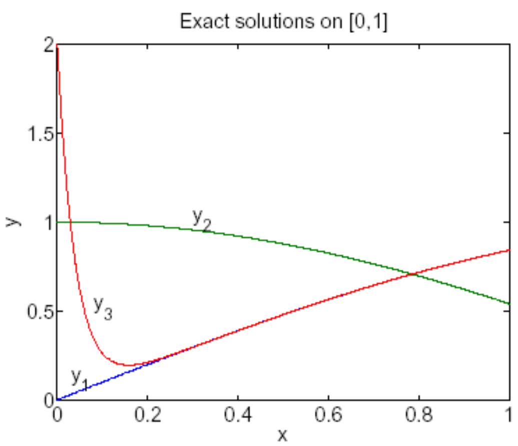

Fig. 1 Analytical Solution of linear Stiff problem

The results of using the STHW and ADM methods for solving this stiff problem on the interval of [0,1] are presented in Fig. 2-7. The Fig. 2-4 show that the STHW method definitely seems to have difficulty approximating y_3 while y_1 and y_2 are computed without difficulties. Especially the approximations with *n*=10 and *n*=15 are hopeless. However, the ADM method performs perfectly well even for *n* as low as 4 as shown in Figures 5-7.

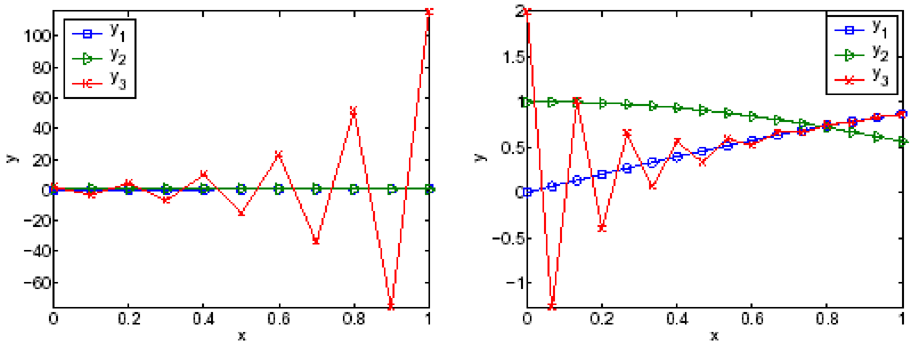

Fig. 2 STHW solution of linear Stiff problem with n = 4 and 8

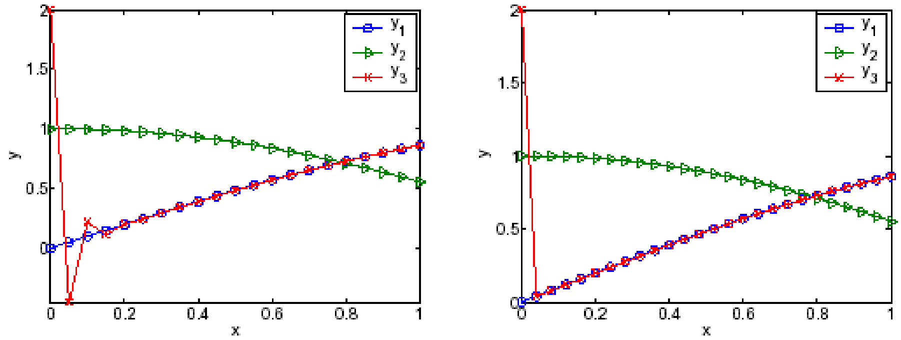

Fig. 3 STHW solution of linear Stiff problem with n = 12 and 16

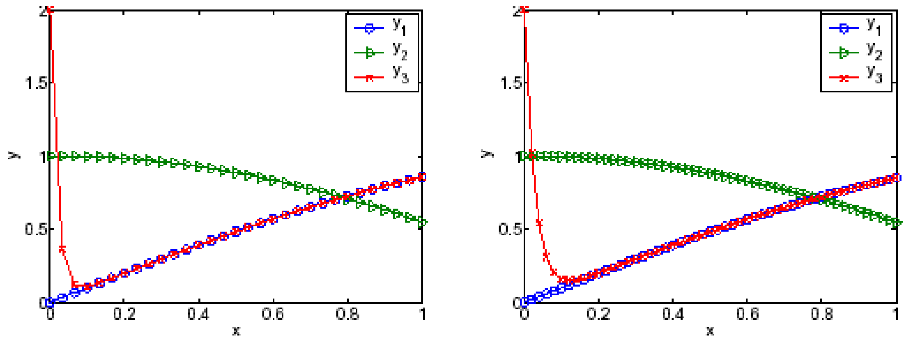

Fig. 4 STHW solution of linear Stiff problem with n = 20 and 30

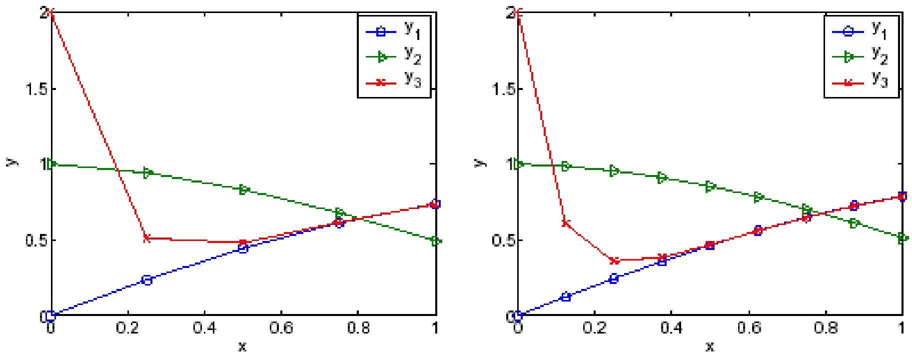

Figure 5 ADM solution of linear Stiff problem with n = 4 and 8

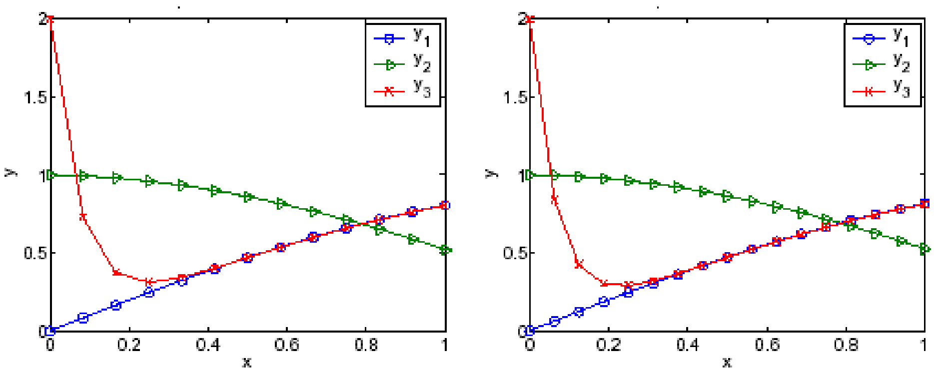

Fig. 6 ADM solution of linear Stiff problem with n = 12 and 16

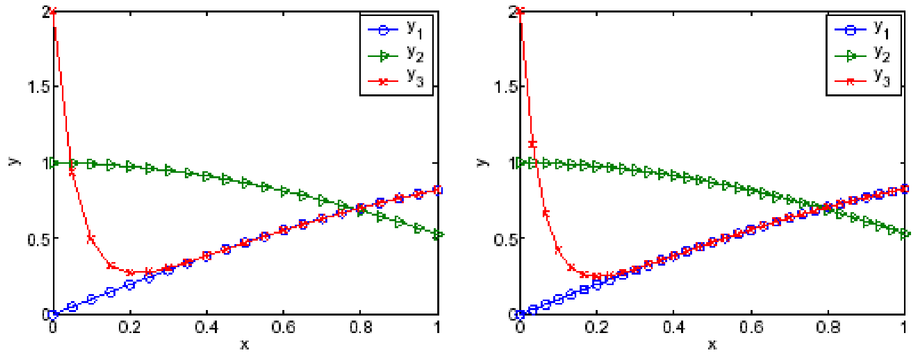

Fig. 7 ADM solution of linear Stiff problem with n = 20 and 30

V. Numerical Example for Nonlinear Stiff Problem

Consider the stiff system of two dimensional Kaps problem with corresponding initial conditions is of the non autonomous form.

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} -1002 \ y_1(x) + 1000 \ y_2(x)^2 \\ y_1(x) - y_2(x)(1 + y_2(x)) \end{bmatrix}, \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Analytic solution is

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} \exp(-2x) \\ \exp(-x) \end{bmatrix}$$

which is drawn in Figure 8.

In Fig. 9-14, the computed solutions of this problem using the STHW and ADM method on the interval of [0,10] are displayed. Even using a large number of steps, the STHW method performs poorly. However the ADM method easily gives a good approximation. From these two examples, it is clearly confirmed that the STHW method is not suitable but the ADM method is appropriate for stiff problems.

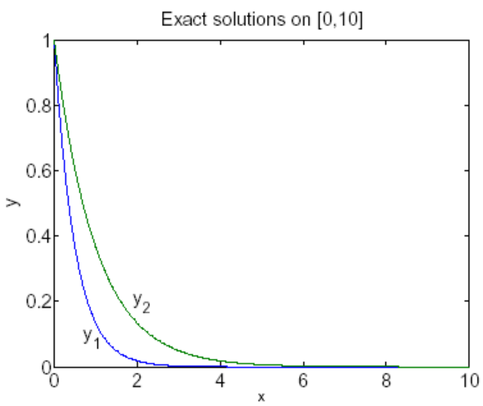

Fig. 8 Analytical Solution of Stiff nonlinear problem.

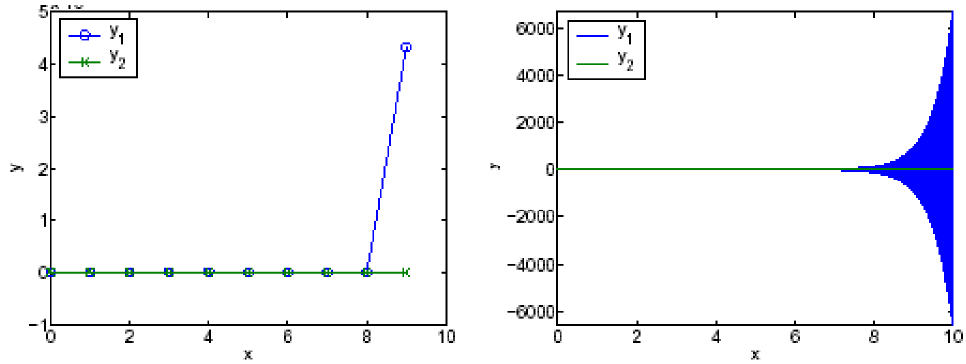

Fig. 9 STHW solution of Stiff nonlinear problem with n = 4 and 8

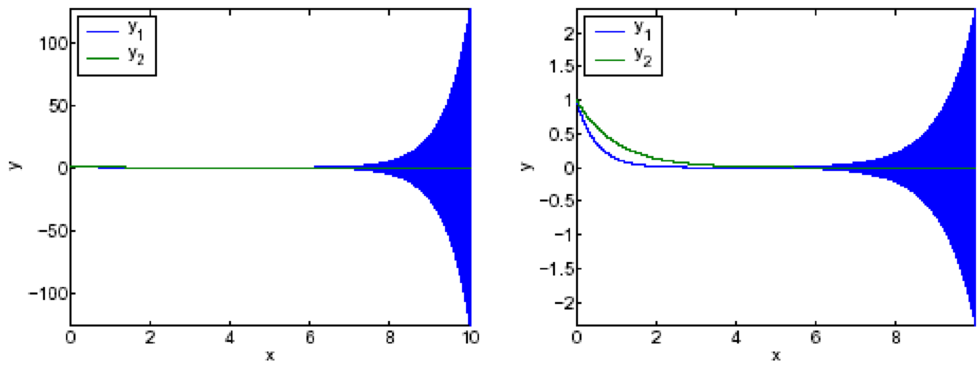

Fig. 10 STHW solution of Stiff nonlinear problem with n = 12 and 16

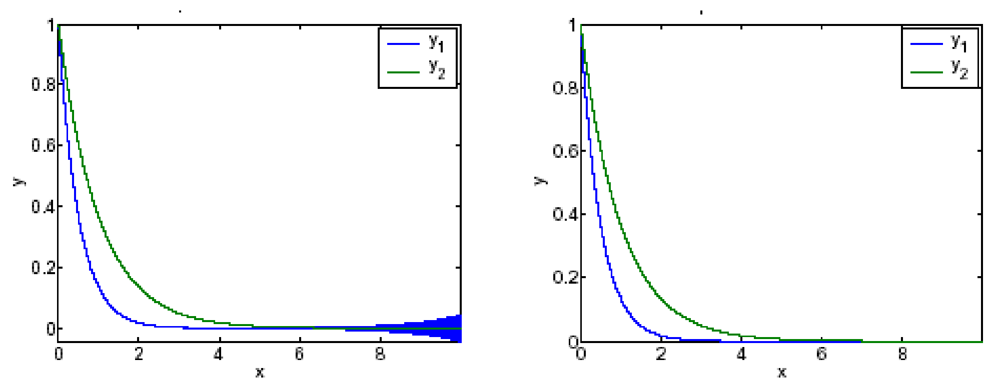

Fig. 11 STHW solution of Stiff nonlinear problem with n = 20 and 30

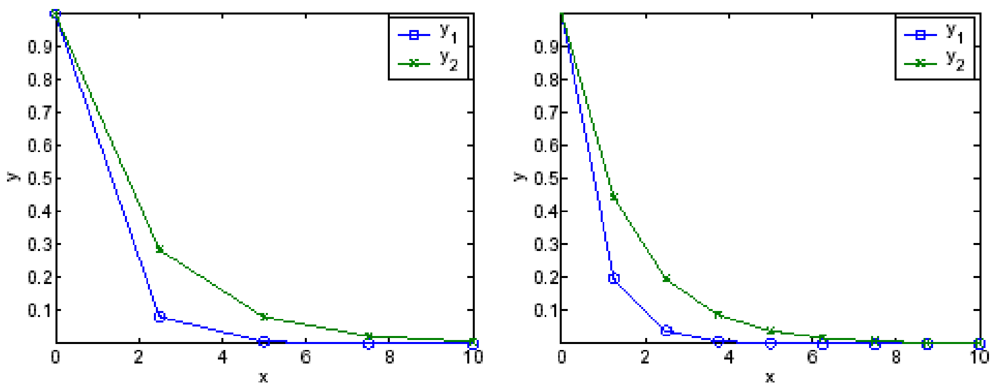

Fig. 12 ADM solution of Stiff nonlinear problem with n = 4 and 8

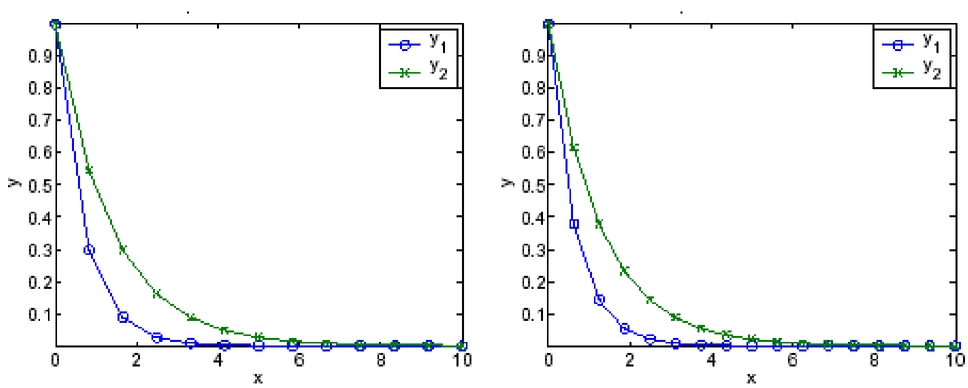

Fig. 13 ADM solution of Stiff nonlinear problem with n = 12 and 16

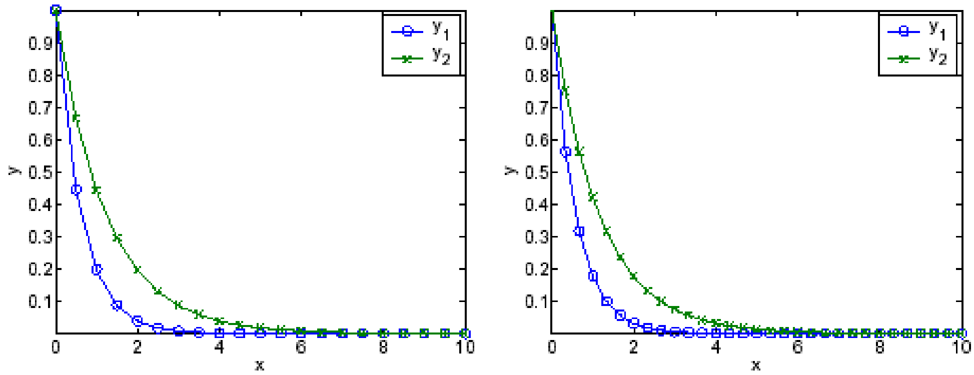

Fig. 14 ADM solution of Stiff nonlinear problem with n = 20 and 30

VI. Conclusion

The STHW method is simple. It uses only four pieces of information from the past and evaluates the driving function only four per step. However, the STHW method is not very practical for computational purpose since considerable computational effort is required to improve accuracy. From the Figures 2-7 and 9-14, one can predict that the error is very less in ADM when compared to the STHW method. This ADM provided a momentum for advancing numerical methods for solving linear and nonlinear stiff problems.

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