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First Edge Wiener Index of Link of Graphs

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Abstract : *One of the most widely known topological descriptors is the Wiener index or (Wiener number) named after American chemist Harold Wiener in 1947. Wiener number of a connected graph G is defined as the sum of the distances between distinct pairs of vertices of G.It correlates between physico- chemical and structural properties. Recently an edge version of Wiener Index was introduced by Ali Iranmanesh.In this paper, we have determined Wiener numbers of link of graphs.*
Keywords: *distance sum , edge Wiener index ,link of graph, topological indices.*

I. Introduction

A Graph G is formally defined to be a pair [V(G,E(G)] where V(G) is a non empty finite set of elements called vertices and E(G) is a finite set of unordered pairs of elements of V(G) called edges. Molecular graphs represent the constitution of molecules[**1**]. They are generated using the following rule: Vertices stand for atoms and edges for bonds. A graph theoretical distance d(u,v) between the vertices u and v of the graph G is equal to the length of the shortest path that connects u and v. An invariant of a graph G is a number associated with G that has the same value for any graph isomorphic to G. If G is a molecular graph then the corresponding invariants are called molecular descriptor or topological indices and they are used in theoretical chemistry for the design of so called Quantitative Structure Property Relations (QSPR) and Quantitative Structure Activity Relations(QSAR).One of the oldest topological index is Wiener index and is defined as the half of the sum of all the distances between every pair of vertices of G.[**2**] ie

$$W(G) = \frac{1}{2}\sum_{u,v} d(u,v) \qquad (1)$$

Equations (1) is called vertex version of Wiener Index .and its chemical and mathematical applications are welldocumented in [**3-6**]the edge versions of Wiener index were introduced by Iranmanesh etc . in 2008 as follow [**7**] and is defined as follows.
The first edge- Wiener number is

$$W_{e0}(G) = \frac{1}{2}\sum_{\substack{e,f\in E(G)\\ e\neq f}} d_0(e,f) \qquad (2)$$

Where $d_0(e,f) = d_1(e,f) + 1 \quad e\neq f$
$= 0 \quad e=f$
$d_1(e,f) = \min\{d(x,u), d(x,v), d(y,u), d(y,v)\}$ such that e=xy and f = uv and $W_{e0} = W(L(G))$.
The second edge - Wiener number is

$$W_{e4}(G) = \frac{1}{2}\sum_{\substack{e,f\in E(G)\\ e\neq f}} d_4(e,f)$$

Where $d_4(e,f) = \{ d_2(e,f) \quad e\neq f$
$= 0 \quad e=f$
$d_2(e,f) = \max\{d(x,u), d(x,v), d(y,u), d(y,v)\}$ such that e=xy and f = uv.
Since all these topological indices are depends on distance between every pair of vertices of a given graph G. Let G_1 and G_2 be two simple and connected graph with disjoint vertex sets. For given vertices $x_1 \in V(G)$ and $y_2 \in V(G)$, a link of two graphs G_1 and G_2 is defined as the graph $G_1\sim G_2(x, y)$ obtained by joining x and y by an edge . For simply we show the link of G_1 and G_2 by $G_1\sim G_2$. The link of graphs $G_1,G_2,\ldots\ldots,G_n$ is $G_1\sim G_2\sim \ldots\sim G_n$.Furthermore if $G_1 = G_2 = \ldots = G_n = G$ we use of the notation $G\sim^n G$. Balakrishnan [**8**] compute the Wiener
index of a graph with exactly *n* cut – edges.In this paper we have determined first edge Wiener index of link of graph.

Some Known Results:

For K_n

1. $W(K_n) = \frac{n(n-1)}{2}$
2. $W(K_n) = \frac{n(n-1)}{2}$
3. $W_{e0}(K_n) = \frac{n(n-1)^2(n-2)}{4}$
4. $WW_{e0}(K_n) = \frac{n((n-1)(n-2)(3n-5)}{4}$
5. $W_{e4}(K_n) = \frac{n(n-1)(n-2)(n+1)}{8}$
6. $W_{e4}(K_n) = \frac{n(n-1)(n-2)(n+1)}{4}$

For $K_{n,m}$

1. $W(K_{n,m}) = (m+n)^2-(m+n)-mn$
2. $WW(K_{n,m}) = 3(m+n)^2-3(m+n)-4mn$
3. $W_{e0}(K_{n,m}) = \frac{nm}{2}(2nm-n-m)$
4. $WW_{e0}(K_{n,m}) = nm(3nm-2n-2m+1)$
5. $W_{e4}(K_{n,m}) = nm(nm-1)$
6. $WW_{e4}(K_{n,m}) = 3nm(nm-1)$

II. Main results

Theorem 2.1

$$W_{e0}(G_1 \sim G_2) = W_{e0}(G_1) + W_{e0}(G_2) + (m_2+1)(D_0)G_1(e_1) + (m_1+1)(D_0)G_2(e_1)$$

Proof:

Let $E(G_1) = \{e_i(1) \mid i = 1 \text{ to } m_1\}$

$E(G_2) = \{e_i(2) \mid i = 1 \text{ to } m_2\}$

and $E(G_1 \sim G_2) = E(G_1) \cup E(G_2) \cup \{e_1\}$

Hence $|E(G_1 \sim G_2)| = m_1 + m_2 + 1$. In General

$$(D_0)_G(e) = \sum_{f \in E(G)} d_0(e,f)$$

$$W_{e0}(G_1 \sim G_2) = \sum_{\substack{i<j \\ i,j=1}}^{m_1+m_2+1} d_0(e_i,e_j) = \sum_{\substack{i,j=1 \\ i<j}}^{m_1} d_0(e_i(1),e_j(1)) + \sum_{\substack{i,j=1 \\ i<j}}^{m_2} d_0(e_i(2),e_j(2)) + + \sum_{i=1}^{m_1} d_0(e_1,e_i(1)) +$$

$$+ \sum_{i=1}^{m_2} d_0(e_1,e_i(2)) + \sum_{i=1}^{m_1}\sum_{j=1}^{m_2} d_0(e_i(1),e_j(2))$$

$$= W_{e0}(G_1) + W_{e0}(G_2) + (D_0)_{G_1}(e_1) + (D_0)_{G_2}(e_1) + \sum_{i=1}^{m_1}\sum_{j=1}^{m_2} d_0(e_i(1),e_j(2))$$

$$\sum_{i=1}^{m_1}\sum_{j=1}^{m_2} d_0(e_i(1),e_j(2)) = \sum_{i=1}^{m_1}\sum_{j=1}^{m_2} (d_0(e_1,e_i(1)) + d_0(e_1,e_i(2)) = m_2(D_0)G_1(e_1) + m_1(D_0)G_2(e_1)$$

Therefore $W_{e0}(G_1 \sim G_2) = W_{e0}(G_1) + W_{e0}(G_2) + (m_2+1)(D_0)G_1(e_1) + (m_1+1)(D_0)G_2(e_1)$

Theorem 2.2

$$W_{e0}(G_1 \sim G_2 \sim G_3) = W_{e0}(G_1) + W_{e0}(G_2) + W_{e0}(G_3) + (m_2+m_3+2)(D_0)_{G_1}(e_1) + (m_1+1)(D_0)_{G_2}(e_1) + (m_3+1)(D_0)_{G_2}(e_2) + (m_1+m_2+2)(D_0)_{G_3}(e_2) + (m_3+1)(m_1+1)d_0(e_1,e_2)$$

Proof:

$$W_{e0}(G_1\sim G_2\sim G_3)=W_{e0}\big((G_1\sim G_2)\sim G_3\big)=W_{e0}(G_1\sim G_2)+W_{e0}(G_3)+(m_3+1)(D_0)_{G_1\sim G_2}(e_2)+$$
$$+(m_1+m_2+2)(D_0)_{G_3}(e_3)$$

By using previous Theorem

$$W_{e0}(G_1\sim G_2\sim G_3)=W_{e0}(G_1)+W_{e0}(G_2)+(m_1+1)(D_0)_{G_2}(e_1)+(m_2+1)(D_0)_{G_1}(e_1)+W_{e0}(G_3)$$
$$+(m_3+1)(D_0)_{G_1\sim G_2}(e_2)+(m_1+m_2+1)(D_0)_{G_3}(e_3) \qquad (3)$$

Consider

$$(D_0)_{G_1\sim G_2}(e_2)=\sum_{i=1}^{m_1+m_2+1} d_0(e_2,e_i)=\sum_{i=1}^{m_2} d_0\big(e_2,e_i(2)\big)+d_0(e_1,e_2)+\sum_{i=1}^{m_1} d_0\big(e_2,e_i(1)\big)$$

$$=(D_0)_{G_2}(e_2)+d_0(e_1,e_2)+\sum_{i=1}^{m_1}\Big(d_0\big(e_1,e_i(1)\big)+d_0(e_1,e_2)\Big)$$

$$=(D_0)_{G_2}(e_2)+(m_1+1)+d_0(e_1,e_2)+(D_0)_{G_1}(e_1) \qquad (4)$$

Substitute (3) in (4) and simplify we get

$$W_{e0}(G_1\sim G_2\sim G_3)=W_{e0}(G_1)+W_{e0}(G_2)+W_{e0}(G_3)+(m_2+m_3+2)(D_0)_{G_1}(e_1)+(m_1+1)(D_0)_{G_2}(e_1)$$
$$+(m_3+1)(D_0)_{G_2}(e_2)+(m_1+m_2+2)(D_0)_{G_3}(e_2)+(m_3+1)(m_1+1)d_0(e_1,e_2)$$

Theorem 2.3

$$W_{e0}(G_1\sim G_2\sim G_n)=\sum_{i=1}^{n} W_{e0}(G_i)+\sum_{j=1}^{n-1}[(D_0)_{G_j}(e_j)\sum_{i=j+1}^{n}(m_j+1)]+(D_0)_{G_n}(e_{n-1})(m_1+m_2+\cdots$$
$$+m_{n-1}+n-1)+\sum_{j=1}^{n-2}\left\{\left[(D_0)_{G_{j+1}}(e_j)+d_0(e_j,e_{j+1})\sum_{i=j+2}^{n}(m_i+1)\right]\sum_{i=1}^{j}(m_i+1)\right\}$$

Proof: Let us use induction method on n to prove this result. By previous Theorems result is true for n=2 and 3. Assume that result is true for n=k.

$$W_{e0}(G_1\sim G_2\ \ldots..\sim G_k)=\sum_{i=1}^{k} W_{e0}(G_i)+\sum_{i=1}^{k-1}(D_0)_{G_i}(e_j)\left[\sum_{i=j+1}^{k}(m_j+1)\right]+(D_0)_{G_k}(e_{k-1})(m_1+m_2+..$$
$$+m_{k-1}+k-1)+\sum_{j=1}^{k-2}\left\{\left[(D_0)_{G_{j+1}}(e_j)+d_0(e_j,e_{j+1})\sum_{i=j+2}^{k}(m_i+1)\right]\sum_{i=1}^{j}(m_i+1)\right\}$$

We have to prove that result is true for n=k+1 .Consider

$$W_{e0}(G_1\sim G_2\sim\ldots\sim G_k\sim G_{k+1})=W_{e0}\big((G_1\sim G_2\sim\ldots..\sim G_k)\sim G_{k+1}\big)=W_{e0}(G_1\sim\ldots.\sim G_k)+W_{e0}(G_{k+1})$$
$$+(m_{k+1}+1)(D_0)_{G_1\sim\ldots\sim G_k}(e_k)+(m_1+m_2+\cdots+m_k+k-1+1)(D_0)_{G_{k+1}}(e_k)$$

Now

$$(D_0)_{G_1-\ldots-G_k}(e_k)=\sum_{j=1}^{m_1+m_2+\cdots+m_k+k-1} d_0(e_k,e_j)=\sum_{i=1}^{mk} d_0\big(k,e_i(k)\big)+d_0(e_k,e_{k-1})+\sum_{i=1}^{m_{k-1}} d_0\big(e_k,e_i(k-1)\big)$$

$$=(D_0)_{G_k}(e_k)+d_0(e_k,e_{k-1})+(D_0)_{G_{k-1}}(e_{k-1})+m_{k-1}d_0(e_k,e_{k-1})+d_0(e_k,e_{k-1})+d_0(e_{k-1},e_{k-2})$$

$$+(D_0)_{G_{k-2}}(e_{k-2})+m_{k-2}d_0(e_k,e_{k-1})+m_{k-2}d_0(e_{k-1},e_{k-2})+\cdots+d_0(e_k,e_{k-1})+d_0(e_{k-1},e_{k-2})$$

$$+\cdots+d_0(e_2,e_3)+d_0(e_1,e_2)+(D_0)_{G_1}(e_1)+m_1d_0(e_1,e_2)+m_1d_0(e_2,e_3)+..+m_1(e_{k-1},e_k)$$

$$=\sum_{i=1}^{k}(D_0)_{G_i}(e_i)+\sum_{i=1}^{k-1}\left\{d_0(e_i,e_{i+1})\left(\sum_{j=1}^{i}(m_j+1)\right)\right\}$$

Therefore

$$W_{e0}(G_1\sim\ldots..\sim G_{k+1})=\sum_{i=1}^{k+1}W_{e0}\,(G_i)+\sum_{j=1}^{k}\left[(D_0)_{G_j}(e_j)\left(\sum_{i=j+1}^{k+1}(m_i+1)\right)\right]+(m_1+m_2+$$

$$+\cdots+m_k+k)(D_0)_{G_{k+1}}(e_k)+\sum_{j=1}^{k-1}\left[(D_0)_{G_{j+1}}(e_j)+d_0(e_j,e_{j+1})\left(\sum_{i=j+2}^{k+1}(m_i+1)\right)\right]\left[\sum_{i=1}^{j}(m_i+1)\right]$$

Hence the result is true for n=k+1. Therefore by induction. Hypothesis result is true for all positive integers n.

Corollary 1

If $G_1=G_2=\cdots..=G_n=G$ then

$$\sum_{i=a}^{b}(m_i+1)=(b-a+1)(m+1)$$

$$W_{e0}(G\overset{n}{\sim}G)=nW_{e0}(G)+(m+1)\sum_{j=1}^{n-1}\{(D_0)_{G^{(j)}}(ej)(n-j)+(D_0)_{G^{(j+1)}}(e_j)j\}$$

$$+(m+1)^2\sum_{j=1}^{n-2}d_0\,(e_j,e_{j+1})(n-j-1)j$$

Where $G^{(j)}$ is j^th image of G

Corollary 2

If $G_1=G_2=\cdots..=G_n=C_m$ then

$(D_0)_{cm^{(j)}}(e_j)=\frac{2W_{e0}(cm)}{m}+\frac{m+1}{2}$ If m is odd

$=\frac{(m+1)^2}{4}$

$(D_0)_{cm^{(j)}}(ej)=\frac{2W_{e0}(cm)}{m}+\frac{m}{2}$ If m is even

$=\frac{m}{4}(m+2)$

$(D_0^2)_{cm^{(j)}}(ej)=\frac{2WW_{e0}(cm)}{m}+\frac{(m+1)}{2}\left(\frac{m+1}{2}+1\right)=\frac{(m+1)}{12}(m+2)(m+3)$ if m is odd.

$=\frac{2WW_{e0}(cm)}{m}+\frac{m}{2}\left(\frac{m+1}{2}\right)=\frac{m(m+2)(m+4)}{12}$ if m is even.

Hence

$W_{e0}(C_m\overset{n}{\sim}C_m)=\frac{n(m+1)}{8}\left(m(m-1)+2(m+1)^2(n-1)\right)+(m+1)^2(S_0)$ if m is odd

Where

$$S_0=\sum_{j=1}^{n-2}d_0\,(e_j,e_{j+1})(n-j-1)j$$

Corollary 3
In $\left(C_m \overset{n}{\sim} C_m\right)$ if $d_0(e_i, e_{i+1}) = k$ *for all* i *then*

$$S_0 = \sum_{j=1}^{n-2} d_0\left(e_j, e_{j+1}\right)(n-j-1)j = \mathcal{K}\sum_{j=1}^{n-2}(n-j-1)j = \frac{\mathcal{K}\, n(n-2)(n-1)}{6}$$

$$S_1 = \sum_{j=1}^{n-2} d_0^2\left(e_j, e_{j+1}\right)(n-j-1)j = \frac{\mathcal{K}^2 n(n-2)(n-1)}{6}$$

$$S_2 = \sum_{i=1}^{n-3} d_0\left(e_i, e_{i+1}\right)i \sum_{j=i+1}^{n-2} d_0\left(e_j, e_{j+1}\right)(n-j-1)j = \frac{\mathcal{K}^2(n-3)(n-2)n(n-1)}{24}$$

Hence

$$W_{e0}\left(C_m \overset{n}{\sim} C_m\right) = \frac{n(m+1)}{8}\left[3m(m-1) + 6(m+1)^2(n-1) + 4(m+1)k(n-1)(n-2)\right] \quad if\ \ m\ is\ \ odd$$

$$W_{e0}\left(C_m \overset{n}{\sim} C_m\right) = \frac{nm^3}{8} + \frac{(m+1)(n-1)}{12}\left[3m(m+2) + 2(m+1)k(n-2)\right] \quad if\ \ m\ is\ \ even$$

III. Conclusion

Using this link relation we can fine the edge Wiener index of some dentrimer molecules.

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