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A Study on Category of Graphs

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***Abstract**: Let G = (V (G),E(G)) and G_1= (V (G_1),E(G_1)) be graphs. In this paper werepresent a homomorphism f:G → G_1as a pair $f = (f^*, \tilde{f})$ where f^* : V (G) → V (G_1)and $\tilde{f}$: E(G) → E(G_1) are maps such that $\tilde{f}$(x,y) = (f^*(x), f^*(y)) for all edges(x,y) in G. With this representation we characterize some specialmorphisms likemonomorphism, epimorphism, coretraction, retraction etc in terms of set functions.Finally we show that the Category of Graphs is not balanced.*

***Keywords:**monomorphism, epimorphism, coretraction, injective, surjective, retraction*

I. Introduction

Category theory provides a general idea that has proved fruitful in subjects as diverse as geometry, topology and foundational mathematics. In this paper, the basic notions and lemma involving finite diagram are given. These notions are injective, coretraction, monomorphism and duals of these as studied by various authors [1, 2, 3, 4]. Also we undertake the necessary task of introducing some of the basic notations for graphs and as studied earlier by various authors [5, 6] discussed mapping between graphs as homomorphism. In the last section, we concluded that the category of graphs is not balanced [7, 8].

Defnition1.1A graph G consists of a vertex set V (G) and an edge set E(G) where E(G) is a set of unordered pairs of distinct elements in V(G). The above graph G is denoted as G = (V (G), E (G)) or simply as (V, E) whenever the context is clear [5].

The graphs as we have defined above are called simple graphs. The elements in V (G) are called vertices and those in E(G) are called edges. In all our discussions, V(G) is taken to be a finite set.
If e= (u, v) $\in$ E (G) (i.e. an edge in G) then we say that u and v are adjacent in G or that v is a neighbor of u and denote this u ~ v. Moreover u and v are said to be incident with e and e is said to be incident to both u and v.
If v is a vertex in a graph G then the degree or valency of v, denoted as d(v) is the number of edges incident with v. A vertex v in G is said to be an isolated vertex if d(v) = 0 and an end vertex if d(v) = 1.
A graph G is said to be a null graph if V (G) = ϕ, the empty set. In this case E (G) is also empty. If V (G) $\neq \phi$ but E (G) = ϕ then G is called an empty graph.

By a graph we always mean a simple non-null graph only[6].

Definition 1.2 Homomorphism between graphs
Let G and G_1 be graphs. Then a function
$\alpha: V(G) \to V(G_1)$
is called a homomorphism from G into G_1 if α (u) and α (v) are adjacent in G_1
whenever u and v are adjacent in G. Thus α induces a function say
$\tilde{\alpha}: E(G) \to E(G_1)$
Such that
$\tilde{\alpha}((u,v)) = (\alpha(u), \alpha(v))$.
Thus the above homomorphism from G into G_1 can be represented by a pair $(\alpha, \tilde{\alpha})$ where
$\alpha: V(G) \to V(G_1)$
and
$\tilde{\alpha}: E(G) \to E(G_1)$
are functions such that $\tilde{\alpha}((u,v)) = (\alpha(u), \alpha(v))$ for all edges (u, v) $\in$ E (G).
Conversely given any pair of function (h, k) where $h: V(G) \to V(G_1)$ and $k: E(G) \to E(G_1)$ are such that $k((u,v)) = (h(u), h(v))$ for all edges (u, v) $\in$ E (G).
Then the above pair of functions induces a homomorphism from G into G_1[5].

This observation enables us to represent a homomorphism between graphs by a pair of functions subject to some conditions.

Definition 1.3

Let G and G_1 be graphs. A homomorphism f: G $\rightarrow G_1$ is a pair $f=(f^*,\tilde{f})$ where $f^*: V(G)\rightarrow V(G_1)$ and $\tilde{f}: E(G)\rightarrow E(G_1)$ are functions such that

$\tilde{f}((u,v)) = (f^*(u), f^*(v))$ for all u, v $\in$ V(G). In the following discussions, unless stated otherwise, if f, g etc are homomorphisms between graphs then we represent $f=(f^*,\tilde{f})$, $g=(g^*,\tilde{g})$… and so on. Moreover for convenience, if (u, v) $\in$ E (G) then $f^*((u,v))$ is denoted simply by $f^*(u,v)$.

Definition 1.4 Category of graphs:

Let G , G_1 and G_2 be graphs and $f=(f^*,\tilde{f})$: G $\rightarrow$ G_1 and $g=(g^*,\tilde{g})$: $G_1\rightarrow G_2$

Be homomorphisms(or simply morphisms) of graphs with the usual meaning. Then their composition

(gf) = ((gf) *, $(\tilde{gf})$)definedas follows. $(gf)^*=g^*f^*: V(G)\rightarrow V(G_2)$

Moreover for all (u, v) $\in$ E (G)

$$
\begin{aligned}
(\tilde{gf})(u,v) &= ((gf)^*(u),(gf)^*(v)) \\
&= (g^*f^*(u), g^*f^*(v) \\
&= \tilde{g}(f^*(u), f^*(v)) \\
&= \tilde{g}(\tilde{f}(u,v)) = \tilde{g}.\tilde{f}(u,v)
\end{aligned}
$$

so that $(\tilde{gf}) = \tilde{g}.\tilde{f}$. Thus $gf=(g^*f^*,\tilde{g}\tilde{f})$ is a homomorphism from G into G_2.This defines the composition of homomorphism of graphs[1]

Existence of Identity

Let G = (V(G) , E(G)) be a given graph. Let $1_{V(G)}$:V(G) $\rightarrow$V(G) and $1_{E(G)}$:E(G) $\rightarrow$E(G) be the respective identity functions. Let $1_G = (1_{V(G)}, 1_{E(G)})$ be the homomorphism fromG $\rightarrow$ G.

Then it is easy to verify that for all g: G $\rightarrow$ G_1 and h: $G_2\rightarrow$ G, 1_G .h = h andg . 1_G = g

And hence $1_G$is the identity homomorphism in G .

Thus we have a category g called category of graphs, whose objects are graphs and morphisms are homomorphism of graphs[1].

Note: This category of graphs is identically the same as the category of graphs usually defined, except for the "representation" of homomorphism.

Definition 1.5:

Let f, g : G $\rightarrow$ G_1 be a homomorphism of graphs. Then we say thatf = g if and only if $f^*=g^*$ and$\tilde{f} = \tilde{g}$

Lemma 1.6: Let f, g : G → G_1 be a homomorphism of graphs. Then f = g if and only if $f^*=g^*$.

Proof:f = gimplies that $f^*=g^*$ by definition.

Conversely if $f^*=g^*$, then for all (u , v)$\in$ E (G)

$$
\begin{aligned}
\tilde{f}((u,v)) = (f^*(u), f^*(v)) &= (g^*(u), g^*(v)) \\
&= \tilde{g}(u,v)
\end{aligned}
$$

Implies that $\tilde{f} = \tilde{g}$.Hence f = g.

As in any category we wish to define some special morphisms in the category of graphs and characterize them in terms of (set) functions.

Definition 1.7:

Let G and G_1 be graphs. A homomorphism f: G $\rightarrow G_1$, is called a coretraction (left invertible) if there is a homomorphism g: $G_1 \rightarrow$ G such that gf = 1_G.
Dually a homomorphism f: G $\rightarrow G_1$, is called a retraction if there is a homomorphism g: $G_1 \rightarrow$ G such that fg = 1_{G1}.(right invertible).
A morphismf: G $\rightarrow$ G_1 is said to be an isomorphism if it is both a coretraction and a retraction[1] .
Note: A coretraction is also called as a section [4].

Remark 1.8:

Let us denote the empty set as well as the empty morphisms by the symbol ϕ .Suppose G and G_1 are graphs if E (G) $\neq \phi$ and E (G)= ϕ , then there exists no homomorphism from G into G_1. In particular for homomorphism f: G $\rightarrow$ G_1 and g: $G_1 \rightarrow$ G to exist, it is necessary that both E(G) and E(G_1) are empty or both E(G) and E(G_1) are nonempty. We denote the identiy function from $\phi \rightarrow \phi$ also as1_ϕ . Now we proceed to characterize some of the special morphisms in the category of graphs by set functions.

Proposition 1.9: Let $f=(f^*,\tilde{f})$: G $\rightarrow$ G_1be a homomorphism of graphs. Then f is a coretraction if and only if

i) $f^*:V(G)\rightarrow V(G_1)$ is injective , and

ii) For all $u,v\in V(G),(f^*(u),f^*(v))\in E(G_1)$ implies that $(u,v)\in E(G)$.

Proof: Suppose f is a coretraction. Then there exists a homomorphism $g=(g^*,\tilde{g})$: $G_1 \rightarrow$ Gsuch that $gf=1_G$ .i.e $(g^*f^*,\ \tilde{g}\tilde{f})=(1_{V(G)},1_{E(G)})$.

Hence $g^*f^* = 1_{V(G)}$ and $\tilde{g}\tilde{f} = 1_{E(G)}$, the respective identity functions which are bijections.

Therefore $f^*:V(G)\rightarrow V(G_1)$ and $\tilde{f}: E(G)\rightarrow E(G_1)$ are injective.

Again if $u,v\in V(G)$, $(u,v)\notin E(G)$ but $(f^*(u),f^*(v))\in E(G_1)$.

Then $\tilde{g}(f^*(u),f^*(v))\in E(G)$, (since g is a homomorphism of graphs) implies that

$(g^*f^*(u),g^*f^*(v))\in E(G))$

Hence $(u,v)\in E(G)$ since $g^*f^* = 1_{V(G)}$. This completes the proof for the "if " part.
Conversely assume conditions (i) and (ii) as stated above.

Case 1: Suppose both E (G) and E (G_1) are empty. Then $\tilde{f}: E(G)\rightarrow E(G_1)$ is the empty function which is also denoted as ϕ .

We define functions $h:V(G_1)\rightarrow V(G)$ and $k: E(G_1)\rightarrow E(G)$ as follows. $k=\phi$,
The empty function …….(1).

Now fix a vertex $u_0\in V(G)$. Since f^* is injective, for each $v\in \mathrm{Im}age f^*$ there exists a unique element $u\in V(G)$ such that $f^*(u)=v$. So define a function $h:V(G_1)\rightarrow V(G)$

by $$h(v)=\begin{cases} u\, if\ f^*(u)=v \\ u_0\ \ otherwise \end{cases}$$

From the construction it follows that $h:V(G_1)\rightarrow V(G)$ and $k: E(G_1)\rightarrow E(G)$ are functions such that $hf^* = 1_{V(G)}$ *and* $k\tilde{f}=\phi = 1_{E(G)}$. If we denote $g^* = h$ and $k = \tilde{g}$ then $g=(g^*,\tilde{g})$: $G_1 \rightarrow$ G is a homomorphism such that $(g^*f^*\ \ \tilde{g}\tilde{f}) = (1_{V(G)},1_{E(G)})$ *i.e* $gf=1_G$. Thus f is a coretraction.

Case2: Suppose both E(G) and E(G_1) are nonempty; we define functions, $h:V(G_1)\rightarrow V(G)$ and $k: E(G_1)\rightarrow E(G)$ such that , for all $v,w\in V(G_1)$,

$k(v,w)=(h(v),\ h(w))$ so that the pair (h , k) represents a homomorphism from

G_1 into G. We have various cases to consider. First let us fix a vertex $u_0 \in V(G)$ and an edge $(u', u'') \in E(G)$.

Case 2a: Suppose $v \in V(G_1)$ and v is incident with an edge say v_2 such that $(v_1, v_2) \in \mathrm{Im}age\, \tilde{f}$. Let u_1, u_2 be unique elements (since f^* is injective) in V (G) such that $f^*(u_1) = v_1$ and $f^*(u_2) = v_2$. Then $(v_1, v_2) = (f^*(u_1), f^*(u_2)) \in E(G_1)$ implies that $(u_1, u_2) \in E(G)$ by condition (ii). In this case define $k(v_1, v_2) = (u_1, u_2) \in E(G_1)$ ……. (1).

Case 2b: Suppose v_2 is incident with a vertex say $v_3 \in V(G_1)$. *If* $v_3 \in \mathrm{Im}\, f^*$ then this reduces to case 2a. So assume that $v_3 \notin \mathrm{Im}\, f^*$. Then define $h(v_3) = u_1$. In this case let us define $k(v_2, v_3) = (u_1, u_2)$ (see Figure 1).

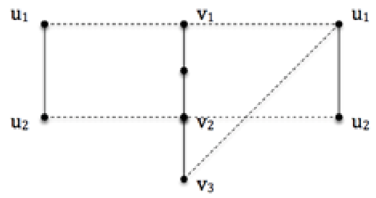

Figure 1

Case 2c: Suppose $v' \in V(G_1)$ and is incident with an edge $v'' \in V(G_1)$ i.e. $(v', v'') \in E(G_1)$ such that v' and v'' does not belong to Imf*. Then define $h(v') = u'$, $h(v'') = u''$ *and* $k(v', v'') = (u', u'')$.

Case 2d: Suppose $v \in V(G_1)$ *and* $f^*(u) = v$ and v is not incident with an edge in E(G_1).
Define h (v) = u.

Case 2e: Suppose $v_4 \in V(G)$ is an isolated vertex and v does not belongs to Image f*, define $h(v_4) = u_0$. (See Figure 2)

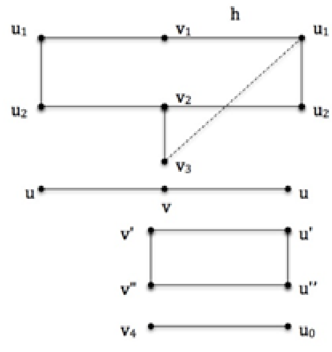

Figure 2

Then we have functions $h: V(G_1) \to V(G)$ and $k: E(G_1) \to E(G)$ such that for all edges $(v', v'') \in E(G_1)$, $k(v', v'') = (h(v'), h(v''))$. Moreover from the construction it follows that $hf^* = 1_{V(G)}$ *and* $k\tilde{f} = 1_{E(G)}$. By taking $g^* = h$ *and* $\tilde{g} = k$, we have a homomorphism $g = (g^*, \tilde{g})$: $G_1 \to$ G such that $g\, f = 1_G$. Thus f is a coretraction.

Remark: 1.10 we observe that $f: G \to G_1$ is a homomorphism of graphs such that $f^*: V(G) \to V(G_1)$ is injective , then $\tilde{f}: E(G) \to E(G_1)$ is also injective. For if (u, v) and (u_1, v_1) belongs to E(G) such that $\tilde{f}(u,v) = \tilde{f}(u_1, v_1)$. This implies that $(f^*(u), f^*(v)) = (f^*(u_1), f^*(v_1))$ so that

$$f^*(u) = f^*(u_1) \text{ and } f^*(v) = f^*(v_1) \text{ or } f^*(u) = f^*(v_1) \text{ and } f^*(v) = f^*(u_1).$$

In either case $edge\,(u,v) = edge\,(u_1, v_1)$. Hence $\tilde{f}$ is also injective.

Remark 1.11: Again condition (ii) in the proposition is also necessary. For if f: G $\to$ G_1 is a Coretraction with a homomorphism g: $G_1 \to$ G such that $g\,f = 1_G$. Now if $(u,v) \notin E(G)$ but $(f^*(u), f^*(v)) \in E(G_1)$. Then $\tilde{g}(f^*(u), f^*(v)) \in E(G)$ i.e $(g^* f^*(u), g^* f^*(v))$ belongs to $E(G)$ i.e $(u,v) \in E(G)$ which is a contradiction. Therefore $(f^*(u), f^*(v)) \in E(G_1)$ implies that $(u,v) \in E(G)$.

Remark 1.12: Let $f: G \to G_1$ be a coretraction. Then f^* and $\tilde{f}$ are injective. Moreover, since f is a homomorphism then for all edges $(u,v) \in E(G)$, $((f^*(u), f^*(v)) \in E(G_1)$. On the otherhand by condition ii) $(f^*(u), f^*(v) \in E(G_1)$ implies that (u,v) belongs to E(G_1). Thus f is a coretraction if and only if

i) $f^*: V(G) \to V(G_1)$ is injective and

ii) $(u,v) \in E(G)$ *if and only if* $(f^*(u), f^*(v)) \in E(G_1)$.

Remark 1.13: f is a coretraction does not imply that $\tilde{f}: E(G) \to E(G_1)$ is surjective. For consider the homomorphism $f: G \to G_1$ given by the following diagram (*See Figure 3*).

Then f is a coretraction but $\tilde{f}$ is not surjective.

Proposition 1.14: Let $f: G \to G_1$ be a retraction. Then both $f^*: V(G) \to V(G_1)$ and $\tilde{f}: E(G) \to E(G_1)$ are surjective.

Proof: $f: G \to G_1$ is a retraction implies that there exists a homomorphism $g: G_1 \to G$ such that $f\,g = 1_G$. Hence $f^*\,g^* = 1_{V(G_1)}$ *and* $\tilde{f}\,\tilde{g} = 1_{E(G_1)}$. This shows that both f * and $\tilde{f}$ are surjective.

Remark 1.15: The converse of the above result is not true. For consider the following diagram of a homomorphism $f: G \to G_1$ (*See Figure 4*).

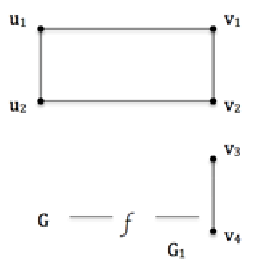

Figure 3

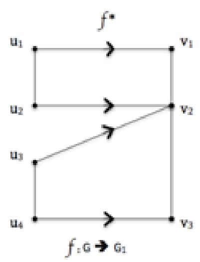

Figure 4

Both $f^*: V(G) \to V(G_1)$ and $\tilde{f}: E(G) \to E(G_1)$ are surjective. However there is no homomorphism $g: G_1 \to G$ such that $f\,g = 1_G$, since the preimage of v_2 is not unique and $(v_2, v_3) = (f^*(u_2), f^*(u_4))$ is an edge in G_1 but (u_2, u_4)is not an edge in G .
(Since $u_4 \neq u_1$).

However we have the following proposition.

Proposition 1.16: Let $f: G \to G_1$ be a homomorphism of graphs such that

i) both $f^*: V(G) \to V(G_1)$ and $\tilde{f}: E(G) \to E(G_1)$ are surjective, and

ii) For a all $u, v \in V(G), (f^*(u), f^*(v))$ belong to E (G₁) implies that $(u,v) \in E(G)$.
Then f is a retraction.

Proof: We construct functions $h: V(G_1) \to V(G)$ and $k: E(G_1) \to E(G)$
such that $k(v_1, v_2) = (h(v_1), h(v_2))$ and such that $f^* h = 1_{V(G_1)}$ *and* $\tilde{f} k = 1_{E(G_1)}$.

Case1: Suppose E(G) and E(G_1) are both empty. Since $V(G_1) \neq \phi$ *and* $f^*: V(G) \to V(G_1)$ is surjective, for each $v \in V(G_1)$ fix a vertex say $u \in V(G)$ such that $f^*(u) = v$. Then define $h: V(G_1) \to V(G)$ *as* $h(v) = u$.

Then $f^* h(v) = f^*(u) = v$ *for all* $v \in V(G_1)$

so that $f^* h = 1_{V(G_1)}$. Further since E(G)= ϕ = E(G_1) $\tilde{f}: E(G) \to E(G_1)$ is the empty function ϕ . Take k : E(G) → E(G_1)as the empty function. Trivially
$f^* k = 1_{E(G_1)}$.(also the empty function). Take $g^* = h$ *and* $\tilde{g} = k$. Then
$f^* g^* = 1_{V(G_1)}$ *and* $\tilde{f}\ \tilde{g} = 1_{E(G_1)}$. So that
$f\ g = 1_{G_1}$ i.ef is a retraction .

Case 2: Assume that both E(G) and E(G_1) are nonempty.

Case 2a: Let $v_1 \in V(G_1)$. Suppose $(v_1, v_2) \in E(G_1)$. Since $\tilde{f}$ is surjective, there exists an edge $(u_1, u_2) \in E(G_1)$ such that $\tilde{f}(u_1, u_2) = (v_1, v_2)$ i.e. such that $(f^*(u_1), f^*(u_2)) = (v_1, v_2)$.
Choose and fix one such(u_1 , u_2) ∈ E(G).
Define $h(v_1) = u_1$ *and* $k(v_2) = u_2$ *and* $k(v_1, v_2) = (u_2, u_2) = (h(v_1), h(v_2))$.

Case 2b: Suppose $v_2$is also adjacent to a vertex say $v_3 \in V(G_1)$. since f * is surjective, there exists say $u_3 \in V(G)$ such that $f^*(u_3) = v_3$. Fix one such$u_3 \in V(G)$. Then (v_2 ,v_3) = ((f*(u_2),f*(u_3)) ∈ E(G_1)) and so by assumption (u_2, u_3) ∈ E(G). Defineh(v_3) = $u_3$and $k(v_2, v_3) = (u_2, u_3) = (h(v_2), h(v_3))$.

Case 2c: Suppose $v_4 \in V(G_1)$ and v_4 is adjacent to say $v_5 \in V(G_1)$ such that neither $v_4$nor$v_5$is adjacent to any other vertex in G_1. Since $\tilde{f}$ is surjective , choose and fix an edge(u_4 , u_5) ∈ E(G),such that
$\tilde{f}(u_4, u_5) = (v_4, v_5)$ .i.e. $(f^*(u_4), f^*(u_5)) = (v_4, v_5)$.
Define $h(v_4) = u_4$ *and* $h(v_5) = u_5$ *and* $k(v_4, v_5) = (u_4, u_5) = (h(v_4), h(v_5))$.

Case 2d: Suppose $v \in V(G_1)$is an isolated vertex. Since f * is surjective choose and fix $u \in V(G)$ Such that $f^*(u) = v$. Define $h(v) = u$. Then it is clear that the pair $(h, k): G_1 \to G$ is a homomorphism of graphs. Denoting $g^* = h$ *and* $\tilde{g} = k$
We have a homomorphism
$g = (g^*, \tilde{g}): G_1 \to G$. Moreover from the construction it follows that $f^* g^* = f^* h = 1_{V(G)}$ and $\tilde{f}\,\tilde{g} = \tilde{f} k = 1_{E(G_1)}$ *Thus* $fg = 1_{G_1}$ so that f is a retraction.

Remark 1.17: We have seen earlier that if $f:G \to G_1$ is a homomorphism of graphs then f* is injective imples that $\tilde{f}$ is injective. However if f * is surjective then $\tilde{f}$ need not be surjective.
Consider the homomorphism of graphs $f:G \to G_1$
Given by the diagram. (See Figure 5)

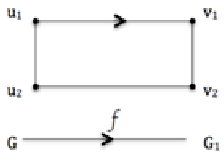

Figure 5

Clearly f * is surjective. However $\tilde{f}:\phi \to E(G_1) \neq \phi$ is not surjective[8].

Remark 1.18: As mentioned earlier, condition (ii) is equivalent to saying that $(u,v) \in E(G)$
If and only if $(f^*(u), f^*(v) \in E(G_1)$. Howeverf * and $\tilde{f}$ are surjective does not imply condition (ii) as shown by the following example (See Figure 6).

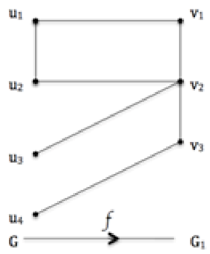

Figure 6

Here both f * and $\tilde{f}$ are surjective $(v_2, v_3) = (f(u_3), f(u_4))$
belongs to $E(G_1)$ but there is no edge in G joining u_3 and u_4.

Definition 1.18: Let $f:G \to G_1$ be a homomorphism of graphs. Then f is said to be a monomorphism if for all morphism $g_1, g_2 : G_2 \to G$. $f\, g_1 = f\, g_2$ *implies that* $g_1 = g_2$ (left cancellation).
Dually f is said to be an epmorphism if for all morphisms $g_1, g_2 : G_1 \to G_2$. $g_1 f = g_2\, f$ implies that $g_1 = g_2$ (right cancellation).

Proposition 1.19: Let $f:G \to G_1$ be a homomorphism of graphs. Then f is a monomorphism if and only if f * is injective.

Proof: Let $f:G \to G_1$ be a monomorphism. Suppose f * is not injective. Then there exists $u_1, u_2 \in V(G), u_1 \neq u_2$ *but* $f^*(u_1) = f^*(u_2) = v(say)$. Consider the empty graph $G_2$with $V(G_2) = \{w\}$ *and* $E(G_2) = \phi$. Define homomorphisms $g_1, g_2 : G_2 \to G$ *as* $g_1^*(w) = u_1$ *and* $g_1^*(w) = u_2$. Clearly $g_1^* \neq g_2^*$ *and so* $g_1 \neq g_2$. However $f^* g_1^*(w) = f^*(u_1) = v$ *and* $f^* g_2^*(w) = f^*(u_2) = v$ and hence $f^*\, g_1^* = f^*\, g_2^*$, *i.e.* $(fg_1)^* = (fg_2)^*$ and hence $fg_1 = fg_2$ (*by lemma*1.6) *but* $g_1 \neq g_2$. Hence f cannot be a monomorphism.

Conversely let f * be injective. Suppose $g_1, g_2 : G_2 \rightarrow G$ are homomorphisms such that $f\, g_1 = f\, g_2$ .Then $(fg_1)^* = (fg_2)^*$ i.e. $f^* g_1^* = f^* g_2^*$ .This implies that $g_1^* = g_2^*$ since f * is injective. Therefore $g_1 = g_2$.(by lemma 1.6).

Proposition 1.20:A homomorphism $f : G \rightarrow G_1$ is an epimorphism if and only if $f^* : V(G) \rightarrow V(G_1)$ is surjective**.**

Proof: Suppose f * is surjective. Let $g_1, g_2 : G_1 \rightarrow G_2$ be graph homomorphisms such that $g_1 f = g_2\, f$. Then $g_1^{\;*} f^* = g_2^{\;*} f^*$ *and* $\tilde{g}_1\, \tilde{f} = \tilde{g}_2\, \tilde{f}$ .This implies that $g_1^* = g_2^*$ since f * is surjective. Hence $\tilde{g}_1 = \tilde{g}_2$ $(by\, 2)$ and $g_1 = g_2$.

Conversely suppose f * is not surjective. Then there exists an element $v_0 \in V(G_1)$ such that v_0 has no preimage under f *. Fix such a $v_0 \in V(G_1)$. Choose an element Z_0 such that $Z_0 \notin V(G_1)$. Let G_2 be the graph where $V(G_2) = V(G_1)$ $\cup$ $\{Z_0\}$. $E(G_2)$ is defined as follows.

If $v_1 \neq Z_0$ and $v_2 \neq Z_0$ then $(v_1, v_2) \in E(G_2)$ *if and only if* $(v_1, v_2) \in E(G_1)$.

Also $(v,\ Z_0) \in E(G_2)$ if and only if $(v\ , v_0) \in E(G_1)$.

Define homomorphisms $g_1, g_2 : G_1 \rightarrow G_2$ as follows.

$g_1^*(v) = v$ for all $v \in V(G_1)$ and $\tilde{g}_1(u, v) = (u, v)$ *for all* $(u, v) \in E(G_1)$. Then $g_1 : G_1 \rightarrow G_2$ is a homomorphism of graphs. Similarly define $g_2 : G_1 \rightarrow G$ as follows $g_2^* : V(G_1) \rightarrow V(G_2)$ is defined as

$$g_2^*(v) = \begin{cases} v \text{ if } v \neq v_0 \text{ and} \\ Z_0 \text{ if } v = v_0 \end{cases}$$

$$\tilde{g}_2(u, v) = \begin{cases} (u, v) \text{ if } u \neq v_0 \text{ and } v \neq v_0 \\ (u, Z_0) \text{ if } (u, v_0) \in E(G_1) \end{cases}$$

Then clearly g: $G_1 \rightarrow G_2$ is also a graph homomorphism. From the construction it follows that $g_1 f = g_2\, f$ but $g_1 \neq g_2$. Thus f is not an epimorphsm. This completes the proof.

Proposition 1.21: Let $f : G \rightarrow G_1$ be a homomorphism of graphs. Then f is an isomorphism if and only if f * and $\tilde{f}$ are bijections.

Proof: Let fbe an isomorphism. Then f is a coretraction and a retraction. f is a coretraction implies that f * and (hence $\tilde{f}$) are injective and $(f^*(u), f^*(v))$ is an edge in G_1 if and only if (u , v) is an edge in G ……..(A).

f is a retraction implies that f * and $\tilde{f}$ are surjective ……(B) . (A) and (B) together implies that f * and $\tilde{f}$ are bijections.

Conversely if f * and $\tilde{f}$ are bijections thenf * is injective and condition ii) of (Proposition 1.9) is true so that f is a coretraction. Again if f *and $\tilde{f}$ *are bijections then* f^* *and* $\tilde{f}$ are surjective and condition ii) of (proposition 1.16) is true and so f is a retraction. Thus f is an isomorphism.

Remark 1.22: From the above proposition it follows that f is an isomorphism if and only if

i) f* is a bijection and

ii) (u, v) is an edge in G if and only if (f* (u) , f*(v)) is an edge in G_1 which is the usual definition given in texts.

Theorem 1.23: The category of graphs is not balanced [7, 8].
Proof: We recall that a category C is said to be balanced if every morphism which is both a monomorphism and an epimorphism is an isomorphism. However the category G of graphs is not balanced as seem from the following example(See Figure 7).

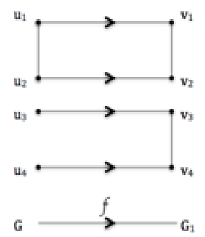

Figure 7

f * is injective implies that f is a monomorphism (by proposition 1.9) f * is surjective implies that f is an epimorphism (by 1.16).

However f is not an isomorphism, since $(v_3 , v_4) = (f*(u_3) , f*(u^4))$
is an edge in G_1 but (u_3 , u_4) is not an edge in G.

II. Conclusion

Hence with this representation we characterize some special morphisms likemonomorphism, epimorphism, coretraction, retraction etc in terms of set functions. Also, Finally we show that the Category of Graphs is not balanced

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