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Evaluation of the Positioning and the Orientation of the Edge and the Corner Cubies of the Rubik's Cube

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Abstract: *In this paper we aimed at defining a concatenation of rotations of the Rubik's cube and one finite group of scrambling of facets. Through observations of Rubik's Cube and group theoretic arguments we describe the positioning and the orientation of the edge and the corner cubies. Since the cube has 54 facets, it has a subgroup of* S_{54} [3]. *We have seen that the permutation of the corner cubies must have the same sign as the permutation of the edge cubies. We have also seen that we cannot change the orientation of a single cubie without changing the orientation of another cubie of the same kind.*

Keywords: *Homomorphism, permutation, orientation, position, edge, corner*

I. Introduction

Rubik's Cube is a three dimensional mechanical puzzle. The goal of the puzzle is to rotate the faces of the cube in a sequence such that at the end of the sequence each face is coloured in a single, distinct colour. The cube consists of 26 smaller cubes which are referred to as cubies. The cubies are grouped in sets of 8 or 9 that can be rotated together. These sets will be referred to as layers. Each face of the cube consists of nine small, coloured squares. We will refer to these squares as facets. Instead of referring to the faces of the cube by their colours we choose to fix the cube in space, with one face facing us, and call that side the front face. The other sides are then called the back face, up face, down face, right face and left face. A 90^o clockwise rotation of the front face is denoted by F. Rotations of the other faces are similarly denoted $\mathrm{B, U, D, R}$ and L as seen from the respective faces. A 90^o counter-clockwise rotation will be denoted by a supercript $'$. For instance, a 90^o counter-clockwise rotation of the front face is denoted F'.

II. Preliminary

Definition 1: We define a concatenation of operators of two or more rotations as $\mathbb{P}\mathbb{Q}$ which denotes the sequence of rotations of first rotating $\mathbb{Q}$ then $\mathbb{P}$.

Definition 2: If $\mathrm{x} = \mathrm{a_1 a_2 a \dots a_n}$ is a sequence of rotations the corresponding reduced sequence $\hat{\mathrm{x}}$ is the sequence obtained from x by removing all partial sequences of two elements where an element is adjacent to its inverse [3].

Definition 3: Let X be a non-empty set. A bijective mapping of $\alpha\colon \mathrm{X} \to \mathrm{X}$ is called a permutation of X [1].

Definition 4: Let G, G' be two groups; then the mapping $\varphi : \mathrm{G} \to \mathrm{G}'$ is a homomorphism if $\varphi(\mathrm{ab}) = \varphi(\mathrm{a})\varphi(\mathrm{b})$ for all , $\mathrm{b} \in \mathrm{G}$ [2].

Definition 5: Let X be a set and let G be a group. We call X a G-set and we say G act on X provided the following conditions hold:

1 Each g belonging to G gives rise to a function $\varphi_{\mathrm{g}}\colon \mathrm{X} \to \mathrm{X}$. That is $\mathrm{X} \times \mathrm{G} \to \mathrm{X}$ (given $\mathrm{x} \in \mathrm{X}$ and $\mathrm{g} \in \mathrm{G}$, we produce another element of X, which we write $\mathrm{x} \cdot \mathrm{g}$)
2 The identity 1 of a group G defines the identity function on X. That is $\mathrm{x} \cdot \mathrm{e} = \mathrm{e} \cdot \mathrm{x}$ for $\mathrm{x} \in \mathrm{X}$ (where e is the identity element of G)
3 If g, h belongs to G then, the composite $\varphi_{\mathrm{gh}}\colon \mathrm{X} \to \mathrm{X}$, satisfies $\varphi_{\mathrm{gh}}(\mathrm{x}) = \varphi_{\mathrm{h}}\left(\varphi_{\mathrm{g}}(\mathrm{x})\right)$. That is $(\mathrm{x} \cdot \mathrm{g}) \cdot \mathrm{h} = \mathrm{x} \cdot (\mathrm{g} \cdot \mathrm{h})$ for all $\mathrm{g, h} \in \mathrm{G}$ and $\mathrm{x} \in \mathrm{X}$.

We call this action a left action since the left-most element (namely, g) in the product gh acts first [5].

Definition 6: Let X be a set and let G be a group. We say G acts on X on the right provided the following conditions hold:

1 Each g belonging to G gives rise to a function $\varphi_{\mathrm{g}}\colon \mathrm{X} \to \mathrm{X}$
2 The identity 1 of a group G defines the identity function on X

3 If g, h belongs to G then, the composite $\varphi_{gh}: X \to X$, satisfies $\varphi_{gh}(x) = \varphi_g(\varphi_h(x))$.
We call this action a right action since the right-most element (namely, h) in the product gh acts first [5].

Definition 7: Let G act on a set X. We call the action transitive if for each pair x, y belonging to X there is a $g \in G$ such that $y = \varphi_g(x)$.
In other words, a group G acts transitively on a set X if any element x of X can be send to any other element y of X by some element of G [5].

Definition 8: If $G_1, G_2, \ldots, G_n$ are n groups, then their (external) direct product $G_1 \times G_2 \times G_3 \times \cdots \times G_n$ is the set of all ordered n-tuples $(a_1, a_2, \ldots, a_n)$ where $a_i \in G_i$, for $i = 1, 2, \ldots, n$, and where the product in $G_1 \times G_2 \times G_3 \times \cdots \times G_n$ is defined component-wise, that is, $(a_1, a_2, \ldots, a_n)(b_1, b_2, \ldots, b_n) = (a_1 b_1, a_2 b_2, \ldots, a_n b_n)$ [2].

Theorem 9: (Second fundamental theorem of cube theory) A 4-tuple (v, r, w, s) such that $r \in S_8, s \in S_{12}, v \in C_3^8, w \in C_2^{12}$) corresponds to a possible position of the Rubik's cube if and only if
(a) $\mathrm{sgn}(r) = \mathrm{sgn}(s)$, ("equal parity as permutations")
(b) $v_1 + \cdots + v_8 \equiv 0(\mathrm{mod}3)$, ("conservation of total twists")
(c) $w_1 + \cdots + w_{12} \equiv 0(\mathrm{mod}2)$, ("conservation of total flips") [5].

Definition 10: Let cubie A and cubie B be two cubies in the cube and let M be any non-identity move on the cube that brings cubie A and cubie B back to their original positions in the cube. Also, let $p_1, p_2, p_3, \ldots, p_n$ be the $n \geq 2$ labels for the positions that cubie A assumes in consecutive order for consecutive rotations in M, where p_1 is the label marking the first position in the cube that cubie A assumes by the first rotation in M and p_2 the label marking the second position in the cube that cubie A assumes by the second rotation in M etc, and lastly, p_n the label marking the original position in the cube that cubie A assumes by the last rotation in M. Note that different labels may mark the same position in the cube. We will here introduce a new terminology; by saying that a cubie assumes a label p we mean that the cubie assumes the position in the cube that the label p is marking. Now, if p_i for some $i \in \{1,2,3, \ldots, n\}$ is a label marking the original position of cubie B and if cubie B assumes all the n labels of cubie A in consecutive order for consecutive rotations in M, starting with assuming p_{i+1} by the first rotation in M and then p_{i+2} by the second rotation in M etc, ending by assuming the label p_i by the last rotation in M, then we call the set $P = \{p_1, p_2, \ldots, p_n\}$ the path through the cube of cubie A and cubie B under the move M and we say that cubie A and cubie B have the same path through the cube under the move M [4].

Definition 11: Two cubies have the same path through the cube under a move M if the two cubies have the same movement through the cube when moved by a move M. So, to determine that two cubies have the same path through the cube under a move M, you first check that the cubies are of the same kind, that is, that both cubies are either corner cubies or edge cubies, since corner cubies always move to corner cubies and edge cubies move to edge cubies. Then, you check that the cubies assume the same positions in the cube when applying the move M. Finally, you check that the cubies assume these same positions in the same order. If all these three conditions apply to your cubies, then they have the same path through the cube under the move M [4].

Lemma 12: Let $c_1, c_2, \ldots, c_n$ be n cubies in the cube, with $n \geq 2$. Let M be any non-identity move on the cube that brings all the n cubies back to their original positions in the cube. If all n cubies have the same path through the cube under the move M and if one of the cubies assumes its original orientation when the move M is made, then all the cubies assume their original orientation [4].

III. Positions of Edge Pieces and Corner Pieces

We Consider the set of edge pieces, E, and the set of corner pieces, C, of the cube. We see that each sequence of rotations will map corner pieces to corner pieces, and edge pieces to edge pieces. Hence, all facet permutations of S_{54} are not allowed, since one cannot map a corner facet to an edge facet. With this, we assign each edge piece an index between 1 and 12, and each corner piece an index between 1 and 8. That is $E = \{e_1, e_2, \ldots, e_{12}\}$ and $C = \{c_1, c_2, \ldots, c_8\}$, as shown in figure 1

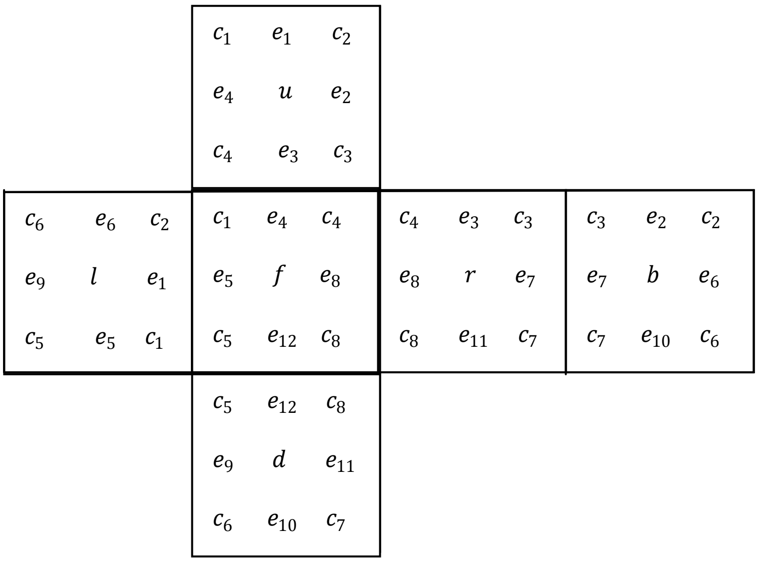

Figure 1: Each edge given a number 1, ... , 12 and each corner given a number 1, ... , 8.

$\mathfrak{G}_R$ is defined to act on E by $g.x = y, g \in \mathfrak{G}_R, x \in E$, where y is the index of the edge position that x is brought to by g. And $\mathfrak{G}_R$ is defined to act on C by $g.x = y, g \in \mathfrak{G}_R, x \in C$
Where y is the index of the corner position that x is brought to by g. Thus, we obtain two homomorphism $\varphi_E: \mathfrak{G}_R \to S_{12}$ and $\varphi_C: \mathfrak{G}_R \to s_8$. With the indexation shown in Figure 1 the rotations generating $\mathfrak{G}_R$ are mapped to the following permutations in S_{12} and s_8, respectively:

$$S_{12}: \varphi_E(U) = (e_1e_2e_3e_4),\ \ S_{12}: \varphi_E(F) = (e_4e_8e_{12}e_5),$$
$$S_{12}: \varphi_E(R) = (e_3e_7e_{11}e_8),\ \ S_{12}: \varphi_E(D) = (e_9e_{12}e_{11}e_{10}),$$
$$S_{12}: \varphi_E(B) = (e_2e_6e_{10}e_7),\ \ S_{12}: \varphi_E(L) = (e_1e_5e_9e_6)$$

$$S_8: \varphi_C(U) = (c_1c_2c_3c_4),\ \ S_8: \varphi_C(F) = (c_1c_4c_8c_5),$$
$$S_8: \varphi_C(R) = (c_3c_7c_8c_4),\ \ S_8: \varphi_C(D) = (c_5c_8c_7c_6),$$
$$S_8: \varphi_C(B) = (c_3c_2c_6c_7),\ \ S_8: \varphi_C(L) = (c_1c_5c_6c_2)$$

IV. Orientation of Edge Pieces

We represent the orientation of the edges of the cube by a 12-tuple of zeros (0) and ones (1) where each coordinate represents an edge position. A coordinate is 0 if the edge piece in that position is correctly oriented and 1 otherwise. Now, we consider how the orientations of the edges change when we perform a rotation of a single face, $\mathbb{P} \coloneqq \pi = (x_1x_2x_3x_4) \in S_{12}$. Before performing $\mathbb{P}$ we have performed some sequence $\mathbb{Q}$ that has permuted the edges to the state described by $\sigma \in S_{12}$ and changed the orientations to the state described by the 12-tuple $(\epsilon_1, \epsilon_2, \ldots, \epsilon_{12}), \epsilon_i \in \{0,1\}$. If we perform $\mathbb{P}$ on a solved cube the change of orientations will be represented by the 12-tuple $(w_1, w_2, \ldots, w_{12})$. We now perform $\mathbb{P}$ on a cube with orientations $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{12})$ and therefore obtain a representation for the orientations of the final state, $(w_1^{'}, w_2^{'}, \ldots, w_{12}^{'})$. Note $w_i^{'} = w_i$ whenever $\epsilon_{x_{i-1}} = 0$ $(\epsilon_{x_{1-1}} = \epsilon_{x_4})$ and that $w_i^{'} = w_i + 1(mod 2)$ if $\epsilon_{x_{i-1}} = 1$.

In general, let $\mathbb{P}$ be a finite sequence of rotations that permutes the edges of a solved cube in a way described by $\pi \in S_{12}$ and changes the orientations in a way described by the 12-tuple w. We perform $\mathbb{P}$ on a solved cube after first performing a sequence $\mathbb{Q}$ that permutes the faces of a solved cube in way described by σ and the orientations in a way described by ϵ. Let the orientations after applying $\mathbb{PQ}$ be described by $w^{'}$. We shall see that $w_i^{'} = w_i$ precisely if $\epsilon_{x_{i-1}} = 0$ and $w_i^{'} = w_i + 1(mod 2)$ precisely if $\epsilon_{x_{i-1}} = 1$. This summarised as $w^{'} = w + \pi.\epsilon$, where π acts by permuting the indices of ϵ according to π^{-1} and the sum is reduced $mod 2$. Thus this shows that the positions and orientations of the edges, and how they change, are described by (some subgroup of) the group

$$S_{12} \ltimes_{\varphi} (Z_2 \times Z_2 \times \cdots Z_2) = S_{12} \ltimes_{\varphi} (Z/2Z)^{12}$$

Where $(Z_2 \times Z_2 \times \cdots Z_2) = 12$ times, $(\pi, w) *_{\varphi} (\sigma, \epsilon) = \big(\pi\sigma, w + \varphi_\pi(\epsilon)\big)$, and $\varphi_\pi(\epsilon)$ is an element of Z_2^{12} obtain from ϵ by permuting its indices according to π^{-1}.

V. Orientation of Corner Pieces

The orientation of the corners is represented by an 8-tuple of zeros (0), ones (1) and twos (2) where each coordinate represents a corner position. A coordinate is 0 if the corner piece in that position is correct, 1 if it is of incorrect orientation of the first type and 2 if it is of incorrect orientation of the second type.
Following similar way in orienting the edge, we let $\mathbb{P}$ be a finite sequence of rotations that permutes the corners of a solved cube in a way described by $\pi \in S_8$ and changes the orientations in a way described by the 8-tuple $w = (w_1, w_2, \ldots, w_8)$, $w_i \in \{0,1,2\}$. We perform $\mathbb{P}$ on a solved cube after first performing a sequence $\mathbb{Q}$ that permutes the corners in way described by σ and change the orientations in a way described by $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_8), \epsilon_i \in \{0,1,2\}$. Let the orientations after applying $\mathbb{PQ}$ be described by w'. We shall see that $w_i' = w_i$ precisely if $\epsilon_{i-1} = 0$ $(\epsilon_{1-1} = \epsilon_8)$, and $w_i' = w_i + 1(mod3)$ precisely if $\epsilon_{i-1} = 1$ and $w_i' = w_i + 2(mod3)$ precisely if $\epsilon_{i-1} = 2$. This summarised as $w' = w + \pi.\epsilon$, where π acts by permuting the indices of ϵ according to π^{-1} and the sum is reduced $mod3$. Thus this shows that the positions and orientations of the edges, and how they change, are described by (some subgroup of) the group

$$S_8 \ltimes_\varphi (Z_3 \times Z_3 \times \cdots Z_3) = S_8 \ltimes_\varphi (Z/3Z)^8$$

Where $(Z_3 \times Z_3 \times \cdots Z_3) = 8$ times, $(\pi, w) *_\varphi (\sigma, \epsilon) = \big(\pi\sigma, w + \varphi_\pi(\epsilon)\big)$, and $\varphi_\pi(\epsilon)$ is an element of Z_3^8 obtain from ϵ by permuting its indices according to π^{-1}.

VI. Results

Theorem 13: Rubik's group, G_R, is described by the set

$$S = (\pi_C, o_C, \pi_E, o_E) \colon \pi_C \in S_8,\ o_C \in (Z/3Z)^8,\ \pi_E \in S_{12}, o_E \in (Z/2Z)^{12}, sgn(\pi_C) = sgn(\pi_E), \sum_{i=1}^{8} (o_C)_i \equiv 0(3), \sum_{i=1}^{12} (o_E)_i \equiv 0(2)$$

And the binary operator $\cdot$ defined by
$(\pi_C, o_C, \pi_E, o_E) \cdot (\pi_C', o_C', \pi_E', o_E') = (\pi_C\pi_C', o_C + \pi_C.o_C', \pi_E\pi_E', o_E + \pi_E.o_E')$ [3].

			1	2	3						
			4	5	6						
			7	8	9						
10	11	12	19	20	21	28	29	30	37	38	39
13	14	15	22	23	24	31	32	33	40	41	42
16	17	18	25	26	27	34	35	36	43	44	45
			46	47	48						
			49	50	51						
			52	53	54						

Figure 2. Numbering each facet of the cube from 1, . . . , 54.

5.1 Given the numbering such as the one in Figure 2, we can define a group action $\varphi \colon \mathfrak{G}_R \times (54) \to (54)$ by $\varphi(g, x) = y, g \in \mathfrak{G}_R, x \in (54)$ where $y \in (54)$ is the index of the facet that x is brought to by the sequence g.

5.2 We define $\varphi_{C,E} \colon \mathfrak{G}_R \to S_8 \times S_{12}$ as $\varphi_{C,E}(\mathbb{P}) = \big(\varphi_C(\mathbb{P}), \varphi_E(\mathbb{P})\big)$. (that is, $\varphi_{C,E}$ is a homomorphism)

5.3 Let s_1 and s_2 be two scrambling of the cube and let them be described as follows by the following 4-tuples:

$$s_1 = (\pi_C, o_C, \pi_E, o_E) \colon \pi_C \in S_8, o_C \in (Z/3Z)^8, \pi_E \in S_{12}, o_E \in (Z/2Z)^{12}$$
$$s_2 = (\pi_C', o_C', \pi_E', o_E') \colon \pi_C' \in S_8, o_C' \in (Z/3Z)^8, \pi_E' \in S_{12}, o_E' \in (Z/2Z)^{12}$$

A binary operation on the set of scrambling, $\cdot$, is defined as follows:

$$s_1 \cdot s_2 = \left(\pi_C \pi_C', o_C + \pi_C . o_C', \pi_E \pi_E', o_E + \pi_E . o_E'\right)$$

Where the first element describes the position of the corner pieces, the second the orientation of the corner pieces, the third element describes the position of the edge pieces and the fourth the orientation of the edge pieces.

5.4 The collection of all moves of the Rubik's cube may be viewed as a subgroup G of S_{48}. The centre of G consists of exactly two elements, the identity and the "super-flip" move which has the effect of flipping over every edge, leaving all the corners alone and leaving all the sub-cubes in their original position. One move for the super-flip of edge and corner cubies is given respectively as follows:

Edge-flip: $R'LB^2RL'D'R'LBRL'ULR'B'L'RDLR'B^2L'RU'$

Corner-flip: $FD^2F'R'D^2RUR'D^2RFD^2F'U'$

5.5 The sum of the orientation of the edge cubies is a multiple of 2.

$$\sum (\epsilon_G + \sigma_G \cdot \epsilon_n)_i = \sum (\epsilon_G + \epsilon_n)_i = \sum (\epsilon_G)_i + \sum (\epsilon_n)_i = 0(mod2)$$

5.6 The sum of the orientation of the corner cubies is a multiple of 3.

$$\sum (\epsilon_G + \sigma_G \cdot \epsilon_n)_i = \sum (\epsilon_G + \epsilon_n)_i = \sum (\epsilon_G)_i + \sum (\epsilon_n)_i = 0(mod3)$$

5.7 The number of ways to orient the edge pieces and the corner pieces are 2^{11} and 3^7, respectively.

5.8 Considering the edges, we find that

$$\varphi_E(UR) = \varphi_E(U)\varphi_E(R) = (e_1e_2e_3e_4)(e_3e_7e_{11}e_8) = (e_3e_7e_{11}e_8e_4e_1e_2)$$

Apply the move $(UR)^7$ on the cube. We observe that all edges have the same path through the cube under the move $(UR)^7$. The move $(UR)^7$ brings all edges back to their original positions in the cube and they all end up right orientated as well, all at once. Here we observe that all edges have the same path through the cube under the move $(UR)^7$. We find that the orientation on one of the edge cubies does not change and conclude that the same holds for the rest of the edges.

5.9 Considering corners,

$\varphi_C(UURR) = \left(\varphi_C(U)\right)^2(\varphi_C(R))^2 = (c_1c_2c_3c_4)^2(c_3c_7c_8c_4)^2 = (c_1c_3)(c_2c_4)(c_3c_8)(c_7c_4) = (c_3c_8c_1)(c_4c_7c_4)$.

So every third move of $UURR$ brings all the corners back to their original position.

VII. Conclusion

As far as we have been able to evaluate, we have found, through observations of Rubik's Cube and group theoretic arguments; that the positioning and the orientation of the edge and the corner cubies cannot change the orientation of a single cubie without changing the orientation of another cubie of the same kind. And that the permutation of the corner cubies must have the same sign as the permutation of the edge cubies.

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