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Some Variable Hybrids Linear Multistep Methods for Solving First Order Ordinary Differential

Equations Using Taylor's Series

[1*]*Olanegan, O. O.,* [2]*Ogunware, B. G.,* [2]*Omole E. O.* [3]*Oyinloye, T. S. and* [3]*Enoch,B. T.*
[1, 3] *Physical Sciences Department, Landmark University, Omu-Aran, Kwara State, Nigeria.*
2Federal University of Technology Akure, Ondo State, Nigeria.
***Corresponding Author:** *ola3yemi@gmail.com*

Abstract: *This paper proposes a family of single step, fifth and fourth order continuous hybrid linear multistep methods (CHLMM). These methods are used to integrate the initial value problems of first order ordinary differential equations. The methods are obtained from the continuous schemes derived via interpolation and collocation procedure. The schemes are consistent, zero stable, convergent and accurate. Taylor's series approximation was adapted as simultaneous numerical integrators over non-overlapping intervals values and for the implementation of the methods. These methods compared favorably with some existing methods because they are efficient and simple in terms of derivation and implementation.*

Keywords: *Accurate, Collocation, Continuous Hybrid Linear Multistep Methods (CHLMM), Efficiency, First Order Ordinary Differential Equations, Interpolation and Taylor's series.*

I. Introduction

This paper considers the development of numerical methods for solving first order ordinary differential equation with initial value problems of the form:

$$y' = f(x, y) \qquad y(x_0) = y_0, \tag{1}$$

f is a real valued function and continuously differentiable within an interval and satisfies Lipchitz's condition which makes the existence and the uniqueness of the solution (1) to be guaranteed. Ordinary differential equations are important tools in solving real-life problems. Various natural phenomena are modelled using odes which are applied to many problems in physics, engineering, biology and social sciences. Thus, much attention have been given to ordinary differential equations in recent years.

Collocation Method is widely considered as a way of generating numerical solution to ordinary differential equation of the form (1). The usual way of solving (1) is to use a one-step explicit method such as Runge-Kutta of the same order of accuracy until enough values have been generated for multistep method to take off. The problem with linear multistep method is that they need help getting started which is encountered in single step methods [1]. The help required in getting started is called a predictor. These starting values are called Predictors for (1) while the equation (1) is called corrector; hence the procedure is called predictor-corrector method. The predictors are explicit while the correctors are implicit methods. The general multistep method of the form (1) includes Simpson method, Adam-Bashforth method and Adam- Moulton's methods. All the Adam-Moulton's methods are regarded as constant coefficient method but in this paper, linear multistep methods with continuous variable coefficient are generated. Such methods are often called continuous collocation methods. Collocation is a projection method for solving integral and differential equations in which the approximate solution is determined from the condition that the equation must be stratified at certain given points. It involves the determination of an approximate solution in a set of functions called the basis function.

Several researchers have developed collocation methods of the form (1) for solving initial value problems, they are [1], [2], [3], [4], [5], [6], [7], and [8]. [9], propose a double hybrid continuous method to solve second order ordinary differential equation. [4], specifically stated the advantages of continuous schemes over the discrete ones, they are;

i. Provision of better global error estimate
ii. Usefulness for further analytical work in a simpler form than the discrete ones.
iii. Provision of approximation at all interior points.

Another added advantage of continuous scheme is that infinite number of schemes could emerge from one continuous scheme [8].

Our interest in this paper is to develop some continuous multistep hybrid methods with **k =** 1 and collocate at all the grids and off grids points. We use Taylor's series approximation to supply the starting values and in the implementation of the methods developed.

II. The Methods Developed

We consider a power series approximation of the form:

$$y(x)=\sum_{j=0}^{(i+c)} a_j x^j \tag{2}$$

i and c represents the number of the interpolation and collocation points respectively and the first derivative of (2) is

$$y'(x)=\sum_{j=0}^{(i+c)} j a_j x^{j-1} \tag{3}$$

Putting (3) into (1) we obtain the differential system

$$\sum_{j=0}^{(i+c)} j a_j x^{j-1} = f(x,y) \tag{4}$$

2.1 Derivation of the First Scheme

Collocating (4) at $x = x_{n+i},\ i = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1$ and interpolating (2) at $x = x_{n+i}, i = \frac{1}{2}$ gives a system of non-linear equation of the form:

$$AX = U \tag{5}$$

where

$$A = [a_0, a_1, a_2, a_3, a_4, a_5, a_6]^T$$

$$U = \left[f_n, f_{n+\frac{1}{4}}, f_{n+\frac{1}{2}}, f_{n+\frac{3}{4}}, f_{n+1}, y_{n+\frac{1}{2}} \right]^T$$ and

$$X = \begin{bmatrix} 1 & x_{n+\frac{1}{2}} & x^2_{n+\frac{1}{2}} & x^3_{n+\frac{1}{2}} & x^4_{n+\frac{1}{2}} & x^5_{n+\frac{1}{2}} & x^6_{n+\frac{1}{2}} \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 \\ 0 & 1 & 2x_{n+\frac{1}{4}} & 3x^2_{n+\frac{1}{4}} & 4x^3_{n+\frac{1}{4}} & 5x^4_{n+\frac{1}{4}} & 6x^5_{n+\frac{1}{4}} \\ 0 & 1 & 2x_{n+\frac{1}{2}} & 3x^2_{n+\frac{1}{2}} & 4x^3_{n+\frac{1}{2}} & 5x^4_{n+\frac{1}{2}} & 6x^5_{n+\frac{1}{2}} \\ 0 & 1 & 2x_{n+\frac{3}{4}} & 3x^2_{n+\frac{3}{4}} & 4x^3_{n+\frac{3}{4}} & 5x^4_{n+\frac{3}{4}} & 6x^5_{n+\frac{3}{4}} \\ 0 & 1 & 2x_{n+1} & 3x^2_{n+1} & 4x^3_{n+1} & 5x^4_{n+1} & 6x^5_{n+1} \end{bmatrix}$$

Using Gaussian elimination for $a_j's$ in (5) gives a continuous method in the form:

$$y(t) = \alpha_\theta(t) y_{n+\theta} + h\left[\sum_{\eta=0}^{\kappa} \beta(t) f_{n+\eta} + \beta_\kappa(t) f_{n+\eta} \right] \tag{6}$$

where $\theta = \frac{1}{2}\ \eta = 0, \frac{1}{4}, \frac{3}{4}\left(\frac{1}{2}\right)1, \quad f_{n+\kappa} = f(x_n + \kappa h)$

Then, using the transformation $t = \frac{x - x_n}{h}$ in (6) we have a continuous scheme and the coefficients as follows:

$$\alpha_{1/2} = 1$$

$$\beta_0 = \frac{h}{360}\left[1 + 180t^2 + 880t^3 + 1440t^4 + 768t^5\right]$$

$$\beta_{\frac{1}{4}} = \frac{h}{90}\left[1 - 240t^2 - 1120t^3 - 1680t^4 - 768t^5\right]$$

$$\beta_{\frac{1}{2}} = \frac{h}{15}\left[1 + 90t^2 + 380t^3 + 480t^4 + 192t^5\right] \tag{7}$$

$$\beta_{\frac{3}{4}} = \frac{h}{90}\left[31 - 720t^2 - 2080t^3 - 2160t^4 - 768t^5\right]$$

$$\beta_1 = \frac{h}{360}\left[29 + 360t + 1500t^2 - 2800t^3 - 2400t^4 - 768t^5\right]$$

Evaluating (7) at $t = 1$ (i.e $x = x_{n+1}$) gives

$$y_{n+1} - y_{n+\frac{1}{2}} = \frac{h}{360}\left[29 f_{n+1} + 124 f_{n+\frac{3}{4}} + 24 f_{n+\frac{1}{2}} + 24 f_{n+\frac{1}{4}} - f_n\right] \tag{8}$$

With order $p = 5$, Error constant $c_7 = -\dfrac{1}{368640}$

2.2 Derivation of the Second Scheme

Collocating (4) at $x = x_{n+i},\ i = 0, \frac{3}{4}, \frac{1}{2}, 1$ and interpolating (2) at $x = x_{n+i}, i = \frac{3}{4}$ gives a system of non-linear equation of the form (5)

where

$$A = \left[a_0, a_1, a_2, a_3, a_4\right]^T$$

$$U = \left[f_n, f_{n+\frac{3}{4}}, f_{n+\frac{1}{2}}, f_{n+1}, y_{n+\frac{3}{4}} \right]^T \text{ and}$$

$$X = \begin{bmatrix} 1 & x_{n+\frac{3}{4}} & x^2_{n+\frac{3}{4}} & x^3_{n+\frac{3}{4}} & x^4_{n+\frac{3}{4}} \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 \\ 0 & 1 & 2x_{n+\frac{1}{2}} & 3x^2_{n+\frac{1}{2}} & 4x^3_{n+\frac{1}{2}} \\ 0 & 1 & 2x_{n+\frac{3}{4}} & 3x^2_{n+\frac{3}{4}} & 4x^3_{n+\frac{3}{4}} \\ 0 & 1 & 2x_{n+1} & 3x^2_{n+1} & 4x^3_{n+1} \end{bmatrix}$$

Using Gaussian elimination method has been done for the first Scheme gives a continuous method in the form (6):

where $\theta = \frac{3}{4}\ \eta = 0, \frac{3}{4}\left(\frac{1}{2}\right)1, \quad f_{n+\kappa} = f\left(x_n + \kappa h\right)$

Then, using the transformation $t = \dfrac{x - x_n}{h}$ we have $x = th + x_n$ we have the coefficients put as follows:

$$\alpha_{\frac{3}{4}} = 1$$

$$\beta_0 = \frac{h}{384}\left[1 - 64t^2 - 256t^3 - 256t^4\right]$$

$$\beta_{\frac{1}{2}} = \frac{h}{192}\left[-7 + 384t^2 + 1280t^3 + 7680t^4\right] \tag{9}$$

$$\beta_{\frac{3}{4}} = \frac{h}{48}\left[9 - 256t^2 - 512t^3 - 256t^4\right]$$

$$\beta_1 = \frac{h}{384}\left[37 + 384t + 1344t^2 1792t^3 768t^4\right]$$

Evaluating (9) at $t = 1$ (i.e $x = x_{n+1}$) gives

$$y_{n+1} - y_{n+\frac{3}{4}} = \frac{h}{384}\left[37 f_{n+1} + 72 f_{n+\frac{3}{4}} - 14 f_{n+\frac{1}{2}} + f_n\right] \tag{10}$$

With order $p = 4,$ Error constant $c_7 = -3.59 \times 10^{-5}$

III. Analysis of the Basic Properties of Method

We verify the accuracy of the methods, examine their basic properties which include Local Truncation Error and order of accuracy, consistency and zero stability.

3.1 Local Truncation Error and Order of Accuracy:

Definition 1: According to [2], Linear Multistep Method (8) and (10) are said to be of order p, if p is the largest positive integer for which $c_0 = c_1 = c_2 = \cdots = c_p = c_{p+1} = 0$ but $c_{p+2} \neq 0.$ Expanding (8) and (10) by Taylor's series and comparing coefficients of the expansion equating it to zero, we get the c_i values for the method:

$c_0 = c_1 = c_2 = \cdots = c_6$

Hence, the two methods are of

order $p = 5$ with principal truncation error $c_{p+2} = -\dfrac{1}{368640}$ and order $p = 4$ with principal truncation error $c_{p+2} = -3.59 \times 10^{-5}$

3.2 Consistency

For (8) and (10) to be consistent, the following criteria must be met.

Condition 1: $p \geq 1$

Condition 2: $\sum_{j=0}^{k} \alpha_j = 0$ where $j = (0 \cdots 2)$

Condition 3: $\rho'(r) = 0$ when $r = 1$

Condition 4: $\rho''(r) = 2!\sigma(r)$ when $r = 1$

Where ρ and σ are the first and second characteristic polynomials of (5) and (7), applying these conditions, the methods was found to be consistent.

3.3 Zero Stability

Definition 2: A linear multistep method is said to be zero-stable if no root $\rho(r)$ has modulus greater than one (that is, if all roots of $\rho(r)$ lie in or on the unit circle). A numerical solution to class of system (1) is stable if the difference between the numerical and the theoretical solutions can be made as small as possible [2]. Hence, (8) and (10) were found to be zero-stable since none of their roots has modulus greater than one.

3.4 Convergence

Definition 3: a linear multistep method of the form (8) and (10) is convergent if it is consistent and zero stable. Hence the necessary and sufficient conditions for the methods to be convergent is that it must both consistent and zero-stable. Since, these conditions are satisfied, then the methods (8) and (10) are said to be convergent.

IV. Implementation of the Method

Methods (8) and (10) are tested on some first order differential equations problems.

Problem 1

$$y' = x - y,\ y(0) = 0, 0 \le x \le 1\ h = 0.1$$

Exact Solution

$$y(x) = x + e^{-x} - 1$$

Problem 2:

$$y'(t) = -\sin t,\ y(0) = 1, h = 0.01$$

Exact Solution

$$y(t) = \cos t$$

V. Results

Table 1: Numerical Results of Problem 1 using the New Methods Developed

X	YEX1	YEX 2	YCOM1	YCOM 2	ERR 1	ERR 2
0.1	0.0048374180	0.0048374180	0.00483741800	0.00483741498	9.6000E-13	3.0469E-09
0.2	0.0187307530	0.0187307530	0.01873075307	0.01873074756	4.8552E-12	5.5130E-09
0.3	0.0408182206	0.0408182206	0.04081822066	0.04081821320	1.4457E-11	7.4810E-09
0.4	0.0703200460	0.0703200460	0.07032004600	0.07032003701	3.1965E-11	9.0232E-09
0.5	0.1065306597	0.1065306597	0.10653065965	0.10653064951	5.9120E-11	1.0202E-08
0.6	0.1488116360	0.1488116360	0.14881163599	0.14881162501	9.7270E-11	1.1074E-08
0.7	0.1965853037	0.1965853037	0.19658530364	0.19658529210	1.4739E-10	1.1685E-08
0.8	0.2493289641	0.2493289641	0.24932896390	0.24932895203	2.1019E-10	1.2078E-08
0.9	0.3065696597	0.3065696597	0.30656965945	0.30656964745	2.8615E-10	1.2287E-08
1.0	0.3678794411	0.3678794411	0.36787944079	0.36787942882	3.7551E-10	1.2345E-08

Table 2: Numerical Results of Problem 2 using the New Methods Developed

X	YEX 1	YEX 22	YCOM 11	YCOM 22	ERR 11	ERR 22
0.1	0.9999500004	0.9999500004	0.999950000417	0.9999500004	0.0000E+00	0.0000E+00
0.2	0.9998000067	0.9998000066	0.999800006667	0.9998000066	0.0000E+00	1.0000E-10
0.3	0.9995500337	0.9995500337	0.999550033749	0.9995500337	0.0000E+00	0.0000E+00
0.4	0.9992001067	0.9992001066	0.999200106661	0.9992001066	0.0000E+00	1.0000E-10
0.5	0.9987502604	0.9987502603	0.998750260395	0.9987502603	0.0000E+00	1.0000E-10
0.6	0.9982005399	0.9982005399	0.998200539935	0.9982005399	0.0000E+00	0.0000E+00
0.7	0.9975510003	0.9975510002	0.997551000253	0.9975510002	0.0000E+00	1.0000E-10
0.8	0.9968017063	0.9968017062	0.996801706302	0.9968017062	1.0000E-12	1.0000E-10
0.9	0.9959527330	0.9959527328	0.995952733011	0.9959527328	1.0000E-12	2.0000E-10
1.0	0.9950041653	0.9950041651	0.995004165277	0.9950041651	1.0000E-12	2.0000E-10

Table 3: Comparison with existing Methods

X	ERR1	ERR 2	ERRJM p=7	ERROD p=9	ERRAR p=6
0.1	9.6000E-13	3.04696E-09	1.7443E-11	0.13201E-14	0.00
0.2	4.8552E-12	5.51305E-09	1.5786E-11	0.10348E-14	0.00
0.3	1.4457E-11	7.48105E-09	1.4283E-11	9.52016E-14	6.00E-10
0.4	3.1965E-11	9.02323E-09	1.2924E-11	8.65280E-14	3.00E-11

0.5	**5.9120E-11**	**1.02026E-08**	**1.1694E-11**	**7.81458E-14**	**0.00**
0.6	**9.7270E-11**	**1.10741E-08**	**1.0581E-11**	**0.14738E-14**	**1.00E-10**
0.7	**1.4739E-10**	**1.16854E-08**	**9.5739E-12**	**0.12312E-14**	**0.00**
0.8	**2.1019E-10**	**1.20780E-08**	**8.6613-E12**	**0.11351E-14**	**0.00**
0.9	**2.8615E-10**	**1.22878E-08**	**7.8396E-12**	**0.10242E-14**	**0.00**
1.0	**3.7551E-10**	**1.23458E-08**	**7.0906E-12**	**9.25926E-14**	**1.00E-10**

NOTE:
YEX 1: Exact Solution for method 1
YEX 2: Exact Solution for method 2
YCOM 1: Computed Solution for Method 1
YCOM 2: Computed Solution for Method 2
ERR 1: Errors in Method 1
ERR 2: Errors in Method 2
ERRJM: Errors in [10]
ERROD: Error in [1]
ERRAR: Error in [11]

VI. Discussion of Results

This paper considered two numerical examples to test the efficiency of the methods. The test problems were solved by [10] and [1]. They individually proposed a hybrid method of order seven and nine respectively which they adopted classical Runge Kutta method to provide the starting values. Our methods compared favourably because the proposed methods are self-starting and does not require starting values as a result Taylor's series which was used as a starting value.

VII. Conclusion

In this paper, a class of hybrid method with the use of Taylor's series for the approximation of **y** variables has enabled us compute the derivatives of the method to any possible order which allows direct solution of Initial Value Problems (IVPs) of ordinary differential equations. Using this new methods with all computations done with the aid of a MATLAB generated codes, has enable us to compute the solution of first order ordinary differentials equations with initial value problems (IVPs). Based on this new approach, it is evident that the new methods are considerably accurate and efficient.

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