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Two Square Determinant Approach for Simplex Method

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Abstract: *Linear Programming Problems have been solved using traditional Algorithms of Simplex Methods. In this paper a modified approach to Simplex Method has been worked upon. This approach involves a determinant at every step to perform the next iteration till the optimum solution is obtained.*

Keywords: *Constraints, Determinant, Feasible Solution, Linear Programming Problem, optimum Solution, Pivot, Simplex Method , Slack Variables, Surplus Variables.*

I. Introduction

Linear Programming Problems deal with optimization i.e maximization or minimization of a function known as objective function. This objective function has to be optimized subject to certain constraints [1],[2],[3]. All expressions involved in the problems are linear. The constraints are determined by certain restrictive conditions such as availability, demand, supply, storage, availability, etc [4],[5]. Solutions to such type of problems are of prime importance in the field of business. A powerful tool to solve such problems is a simplex method.

II. Main Idea Of The Proposed Algorithm

The traditional simplex method, on identifying the pivotal element, uses a chain of elementary row operations only to obtain further iterates until a optimum solution is reached. We propose an algorithm which uses a determinant at every step, to perform the next iterate, depending on the position of the pivot and the elements to be changed to obtain optimum solutions. This proposed algorithm is very efficient and time-saving. It is much simpler as compared to the already existing traditional simplex method. We shall now propose the algorithm involving constraints of "less than or equal to" type. The same algorithm works if the constraints are either "equal to" type or "greater than or equal to" type.

Algorithm for the linear programming problem

Max $Z = d_1 x_1 + d_2 x_2 + \ldots + d_n x_n$

Subject to

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n \le l_1$$
$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n \le l_2$$
$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n \le l_m$$
$$x_i \ge 0 \quad \text{for all } i \in \{1,2,\ldots,n\}$$

Step 1: Writing the problem in standard form.

We convert the constraints above into equalities by introducing slack variables $s_{11}, s_{22}, \ldots s_{mm}$. The problem in standard form is expressed as

Max $Z = d_1 x_1 + d_2 x_2 + \ldots + d_n x_n + s_{11} + s_{22} + \cdots + s_{mm}$

Subject to

$$a_{11} x_1 + a_{12} x_2 + \cdots + a_{1n} x_n + s_{11} = l_1$$
$$a_{21} x_1 + a_{22} x_2 + \cdots + a_{2n} x_n + s_{22} = l_2$$

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$$a_{m1} x_1 + a_{m2} x_2 + \cdots + a_{mn} x_n + s_{mm} = l_m$$
$$x_i \ge 0 \quad \text{for all } i \in \{1,2,\ldots,n\}$$

Step 2: Finding an initial basic feasible solution.
The method starts by assuming that the profit is zero. This is when x_i= 0 for all i and $s_{ii} = l_i$

a) The first row indicates coefficients c_j of variables in the objective function.
b) The first column, $C_B -$column represents the coefficients of current basic variables in the objective functions. The second column known as the basis column, y_B, represents the basic variables which are the slack variables of the current solutions.
c) The body of coefficient matrix under non-basic variables x_i represents the coefficients a_{ij} of the constraints.
d) The identity matrix represents the coefficients of the slack variables of the constraints.
e) The third column will be the x_B column which indicates the quantity of the available resources or values of the constraints or values of the basic variables s_i in the initial basic feasible solution found earlier.

Step 3: Optimality Test.

Compute $Z_j = \sum C_B a_{ij}$
Next compute $Z_j - C_j$
If $Z_j - C_j \geq 0$ for all j, then an optimum solution has been attained.
If for at least one j, $Z_j - C_j < 0$, proceed to Step 4.

Step 4: Iteration towards a Optimal Solution using Determinants:

a) Selection of the entering variable:
Select that column for which $Z_j - C_j$ is most negative. This column is called the pivotal column. If two columns have the same maximum negative value, the tie can be broken arbitrarily.

b) Selection of leaving variable:
Compute min $\left\{\frac{x_B}{a_{ij}} \middle| a_{ij} > 0, a_{ij} \in pivotal\ column\right\}$. If there is a tie in the minimum values, arbitrarily break the tie. If all $a_{ij} \leq 0$, then the solution is unbounded. That a_{ij} for which this ratio is minimum is called pivotal element. Let the row in which the pivotal element occurs be called the pivotal row.

c) Updating the new solution/preparation of simplex table using determinants:
Let the pivotal element be a_{ij}. Divide the pivotal row by the pivotal element. All the other elements in the pivotal column must be made zero. Now, we update the simplex table as follows:

Case 1: Consider an element a_{rs}, $r < i, s < j$. This element a_{rs} is updated to $det\begin{pmatrix} a_{rs} & a_{rj} \\ a_{is} & a_{ij} \end{pmatrix}$ where det stands for determinant. The element a_{rs}, is now standardized by dividing it with the pivotal element.

Case 2: Consider an element a_{pq} in the coefficient matrix such that $p > i,\ q < j.$ The element a_{pq} is updated to $(-1).det\begin{pmatrix} a_{ia} & a_{ij} \\ a_{pq} & a_{pj} \end{pmatrix}$. The element a_{pq}, is now standardized by dividing it with the pivotal element.

Case 3: Consider an element a_{uv} in the coefficient matrix such that u$< i, v > j.$ The element a_{uv} is updated to $(-1).det\begin{pmatrix} a_{uj} & a_{uv} \\ a_{ij} & a_{iv} \end{pmatrix}$. The element a_{uv}, is now standardized by dividing it with the pivotal element.

Case 4: Consider an element a_{gh} in the coefficient matrix such that $g > i,\ h > j.$ The element a_{gh} is updated to det $\begin{pmatrix} a_{ij} & a_{ih} \\ a_{gj} & a_{gh} \end{pmatrix}$. The element a_{gh}, is now standardized by dividing it with the pivotal element.

Step 5: Go to Step 3.

III. Application of the Algorithm

We apply the determinantal approach to simplex method to solve the linear programming problem.

Maximize $Z = 4y_1 + 3y_2 + 6y_3$

subject to $2y_1 + 3y_2 + 2y_3 \leq 440,$

$4y_1 + 3y_3 \leq 470$

$2y_1 + 5y_2 \leq 430$

$y_i \geq 0$, i=1,2,3

Solution: We first write the problem in standard form by introducing slack variables $s_i, \; i = 1,2,3$

Maximize $Z = 4y_1 + 3y_2 + 6y_3 + s_1 + s_2 + s_3$

subject to $2y_1 + 3y_2 + 2y_3 + s_1 = 440,$

$4y_1 + 3y_3 + s_2 = 470$

$2y_1 + 5y_2 + s_3 = 430$

$y_i, s_i \geq 0$, i=1,2,3

We know find an initial basic feasible solution by setting $y_i = 0$

		C_j	4	3	6	0	0	0
C_B	y_B	x_B	y_1	y_2	y_3	s_1	s_2	s_3
0	s_1	440	2	3	2	1	0	0
0	s_2	470	4	0	3	0	1	0
0	s_3	430	2	5	0	0	0	1
		$Z_j - C_j$	-4	-3	-6	0	0	0

The pivotal column is y_3 as it has most negative $Z_j - C_j$ value. (By step 4a) of the algorithm)

By step 4b) of the algorithm since $\min\{\frac{440}{2}, \frac{470}{3}\} = \frac{470}{3}$, s_2 is outgoing.

By step 4c) of the algorithm we divide outgoing row by 3, the pivot and make all entries above and below 3 as 0

By determinant technique the entry below y_1 is modified to $\det\begin{pmatrix} 2 & 2 \\ 4 & 3 \end{pmatrix} = -2$

The entry is standardized by dividing it with the pivot, 3, to become $\frac{-2}{3}$

The entry below y_2 is modified to $\det\begin{pmatrix} 3 & 2 \\ 0 & 3 \end{pmatrix} = 9$

The entry is standardized by dividing it with the pivot, 3, to become 3

The entry below s_1 is modified to $(-1).\det\begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix} = 3$

The entry is standardized by dividing it with the pivot, 3, to become 1

The entry below s_2 is modified to $(-1).\det\begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} = -2$

The entry is standardized by dividing it with the pivot, 3, to become $\frac{-2}{3}$

The entry below s_3 is modified to $(-1).\det\begin{pmatrix} 2 & 0 \\ 3 & 0 \end{pmatrix} = 0$

The entry is standardized by dividing it with the pivot, 3, to become 0

The entry in the third row of the coefficient matrix below y_1 is modified to $(-1).\det\begin{pmatrix} 4 & 3 \\ 2 & 0 \end{pmatrix} = 6$

The entry is standardized by dividing it with the pivot, 3, to become 2

The entry in the third row of the coefficient matrix below y_2 is modified to $(-1).\det\begin{pmatrix} 0 & 3 \\ 5 & 0 \end{pmatrix} = 15$

The entry is standardized by dividing it with the pivot, 3, to become 5

The entry in the third row of the coefficient matrix below s_1 is modified to $\det\begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix} = 0$

The entry is standardized by dividing it with the pivot, 3, to become 0

The entry in the third row of the coefficient matrix below s_2 is modified to $\det\begin{pmatrix} 3 & 1 \\ 0 & 0 \end{pmatrix} = 0$

The entry is standardized by dividing it with the pivot, 3, to become 0

The entry in the third row of the coefficient matrix below s_3 is modified to $\det\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} = 3$

The entry is standardized by dividing it with the pivot, 3, to become 1

The first entry in the column of the coefficient matrix below x_B is modified to $\det\begin{pmatrix} 440 & 2 \\ 470 & 3 \end{pmatrix} = 380$

The entry is standardized by dividing it with the pivot, 3, to become $\frac{380}{3}$.

The third entry in the column of the coefficient matrix below x_B is modified to $(-1).\det\begin{pmatrix}470 & 3\\430 & 0\end{pmatrix} = 1290$

The entry is standardized by dividing it with the pivot, 3, to become 430.

With these modifications we get the next iterated table

		C_j	4	3	6	0	0	0
C_B	y_B	x_B	y_1	y_2	y_3	s_1	s_2	s_3
0	s_1	$380/3$	$-2/3$	3	0	1	$-2/3$	0
6	y_3	470/3	4/3	0	1	0	1/3	0
0	s_3	430	2	5	0	0	0	1
		$Z_j - C_j$	4	-3	0	0	2	0

The next pivotal column is $y_2$and s_1 leaves the basis. 3 is the next pivot.

Repeating step 4 of the algorithm we get

		C_j	4	3	6	0	0	0
C_B	y_B	x_B	y_1	y_2	y_3	s_1	s_2	s_3
3	y_2	$380/9$	$-2/9$	1	0	1/3	$-2/9$	0
6	y_3	470/3	4/3	0	1	0	1/3	0
0	s_3	1970/9	28/9	0	0	$-5/3$	10/9	1
		$Z_j - C_j$	10/3	0	0	1	4/3	0

Since all the values of $Z_j - C_j \geq 0$, we stop

We have an optimum solution given by

Maximum $Z = 3200/3$

When $y_1 = 0, y_2 = \frac{380}{9}, y_3 = \frac{470}{3}$

IV. Conclusion.

The determinant approach to simplex method is much more efficient and time-saving than the previous traditional approach to the simplex method which involves row transformations. This is because at each step, we need to compute only a 2 x 2 determinant which can even be done mentally.

The algorithm can also be used for cases where we encounter an unbounded, alternate or non feasible solution. It can also be used to perform the two-phase simplex method.

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