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An application of Volterra integral equation by expansion of Taylor's series in the problem of heat transfer and electrostatics

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Abstract: *The class of Volterra integral equation of second kind with smooth and weakly singular kernel is presented with the help of Taylor's series expansion method. Here we get approximate solution of the integral equation which can be expressed explicitly in a simple closed form. The method of solution of equations studied in the presented paper is very useful to solve the physical problems as radiative heat transfer, radiative equilibrium, potential theory and electrostatics. The numerical examples arising from radiative heat transfer and electrostatics which are accurate and in a good manner.*

Index Term*: Volterra integral equation, Taylor's series, radiative heat transfer etc*

I. Introduction

Here we consider a Volterra integral equation of second and then find its approximate solution by method of expansion of Taylor's series. A number of attempts have been made to use the Taylors series expansion for the solution of second kind Fredholm integral equations [1] , [2], [3]. Volterra integral equation of second kind arise in many physical applications as Dirichlet problem, reactor theory, electrostatics, astrophysics and radiative heat transfer problems [13], [14], [15]. Numerical solution of integral equation have been studied by many authors using variational iterative method, series solution method, quadrature method, discretization method and projection method etc. [2] , [3], [4]. These numerical methods transform the integral equation to a linear system of algebraic equation which can be solved by direct or iterative methods but the matrix of the linear system is very complicated to obtain and expensive computationally.

Therefore a new technique alternative approximate solution has been used to solve the second integral equations associated with radiative heat transfer. This new technique transform the integral equation to a linear ordinary differential equation which can be solved very simply [14]

[15]. One of such approach is Taylor's series expansion method for solving the integral equation of radiative heat transfer within a grey circular tube [10].

II. Mathematical Formulation of the Problem

Let us consider a Volterra integral equation of second kind of the form

$$\phi(x) = f(x) + \lambda \int_0^x K(x,\xi)\phi(\xi)d\xi \qquad (2.1)$$

Where λ is a parameter, K and ϕ are known function f is a solution to be determined. Here we assume that the equation (2.1) has a unique solution and the kernel $K(x,\xi) = K(x-\xi)$ with k continuous in [0, x] and decreasing as $x-\xi$ increases from 0 i.e.

$$K(x,\xi) = |x-\xi|^{-\alpha} \qquad 0<\alpha<x$$

During the study of [10] we find that the radiation integral equation discussed is of the form of equation (2.1) with a smooth kernel $K(x,\xi) = K(x-\xi)$

$$f(\xi) \approx f(x) + (\xi - x)f'(x) + \frac{(\xi-x)^2}{2!} f''(x) + \cdots\cdots\cdots + \frac{(\xi-x)^n}{n!} f^{(n)}(x) \qquad (2.2)$$

$$[1-\lambda\int_0^x K(x,\xi)d\xi]\, f(x) - [\lambda\int_0^x K(x,\xi)(\xi-x)d\xi]\, f'(\xi) \cdots - [\frac{\lambda}{n!}\int_0^x K(x,\xi)\,(\xi-x)^n d\xi]\, f^{(n)}(\xi) \qquad (2.3)$$

In the above integral equation (2.3) the quantities in brackets are functions of x alone if it is carried out analytically and therefore becomes an n-th order linear ordinary differential equation with variable coefficients. This n-th order linear differential equation can be solved by either numerical methods or by analytical methods.

The solution of above integral equation requires the boundary conditions which may be derived from the physical constraints in the system such as symmetry or an overall heat balance. In general it is very difficult to determine and hence this method is much complicated to be extended to the systems of second kind integral equations or general integral equations of the form (2.1) with smooth or weakly singular kernels.

In this paper we provide a technique of Taylor's series expansion method for the integral equation of second kind of the form (2.1) which does not require the use of boundary conditions. Through this technique we obtain the accurate approximate solution of the integral equation.

III. Modified Taylors Series expansion method

In this section we develop a technique one step more towards the Taylors series expansion method. Here it is assumed that for an integer n, $\phi\ \varepsilon$ [0, x] and n times continuously differentiable function on [0, x]. Let the kernel $\mathrm{K}(\mathrm{x},\xi) = a(\mathrm{x},\xi)\ \mathrm{K}(\mathrm{x}-\xi)$ in the smooth kernel case [13] and [16].

$$\mathrm{f}^{(\mathrm{m})}(\xi) \le c_0\ [\xi^{-\alpha-m+1} + (1-\xi)^{-\alpha-m+1}] \qquad 0<\xi<1 \qquad \mathrm{m} = 0,1,2,3,\ldots\ldots,\mathrm{n}.$$

Where c_0 is appositive constant in the case of weakly singular kernel. In Taylors series expansion (2.2) of (2.1) let us suppose f (ξ) is the approximated solution of (2.1). Now first we differentiate both sides of the equation (2.1) so that one may avoid choosing the initial boundary conditions for $0 < \mathrm{x} < 1$.

$$f'(\mathrm{x}) - \lambda\int_0^x \mathrm{K}'_\mathrm{x}(\mathrm{x},\xi)\phi(\xi)\mathrm{d}\xi = \phi'(\mathrm{x}) \qquad (3.1)$$

$$f^{(n)}(\mathrm{x}) - \lambda\int_0^x \mathrm{K}_\mathrm{x}^{(n)}(\mathrm{x},\xi)\phi(\xi)\mathrm{d}\xi = \phi^{(n)}(\mathrm{x}) \qquad (3.2)$$

Where $\mathrm{K}_\mathrm{x}^{(n)}(\mathrm{x},\xi) = \frac{\partial^n \mathrm{K}(\mathrm{x},\xi)}{\partial\xi^n}$

Substituting the value of f(x) for f(ξ) in the above integrals, we obtain

$$\mathrm{f}'(\mathrm{x}) - [\lambda\int_0^x \mathrm{K}'_\mathrm{x}(\mathrm{x},\xi)\mathrm{d}\xi]\ \mathrm{f}(\mathrm{x}) \approx \phi'(\mathrm{x}) \qquad (3.3)$$

$$\mathrm{f}^{(\mathrm{n})}(\mathrm{x}) - [\lambda\int_0^x \mathrm{K}_\mathrm{x}^{(n)}(\mathrm{x},\xi)\mathrm{d}\xi]\ \mathrm{f}(\mathrm{x}) \approx \phi^{(n)}(\mathrm{x}) \qquad (3.4)$$

Now the above equations combined together with equation (2.3) become a linear system of n+1 algebraic equations having n+1 unknowns f(x), $\mathrm{f}'(\mathrm{x}),\ldots\ldots,\mathrm{f}^{(\mathrm{n})}(\mathrm{x})$ which can be solved without use of boundary conditions. On solving above algebraic equations we try to express $\mathrm{f}'(\mathrm{x}),\ldots\ldots,\mathrm{f}^{(\mathrm{n})}(\mathrm{x})$ in terms of f(x) and $\phi'(\mathrm{x})$, …….., $\phi^{(n)}(\mathrm{x})$ which together with (2.3) gives f (x).
Therefore we obtain the approximate solution in terms of simple integrals instead of solving the linear ordinary differential equation (2.3) with boundary conditions.

IV. Numerical Results

Here we present some numerical results from different areas which illustrate the modified Taylor's series expansion method.

Example 1: In this example we consider the equation (1.1) with a weakly singular kernel

$$\mathrm{K}(\mathrm{x},\xi) = (\mathrm{x}-\xi)^{-2/3}.$$

The function $\phi(\mathrm{x}) = [\mathrm{x}(\mathrm{x}-1)]^2$. Computed results with $\lambda = 1/5$ are shown in Fig. 1.

In above figure 1 we compare the exact solution only the approximation with n=0. Here it is clear that approximation with n=0 gives very accurate solution.

Example 2:
Let us consider an integral equation arising in radiative heat transfer between two grey surfaces [14], [15]. The equation is given in non dimensional form by (1.1) having $\lambda = 1 - \sum$ $\mathrm{K}(\mathrm{x},\xi)$ = k l |s-c|

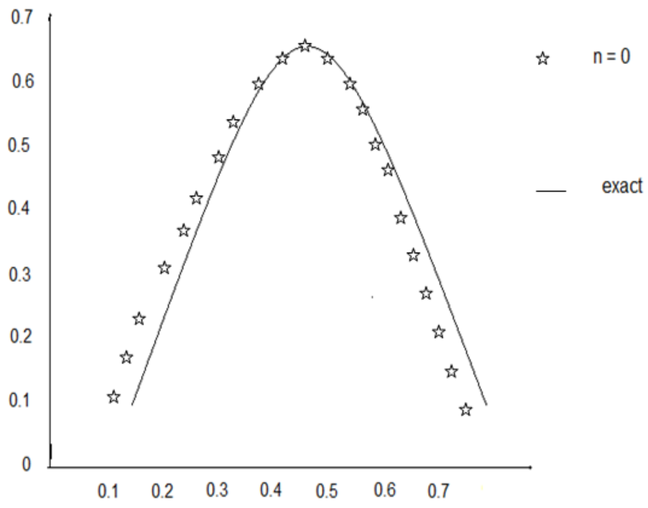
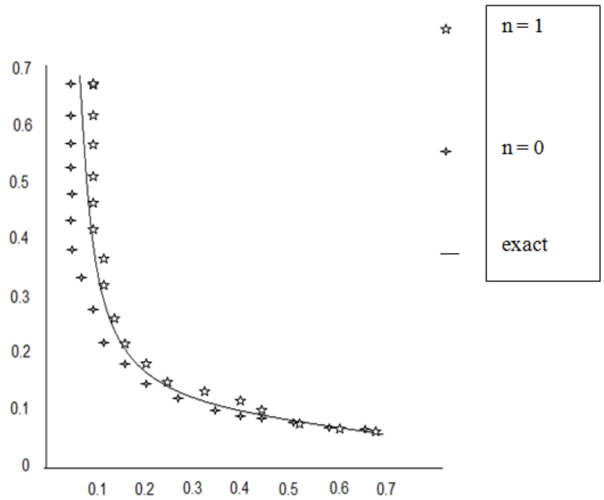


Fig. 1

and $\quad K(\xi) = \frac{\xi(\xi^2 + \frac{3}{2})}{(\xi^2+1)}$

Fig. 2. Comparison of approximation and exact solution in Example 2.



In above example 2(Fig 2), the approximations with n=0, 1 i.e. the Taylors series expansion contains one and two terms are nice acceptable with exact solution.

Example 3: Let us consider the equation (2.1) with $\lambda = \frac{\alpha}{\pi}$ and $K(x, \xi) = [\alpha^2 + (s - t)^2]^{-1}$ which arises in electrostatic problem [8]. When $\phi(x) = 1$ the integral equation takes the form

$$f(x) - \alpha \int_{-x}^{x} K(x, \xi)\phi(\xi)d\xi = 1$$

The above equation can be easily transformed to the form of (1.1)

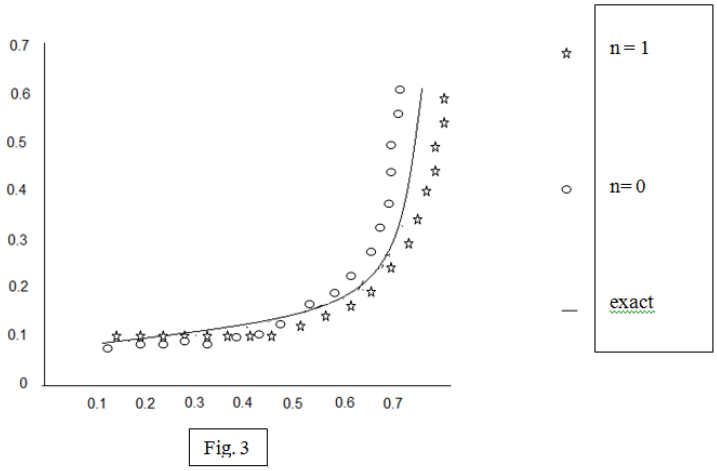


Fig.3: Comparison of approximate and exact solution in Example 3.

In above figure 2 the approximation with n=0, 1 i.e. the Taylors series expansion contains two terms and are very close to exact solution.

V. Conclusion

In this paper we apply Taylor's series expansion method to solve the volterra integral equation of second kind with smooth and weakly singular kernel. On applying Taylor's series expansion method we obtain an approximate solution of the Volterra integral equation which can be computed by symbolic computing codes on any personal computer. The technique describe in this paper can be used not only in radiate heat transfer but also many other applications as potential theory, electrostatics and also for a wide variety of physical and engineering problems. In the support of our approach we presented numerical examples arising from radiative heat transfer and electrostatics with smooth and weakly singular kernel. The graph of comparison of approximations and exact solution show the excellent accuracy of the presented method. We get good agreement between exact solution and our present solution.

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