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Dual Purely Rickart Modules

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***Abstract:**. In this paper, we introduce a dual purely Rickart modules as a dual concept of purely Rickart modules and a proper generalization to dual Rickart modules. Some characterizations are studied. It's shown that a ring R is dual purely Rickart if and only if R is von Neumann regular ring if and only if R is d-Rickart ring. Also, a ring R is a dual purely Rickart ring if and only all R-modules are dual purely Rickart module. In principal right ideal ring R, a ring R is a field if and only if all R-modules are dual purely Rickart. Furthermore, we give a counter example to show that the endomorphism ring of the dual purely Rickart module is not necessarily dual purely Rickart ring. More than, the relation among known modules and dual purely Rickart modules are investigated.*

***Keywords**: Dual purely Rickart modules , purely Rickart, Rickart modules, regular rings, regular modules.*

I. Introduction

A. Hattori, in [4], introduced the right PP (equiv., Rickart[7]) if every principle right ideal is projective (the right annihilator in R of every single element in R is generated by idempotent element). G. Lee, S. T. Rizvi and C.Roman in [5] studied the Rickart modules. A module M is Rickart if the right annihilator in M of each single element in S is generated by idempotent element of S. The same authors in [6], introduced dual Rickart modules. A module M is dual Rickart if the image in M of any single element α of S = $End_R(M)$ is generated by an idempotent of S.

A ring R is von Neumann regular if and only if every principle right ideal is pure [3] (An ideal I of a ring R is right (left) pure in R if for each element a∈I there is b ∈ I such that a= ab(ba)[2]). Following [3], a submodule N a module M is pure if and only if the sequence 0→ N⊗E → M⊗E is exact for all left R-modules E [3]. From the same, a module is Von Neumann regular module if every submodule of M is pure. The authors in [1], introduced purely Rickart module. A module M is purely Rickart if the right annihilator in M of every single element of S = $End_R(M)$ is pure in M. In this work, we introduce the concept of dual purely Rickart module as a dual concept to purely Rickart modules and as a proper generalization to dual Rickart modules. A module is called dual purely Rickart (shortly, d-purely Rickart) if the image of each single element α of S = $End_R(M)$, is pure in M. Also, we will introduce a relatively d-purely Rickart and using it to get useful properties of d-purely Rickart modules. Finally, recall that a module M satisfies the C_2 (respectively D_2)-condition if for any submodule A of M with $A \cong L \le^{\oplus} M$ (resp. $\frac{M}{A} \cong L \le^{\oplus} M$), then $A \le^{\oplus} M$ [8]. We give a purely C_2 as a generalization to this concept. A module M satisfies the purely C_2 condition if $A \cong L \le^{\oplus} M$, then $A \le^{p} M$. It's clear that, every d- purely Rickart module satisfies the purely C_2.

Throughout this paper, R will denotes associative ring with identity and all modules will be unitary right R-modules with S= $End_R(M)$ is the ring of all endomorphism of M. Then M is right R- left S-module. The samples $\le$, $\trianglelefteq$, $\le^{\oplus}$, $\trianglelefteq^{\oplus}$, $\le^{e}$ and ◘ refer to submodule, fully invariant submodule,direct summand, fully invariant direct summand, essential submodule and end the proof.

II. Dual purely Rickart modules

We introduce the following definition:

Definition 2.1. Let M be a right R-module and S = $End_R(M)$. Then M is said to be dual-purely Rickart (shortly, d-purely Rickart) module if the image in M of any single element of S is pure in M (in sense of Cohn). That is for each α ∈ S, Imα $\le^{p}$ M. A ring R is right (left) d-purely Rickart if R_R ($_RR$) is d-purely Rickart module.

Remarks and examples 2.2.

1. Every von Neumann regular module and hence semisimple module is d-purely Rickart module. Note that the Z-module Q is d-purely Rickart (where $End_R(Q)$ is a division ring) which is neither semisimple nor regular module, where Z is neither direct summand nor pure submodule in Q.
2. Z_p as Z-module for each prime integer p is d-purely Rickart module while in general, the Z-module Z_{p^n} is not d-purely Rickart module for each prime integer p and n∈ ℕ (ℕ is a Neutral numbers). In particular, the Z-module Z_4 is not d-purely Rickart. In fact, there is a homomorphism α ∈ S = $End_R(Z_4)$ such that $\alpha(\bar{0})$ =

$\alpha(\bar{2}) = \bar{0}$ and $\alpha(\bar{1}) = \alpha(\bar{3}) = \bar{2}$. It's well known that α is well define and homomorphism. So Imα = $\{\bar{0}, \bar{2}\}$ is not pure submodule in Z_4. Also, the Z-module Z is principle ideal ring of the form nZ, but for each $0 \neq n \in Z$, nZ is not pure in Z. Hence the Z-module Z is not d-purely Rickart (see Proposition (2.3)).

3. Every d-Rickart module is d-purely Rickart modules. But the converse is not true in general. In fact, the ring $R = \prod_{n=1}^{\infty} Z_2$ is von Neumann regular [6, Example 2.28] and so R is d-purely Rickart ring (see Corollary (2.4)). Hence the module $M = R^{(R)}$ is d-purely Rickart module (see Proposition 3.9). But M is not d-Rickart module [6, Example 3.9].
4. If M ≅ N with M is a d-purely Rickart module, then so is the module N.
   Proof. Suppose that M ≅ N with M is a d-purely Rickart module. Let $S = End_R(M)$ and $H = End_R(N)$. Hence there is an isomorphism ψ : S → H such that $\psi(\lambda) = \alpha\lambda\alpha^{-1}$ where α : M → N is an isomorphism and λ∈ S. Now, let β be an arbitrary element in H. Then β(N) = ψ(μ)(N) for some μ ∈ S. Then $\beta(N) = (\alpha\mu\alpha^{-1})(N) = \alpha(\mu(M))$. But $\mu(M) \leq^p M$ where M is d-purely Rickart, hence $\beta(N) = \alpha(\mu(M)) \leq^p N$ where α is an isomorphism. ◘

Firstly, the next result gives a new characterization to d-purely Rickart ring

Proposition 2.3. A ring R is right (left) d-purely Rickart if and only if every principle right (left) ideal is pure in R if and only if every principle right (left) ideal is a direct summand of R.

Proof. Let I = aR be a principle right ideal in a right d-purely Rickart ring R. Then there is an epimorphism α : R→R such that α(r) = ar for each r ∈ R. So, $aR = Im\alpha \leq^p R$. Conversely, for all a ∈ R, aR is principle right ideal. By hypothesis, aR is a pure right ideal in R. Hence R is regular ring and so R is d- Rickart[6]. Therefore R is purely Rickart ring. The last statement follows from ([3, Theorem (2.1), Ch.3]). ◘

It's known that a ring R is von Neumann regular if and only if every principle right (left) ideal is pure [3, Theorem 2.1, Ch.3]. Following [6, Remark 2.2], R_R is d-Rickart module if and only if R is a von Neumann regular ring. From this fact and as a corollary to Proposition (2.3)), the following corollary is an obvious.

Corollary 2.4. The following conditions are equivalent for a ring R:

a. R is a d-purely Rickart ring;
b. R is a von Neumann regular ring;
c. R is a d-Rickart ring.

In general, the purely Rickart modules and d-purely Rickart modules are different. In fact the Z-module Z is purely Rickart which is not d-purely Rickart while Z_{P^∞} is d-purely Rickart Z-module which is not purely Rickart Z-module. It's well known that every regular ring is Rickart and hence is purely Rickart ring, so we have the following remark

Remark 2.5. Every d-purely Rickart ring is purely Rickart but not conversely. In fact, the ring of integers Z is purely Rickart [1] which is not d-purely Rickart.

We mention that every semisimple ring is d-purely Rickart but the converse is not true in general. Here we give a condition under which semisimple and d-purely Rickart rings are equivalent. Before that, a ring R is principle right ideal (simply, pri) if every right ideal in R is principle.

Corollary 2.6. A ring R is right semisimple if and only R is a d-purely Rickart and pri ring.

⟹) Let I be a right ideal in R, by hypothesis, I = eR for some $e^2 = e \in R$. Hence R is pri and $I \leq^p R$. Thus by Proposition (2.3), R is right d-purely Rickart ring and pri.

⟸) Let I be a right ideal in I. By hypothesis and Proposition (2.3), R is a right semisimple ring. ◘

Recall that a right R-module M is an epi-retractable if every submodule of M is a homomorphic image of M [3]. Indeed, the Z-module Z is epi-retractable which is not regular module. We assert that every regular module is d-purely Rickart but the converse is not true in general. The following result answer when the converse is true.

Proposition 2.7. Let M be an epi-retractable module. Then M is d-purely Rickart module if and only if M is von Neumann regular module

Proof. Suppose that M is a d-purely Rickart module and N ≤ M. Since M is an epi-retractable module, then N = α(M) for some $\alpha \in S = End_R(M)$. By hypothesis, M is a d-purely Rickart module, then $N = \alpha(M) \leq^p M$. Therefore, M is a von Neumann regular module. Conversely, follows from the Remarks and examples (2.2(1)). ◘

The Z-module Q is d-purely Rickart which is not epi-retractable module(since $Hom_Z(Q, Z) = 0$) and not regular module(since Z is not pure submodule of Q).

The proof of the following corollary follows from Proposition (2.7) and [1, Remarks and examples (2.2(4))]

Corollary 2.8. If a module M is an epi-retractable and d-purely Rickart then M is purely Rickart module.

Recall that an exact sequence of right R-modules $0 \to N \to M \to F \to 0$ is pure if and only if for every left R-module D, the sequence $0 \to N\otimes D \to M\otimes D \to F\otimes D \to 0$ is an exact [3].

Proposition 2.9. A module M is d-purely Rickart if and only if the short exact sequence

$$₤: 0 \to \mathrm{Im}\alpha \to M \to \frac{M}{\mathrm{Im}\,\alpha} \to 0$$

is pure for each $\alpha \in S = End_R(M)$.

Proof. The short exact sequence ₤ is pure if and only if $\mathrm{Im}\alpha \leq^p M$ for each $\alpha \in S$ and hence if and only if M is d-purely Rickart module. ◘

Corollary 2.10. A module M is d-purely Rickart if and only if $\frac{M}{\mathrm{Im}\,\alpha}$ is a flat submodule of M for each $\alpha \in S = End_R(M)$.

Proof. ⇐) Following [3, Ch.1, Theorem 1.7] and Proposition (2.9).
⇒) Following [3, Ch.1, Corollary p.13] and Proposition (2.9). ◘

Corollary 2.11. In a regular ring, every R-module is d-purely Rickart.

Proof. For each $\alpha \in S = End_R(M)$, since $\frac{M}{Im\alpha}$ can be considered as an R-module, then $\frac{M}{Im\alpha}$ is flat R-module since R is a regular ring [3, Ch.3, p.58]. Hence $\mathrm{Im}\alpha \leq^p M$ [3, Ch.1, Theorem 1.7]. ◘

Recall that a module M is morphic if $\frac{M}{Im\alpha} \cong \ker\alpha$ for each $\alpha \in S = End_R(M)$ [9]. From the fact, every module isomorphic to a flat module is a flat, the proof of the following corollary is clear.

Corollary 2.12. Let M be a morphic module. Then M is a d-purely Rickart module if and only if $\ker\alpha$ is flat in M for each $\alpha \in S = End_R(M)$.

From [3, Ch.3, Theorem 4.2] in an exact sequence $0 \to P \to Q \to F \to 0$ with P finitely generated and Q projective , P is pure if and only if P is a direct summand. We have the following lemma

Lemma 2.13. Let M be a noetherian and projective module, then the short exact sequences ₤: $0 \to \mathrm{Im}\alpha \to M \to \frac{M}{Im\alpha} \to 0$ for each $\alpha \in S = End_R(M)$, is pure if and only if ₤ splits.

Proof. Since M is noetherian module, then Imα is finitely generated submodule of M [5, Proposition 1.16]. So, $\mathrm{Im}\alpha \leq^p M$ if and only if $\mathrm{Im}\alpha \leq^{\oplus} M$ [3, Ch.3, Theorem 4.2]. Therefore ₤ is a pure exact sequence if and only if ₤ splits. ◘

Corollary 2.14. Let M be a noetherian and projective module. Then M is a d-purely Rickart if and only if M is a d-Rickart module.

Recall that an R-module N is pure-injective if and only if any pure exact sequence $0 \to N \to M$ splits [2]. The following proposition shows that the properties of d-purely Rickart and d- Rickart module M are equivalent if Imα is a pure-injective submodule of M for each $\alpha \in S = End_R(M)$.

Proposition 2.15. Let M be a module and $S = End_R(M)$. If Imα is pure-injective submodule of M, then M is d-purely Rickart module if and only if M is d-Rickart module.

Examples 2.16.

1. The Z-module Z_n is d-purely Rickart if and only if its d-Rickart for each $n \in N$. In particular, Z_4 and Z_{12} are not d-purely Rickart.
   Proof. Z_n as Z-module is pure injective since every finite module is pure injective, and then Imα for each $\alpha \in S$ is pure injective for each $n \in N$. By Proposition (2.15), the proof is complete.

2. It's well known that Z_{P^∞} is injective as Z-module and hence is pure-injective. Furthermore, for each α ∈ S = $End_R(Z_{P^\infty})$, Imα = α(Z_{P^∞}) = Z_{P^∞} is pure- injective, hence Z_{P^∞} satisfies the Proposition (2.15).

Recall that a module M is a pure simple if the only pure submodules of M are the trivial submodules [3]. A module M is pure split if every pure submodule of M is a direct summand [2].

Proposition 2.17. Let M be a pure simple (resp. pure split) module. Then, the module M is a d-purely Rickart if We can summarize the conditions under which the concepts of d-purely Rickart modules and d-Rickart modules are equivalent.

Proposition 2.18. A module M is d-Rickart if and only if M is d-purely Rickart such that one of the following conditions holds:

1. $\frac{M}{Im\alpha}$ is a flat submodule of M for each α ∈ S = $End_R(M)$;
2. M is a noetherian and projective module;
3. M is a pure simple (resp. pure split) module;
4. Imα is a pure-injective submodule of M for each α ∈ S = $End_R(M)$.

Proposition 2.19. A module M is d-purely Rickart and pure simple if and only if every nonzero endomorphism α ∈ S = $End_R(M)$ is an epimorphism and M is pure split.

Proof. It's well known that every pure simple is pure split. Now, Let (0 ≠)α ∈ S = $End_R(M)$. By hypotheses, M is d-purely Rickart module, then Imα $\leq^p$ M. But M is a pure simple, so Imα = α(M) = M. Hence α is an epimorphism. Conversely, let α ∈ S, so α(M) = M. Hence Imα $\leq^p$ M and then M is d-purely Rickart module. Furthermore, if (0≠) N $\leq^p$ M, then N $\leq^\oplus$ M. Hence (0≠) jρ ∈ S is an epimorphism, where ρ: M → N is the projection and j: N→M is the injection. But M = jρ(M) = N. Therefore, M is pure simple. ◘

Remarks 2.20.

1. From the proof of Proposition (2.19), one can show that the condition "pure split" is not necessary to prove M is d-purely Rickart module. For example, M=Z_{P^∞} as Z-module is d-purely Rickart which is not pure split but every nonzero endomorphism of S = $End_R(M)$ is an epimorphism.
2. There is a d-purely Rickart module not pure simple has a nonzero endomorphism which is not epimorphism. In fact, the Z-module Z_6 is semisimple module and hence is d-purely Rickart module but there is a homomorphism (0≠) α: $Z_6 \to Z_6$ such that α(Z_6) =$\{\bar{0}, \bar{2}, \bar{4}\}$ is not epimorphism.

Endomorphism ring of d-purely Rickart module needed not domain. In fact, the Z-module $\frac{Q}{Z} \cong \oplus Z_{P^\infty}$. Now, $\frac{Q}{Z}$ is d-purely Rickart module since it is d-Rickart [6, Example 2.3]. But $End_Z(\oplus Z_{P^\infty})$ has an infinite set of nonzero orthogonal idempotent elements, so $End_Z(\oplus Z_{P^\infty})$ is not indecomposable ring. Hence $End_Z(\oplus Z_{P^\infty})$ is not domain.

Endomorphism ring of d-Rickart module is domain if M is an indecomposable [6, Proposition (4.4)]. Here we prove that

Corollary 2.21. If M is a d-purely Rickart module and pure simple then S = $End_R(M)$ is a domain.

Proof. Suppose that αβ = 0 for each α and β ∈ S. If β = 0, then there is nothing to prove. Suppose β ≠0, then by Proposition (2.19), β(M) = M. That is 0= αβ(M) = α(M). Thus α = 0. Therefore S is a domain. ◘

Corollary 2.22. Let M be an epi-retractable module. If M is a d-purely Rickart and pure simple module then S = $End_R(M)$ is a division ring.

Proof. From Proposition (2.19), every nonzero endomorphism of M is an epimorphism. But M is an epi-retractable module, so by [3, Corollary 3.6] S is a division ring.

The converse of Corollary (2.21) is not true in general. In fact, the ring of integers Z is domain and hence is pure simple but by (Remarks and examples (2.1(2)), Z is not d-purely Rickart. Also, the Z-module Z is epi-retractable and pure simple which is not d-purely Rickart. One can easily show that $End_R(Z) \cong Z$ is not division ring. Indeed, $Z_p \oplus Z_p$ is d-purely Rickart since its semisimple Z-module. But $End_Z(Z_p \oplus Z_p)$ is not division ring[6].

Recall that a module M is faithful if and only if $ann_R(M)$ = 0 and M is divisible module if M = rM for each right nonzero-divisor element r ∈ R [12, p.32]. In a domain R, a module M is torsion free whenever mr = 0 then r = 0 for each m ∈ M and r ∈ R [12, p.34].

Corollary 2.23. Let M be a faithful and pure simple module. If M is a d-purely Rickart module, then M is a divisible.

Proof. Let (0≠) a ∈ R and α: M → M defined by α(x) = xa for all x ∈ M. So, α is a homomorphism since M faithful, then α is well-defined. It's clear that Imα = Ma. By Proposition (2.19), α is an epimorphism and hence Ma = Imα = M. Therefore M is divisible. ◘

Corollary 2.24. Let M be an indecomposable faithful module. If M is a d-Rickart module, then M is divisible.
It's well known that if a module M is divisible and torsion free over commutative domain then M is an injective module [12, Proposition 2.7, p.34], from this fact and Proposition (2.19),one can prove the following corollary.
Corollary 2.25. Let M be a pure simple and torsion free over commutative domain R. If M is a d-purely Rickart module, then M is an injective module.

Note that the Z-module Z is pure simple and torsion free over commutative domain Z but Z is not d-purely Rickart and not injective module, while the Z-module Q is pure simple and torsion free and d-purely Rickart over commutative domain Z, hence it is injective.
The authors in [1] defined the following condition (let us called purely D_2) for each N ≤ M if $\frac{M}{N} \cong L \leq^{\oplus} M$ then $N \leq^{p} M$. Every purely Rickart module satisfies the purely D_2 condition [1]. Here we will introduce purely C_2 condition which can be considered as a dual concept to purely D_2 condition.

Definition 2.26. A module M is said to be satisfying the purely C_2 condition if for any submodule $B \cong A \leq^{\oplus} M$ then $B \leq^{p} M$.

Remark 2.27.
1. The class of all modules satisfies the C_2-condition contained in the class of the modules which satisfies the purely C_2-condition.
2. Every regular modules satisfies the purely C_2-condition.
3. The Z-module Z does not satisfy the purely C_2 condition. In fact, the Z-submodule $2Z \cong Z \leq^{\oplus} Z$ but 2Z is not pure in Z.

The following proposition is dual to [1, Proposition 2.12].

Proposition 2.28. Every d-purely Rickart module satisfies the purely C_2 condition. The converse is true if $\mathrm{Im}\alpha \cong A \leq^{\oplus} M$ for each $\alpha \in S = \mathrm{End}_R(M)$.

Proof. Let M be a module and A and B are submodules of M. Let α : A → B be an isomorphism and $A \leq^{\oplus} M$. Then α can be extended to jαρ: M → M where ρ: M → A be the projection and j: B → M be the injection. So $\mathrm{Im}(j\alpha\rho) \leq^{p} M$ since M is a d-purely Rickart module. But $\mathrm{Im}(j\alpha\alpha) = j\alpha\rho(M) = \alpha(A) = B \leq^{p} M$. So, M satisfies the purely C_2-condition. The converse is an obvious since every direct summand submodule is pure and by putting Imα = B for each $\alpha \in S = \mathrm{End}_R(M)$, the proof is complete. ◘

Consider the condition (*): for any N≤M if $\frac{M}{N} \cong L \leq^{p} M$. If a module M satisfies the condition (*), then $N \leq^{p} M$. Then M satisfies the purely D_2 condition.

Proposition 2.29. Let M be a module and $S = \mathrm{End}_R(M)$. Then the following conditions hold:
1. If M is a d-purely Rickart module satisfies the condition (*), then M is purely Rickart module.
2. If M is a d-Rickart module satisfies the purely D_2, then M is purely Rickart module.
3. If M is a Rickart module satisfies the purely C_2, then M is d-purely Rickart module.

Proof. 1. Let α ∈ S, since M is d-purely Rickart module, then $\mathrm{Im}\alpha \leq^{p} M$. But $\frac{M}{\ker\alpha} \cong \mathrm{Im}\alpha \leq^{p} M$, so by (*)-condition, $\ker\alpha \leq^{p} M$. Hence M is purely Rickart module.

3. Similar to proof (1).
4. Let β ∈ S, since M is a Rickart module, then $\ker\beta \leq^{\oplus} M$. Now, $\mathrm{Im}\beta \cong \frac{M}{\ker\alpha} \cong L \leq^{\oplus} M$, so by purely C_2 condition, $\mathrm{Im}\beta \leq^{p} M$. Therefore, M is d-purely Rickart module. ◘

At the end of this section, recall that the endomorphism ring of Rickart module is right Rickart [5] and the endomorphism ring of d- Rickart module is left Rickart [6]. Here we give a counter example to show that the endomorphism ring of d-purely Rickart module needed not d-purely Rickart.

Remark 2.30. The endomorphism ring of d-purely Rickart needed not be d-purely Rickart.

Example 2.31. The Z-module Z_{P^∞} is d-purely Rickart while the S = $End_Z(Z_{P^\infty})$ is not. In fact, S is the p-adic domain and so is pure simple, if S is d-purely Rickart ring then S is a d-Rickart(Proposition (2.17)). But S is a domain, and hence S is an indecomposable. So S is co-hopfian [6], contradiction, where S= $End_Z(Z_{P^\infty})$ is not co-hopfian. Therefore S is not d-purely Rickart ring**.**

III. Direct summand and Direct sums of d-purely Rickart Modules

A submodule of d-purely Rickart module may be not d-purely Rickart in general. The Z-submodule Z of the Z-module Q is not d-purely Rickart while Q is d-purely Rickart module. The following proposition shows that the direct summand of d-purely Rickart module is inherited this property.

Proposition 3.1. Every direct summand of d-purely Rickart module is d-purely Rickart module.

Proof. Let $M = N \oplus K$ be a d-purely Rickart module. Let $\alpha \in End_R(N)$, then α can be extending into $\beta = \alpha \oplus \{0\}$ and $Im\beta = Im(\alpha \oplus \{0\}) = Im\alpha \oplus \{0\} \leq^p M = N \oplus K$. So $Im\alpha \leq^p N$ [3, Ch.1, Proposition 1.5]. Therefore N is a d-purely Rickart module. ◘

Corollary 3.2. If R is a d-purely Rickart ring, then eR is a d-purely Rickart module for any idempotent element e ∈ R.

The Z-module $M = Z_2 \oplus Z_2$ is d-purely Rickart module, since M is a semisimple module. In general, the Z-module $Z_p \oplus Z_p$ is semisimple for each prime integer p and hence is d-purely Rickart module. But in general, we don't know whether if the direct sums of d-purely Rickart module is a d-purely Rickart. The following proposition gives the condition under which the direct sums of d-purely Rickart module is d-purely Rickart.

Proposition 3.3. Let $M = \oplus_{i \in I} M_i$ for arbitrary index set I, i ∈ I. If $M_i \trianglelefteq M$, then M is d-purely Rickart if and only if M_i is d-purely Rickart for all i∈I.

Proof. ⇒) Following (Proposition (3.1)).
⇐) Let $\alpha \in S = End_R(M)$. So $\alpha = \oplus_{i,j \in I} \alpha_{ij}$ where $\alpha_{ij}: M_j \to M_i$. Since each of M_i is d-purely Rickart submodule of M, then $Im\alpha_i = \alpha_i(M_i) \leq^p M_i$ for each i∈I. Now, by hypothesis, $M_i \trianglelefteq M$ for each i∈I. Then, $Hom_R(M_i, M_j) = 0$ for each i ≠ j, i and j ∈I [11]. So, $\alpha_{ji} = 0$ for i≠j, i, j ∈I. Thus $\alpha(M) = \oplus_{i \in I} \alpha_{ii}(M_i) \leq^p \oplus_{i \in I} M_i = M$ [3, Ch.1, Proposition 1.5]. Therefore M is a d-purely Rickart module. ◘

Proposition 3.4. Let $\{R_\alpha\}_{\alpha \in I}$ be a family of rings where I be an arbitrary index set. Then $R = \oplus_{\alpha \in I} R_\alpha$ is a d-purely Rickart if and only if R_α is a d-purely Rickart ring.

Proof. Let $a = (a_\alpha) \in R = \oplus_{\alpha \in I} R_\alpha$ where $a_\alpha \in R_\alpha$ for each α ∈ I, then aR is a principle right ideal in R. Since R_α is d-purely Rickart, then the principle right ideal $a_\alpha R_\alpha \leq^p R_\alpha$ for each α∈I (Proposition 2.3). Since $aR = \oplus_{\alpha \in I} a_\alpha R_\alpha$, then by [3,Ch.1, Proposition 1.5] $aR = \oplus_{\alpha \in I} a_\alpha R_\alpha \leq^p \oplus_{\alpha \in I} R_\alpha = R$. Therefore R is a d-purely Rickart ring. The converse follows (Proposition (3.1)).

Definition 3.5. Let M and N be modules. Then M is said to be N-d-purely Rickart (relatively d-purely Rickart to N) if for every homomorphism α: M→N, $Im\alpha \leq^p N$.

Remarks and examples 3.6.
1. A module M is d-purely Rickart if and only if M relatively d-purely Rickart to itself.
2. If N is von Neumann regular module, then M is N-d-purely Rickart for any right R- module M.
3. If N is Z-simple module, then M is N-d-purely Rickart for any right R-module M. In particular, Z_{P^∞} and Z_{P^n} are Z_p-d-purely Rickart module for all prime integer p and n ∈ while Z_p is not Z_{P^∞} -d-purely Rickart module.
4. Z_4 is not d-purely Rickart module (Remarks and examples 2.2), but Z_4 (= Z_{2^2}) is Z_3-d-purely Rickart module.
5. For any modules M and N, if $r_H(M) = H$ then M is N-d-purely Rickart, where $H = Hom_R(M, N)$. For that: if α ∈ H, then $\alpha \in r_H(M)$. So α(M) = 0. Therefore $Im\alpha \leq^p M$. Since α is arbitrary, then M is N-d-purely Rickart module. In particular, its well known that $Hom_R(Q, Z) = Hom_R(Q, Z_n) = 0$ for each n ∈ ℕ, then Q as Z-module is Z-d-purely Rickart and Z_n-d-purely Rickart.

Proposition 3.7. If $M = M_1 \oplus M_2$ is a d-purely Rickart and M_2 is pure simple where M_1 and M_2 are submodules of M. Then either $Hom_R(M_1, M_2) = 0$ or for each nonzero homomorphism $\alpha : M_1 \to M_2$ is an epimorphism.
Proof. Let $(0 \neq)$ $\alpha : M_1 \to M_2$ be a homomorphism, then $j\alpha\rho(M) = \alpha(M_1) \leq^p M$ where $\rho: M \to M_1$ be the projection and $j: M_2 \to M$ injection. But $\alpha(M_1) \leq M_2$, so $\alpha(M_1) \leq^p M_2$[3, Ch.1, Proposition (1.2)]. But M_2 is pure simple, and α is not zero, then $\alpha(M_1) = M_2$. Hence α is an epimorphism. ◘

Corollary3.8. If $M = M_1 \oplus M_2$ is a d-purely Rickart and M_2 is pure simple. Then M_1 is M_2-d-purely Rickart.
Proof. Let $M = M_1 \oplus M_2$ be a d-purely Rickart module with M_2 pure simple and $\alpha: M_1 \to M_2$ be any homomorphism, then by (Proposition(3.7)) either $\alpha = 0$ so $Im\alpha = 0 \leq^p M_2$ or α is an epimorphism. Hence $Im\alpha = M_2 \leq^p M_2$.

Proposition 3.9. Let M and B be modules. Then M is B-d-purely Rickart if and only if for any $L \leq^{\oplus} M$ and $A \leq B$, L is A-d-purely Rickart module.
Proof. ⇐) Obvious (put M = L and B = A).
⇒) Let $L \leq^{\oplus} M$, $A \leq B$ and $\alpha: L \to A$ be any homomorphism. Hence, $j\alpha\rho \in Hom_R(M, B)$ where $\rho: M \to L$ be the projection and $j: A \to B$ injection map. So $j\alpha\rho(L) = \alpha(L) \leq^p B$. But $\alpha(L) \leq A \leq B$. Hence $\alpha(L) \leq^p A$. ◘

Corollary 3.10. For any module M the following conditions are equivalent:
1. M is a d-purely Rickart module;
2. For any direct summand A and any submodule B of M, A is B-d-purely Rickart module;
3. For any pair of summands A and B of M, A is B-d-purely Rickart module;
4. M is B-d-purely Rickart module for any direct summand B of M.

Proof. (1) ⇒ (2) Follows Proposition (3.9).
(2)⇒(3) and (3)⇒(4) Obvious.
(4)⇒(1) put M = B. ◘

Corollary 3.11. Let $M = M_1 \oplus M_2$ be a d-purely Rickart module, then for each homomorphism $\alpha_{ij}: M_j \to M_i$, $Im\alpha_{ij} \leq^p M_i$ for each $i,j \in \{1,2\}$.

Note : Corollary 3.11 means, if $M = M_1 \oplus M_2$ be a d-purely Rickart module, then M_1 and M_2 are mutually d-purely purely Rickart modules.

Proposition 3.12. Let R be a ring, then the following conditions are equivalent:
1. R is a d-purely Rickart ring;
2. Every right R-module is d-purely Rickart;
3. Every left R-module is d-purely Rickart;
4. Every flat R-module is d-purely Rickart;
5. Every projective R-module is d-purely Rickart;
6. Every free R-module is d-purely Rickart.

Proof. 1⇒ 2) R is a d-purely Rickart ring if and only if R is regular ring (Corollary (2.4) if and only if every right R-module is regular [3, Ch.3, Theorem (3.1)], then every right R-module is d-purely Rickart (Remarks and examples (2.2(1)).
(2) ⇒(3), (3) ⇒ (4), (4) ⇒ (5) and (5) ⇒ (6) are obvious.
(6) ⇒ (1) Let J be a principle right ideal in R, so there is an epimorphism $\alpha: A \to J$ where A is a free R-module. Clear that $i\alpha \in Hom_R(A, R)$, where $i: J \to R$ is the inclusion. Now, $A \oplus R$ is free, so by hypotheses is a d-purely Rickart module. Then by Corollary (3.8), $J = Im\alpha = Im(i\alpha) \leq^p R$. Hence by (Proposition 2.3) R is a right d-purely Rickart ring. ◘

It's known that if R is a principle ideal domain and a von Neumann regular ring then R is a field. The following proposition gives a characterization of a field by using d-purely Rickart module.

Proposition 3.13. If a ring R is a principle ideal domain, then the following conditions are equivalent:
1. All R-modules are d-purely Rickart;
2. All finitely generated flat R-modules are d-purely Rickart;
3. R is a field.

Proof. (1)⇒(2) It's clear.

(2) ⇒ (3) Let I = aR be a principle right ideal in R and α: R → aR such that α(r) = ar for each r ∈ R. Clear that α is an epimorphism. Now R⊕R is a finitely generated flat R-module. So, by hypothesis, R⊕R is a d-purely Rickart module. Since iα ∈ $End_R(R)$, where i: aR → R is the inclusion map, then aR = Imα = Im(iα) $\leq^p$ R (Corollary (3.8)). Hence R is a von Neumann regular ring [3, Ch.3, Theorem 2.1]. Thus R is a principle ideal domain and von Neumann regular ring, then R is field.
3⇒1) Since every field is a von Neumann regular ring, hence is a d-purely Rickart ring, then by (Proposition (3.12)), the proof is complete. ◘

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