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A generalized class of Szegő polynomials from hypergeometric functions

Vikash
Department of Mathematics, Govt. Girls Degree College Behat,
Saharanpur 247 121 Uttar Pradesh, India

Abstract: *In this paper, we considered a generalized class of Szegő polynomials arising from Gauss hypergeometric function using the approach of three term recurrence relation. Formulas for moments and weight function are given explicitly. [2000] Orthogonal polynomials; Hypergeometric Functions; Recurrence relation*

I. Introduction

Szegő started with a distribution function $\mu(z)=\mu(e^{i\theta})$ with infinitely many point of increase defined the following inner product on the unit circle

$$\langle f,g\rangle=\int_0^{2\pi} f(e^{i\theta})\,\overline{g(e^{i\theta})}\,d\mu(\theta). \tag{1}$$

These polynomials which bear the the name of Szegő have been studied extensively by many others and find useful applications in digital signal processing, frequency analysis, and probability theory. We cite for example [3, 5, 8, 9] as some of their useful contribution and application and the classical book[13] of G. Szegő.

If we orthogonalize the sequence $\{z^n\}_{n=0}^{\infty}$ with respect to a positive measure μ on the unit circle $T=\{z\in C:|z|=1\}=\{z=e^{i\theta}:0\leq\theta\leq2\pi\}$ by using the inner product (1), we obtain a sequence of monic polynomials $\{\phi_n(z)\}_{n=0}^{\infty}$ satisfying the orthogonality

$$\int_T \overline{z}^{\,j}\phi_n(z)d\mu(z)=\int_0^{2\pi} e^{-ij\theta}\phi_n(e^{i\theta})d\mu(e^{i\theta})=k_n^{-2}\delta_{nj},\ 0\leq j\leq n,$$

where $k_n^{-2}=|\phi_n|^2=\int_T |\phi_n(z)|^2 d\mu(z)$. This sequence of polynomials is called Szegő polynomials and the orthonormal Szegő polynomials are given by $\psi_n(z)=k_n\phi_n(z),n\geq0$.

Szegő polynomials satisfy the following recurrence relations for $n\geq0$,

$$\phi_n(z)\quad =z\phi_{n-1}(z)+a_n\phi^*_{n-1}(z) \tag{2}$$

$$\phi^*_n(z)\quad =\phi^*_{n-1}(z)+\overline{a}_n z\phi_{n-1}(z). \tag{3}$$

Eliminating $\phi^*_{n-1}(z)$ from the above recurrence relation, we can get

$$\phi_n(z)\quad =a_n\phi^*_n(z)+(1-|a_n|^2)z\phi_{n-1}(z), \tag{4}$$

where $\phi_0(z)=1$, $a_n=\phi_n(0)$ and $\phi^*_n(z)=z^n\,\overline{\phi}_n(\frac{1}{z})$. The numbers $\{a_n\}$ are called reflection coefficients. These coefficients satisfy the following conditions

$$|a_n|<1 \text{ and } \mu_0\prod_{m=1}^{n}(1-|a_m|^2)=k_n^{-2}=\frac{D_n}{D_{n-1}} \text{ for } n\geq0, \tag{5}$$

where, the Toeplitz determinates D_n are such that

$$D_0 = \mu_0 \text{ and } D_n = \begin{vmatrix} \mu_0 & \mu_{-1} & \cdots & \mu_{-n} \\ \mu_1 & \mu_0 & \cdots & \mu_{-n+1} \\ \vdots & \vdots & \cdots & \vdots \\ \mu_n & \mu_{n-1} & \cdots & \mu_0 \end{vmatrix}$$

Here μ_n are called moments and defined by $\mu_n = \int_0^{2\pi} e^{-in\theta} d\mu(e^{i\theta})$ and satisfy $\mu_{-n} = \overline{\mu}_n$, $n \geq 1$.

Szegő polynomials and associated Szegő polynomials have the following determinantal representation

$$\phi_n(z) = \frac{1}{D_{n-1}} \begin{vmatrix} \mu_0 & \mu_{-1} & \cdots & \mu_{-n} \\ \mu_1 & \mu_0 & \cdots & \mu_{-n+1} \\ \vdots & \vdots & \cdots & \vdots \\ \mu_n & \mu_{n-1} & \cdots & \mu_0 \\ 1 & z & \cdots & z^n \end{vmatrix}, \ n \geq 1, \ \phi_0(z) = 1,$$

and

$$\phi^*_n(z) = \frac{1}{D_{n-1}} \begin{vmatrix} \mu_0 & \mu_{-1} & \cdots & \mu_{-n} \\ \mu_1 & \mu_0 & \cdots & \mu_{-n+1} \\ \vdots & \vdots & \cdots & \vdots \\ \mu_n & \mu_{n-1} & \cdots & \mu_0 \\ z^n & z^{n-1} & \cdots & 1 \end{vmatrix}, \ n \geq 1, \ \phi^*_0(z) = 1.$$

From the above representation, the following orthogonality relation can be easily obtained

$$< \phi_n(z), z^j >_{d\mu} = \begin{cases} 0 & j = 0,1,2,3 \ldots\ldots n-1; \\ \dfrac{D_n}{D_n - 1} & j = n \end{cases}$$

and

$$< \phi^*_n(z), z^j >_{d\mu} = \begin{cases} \dfrac{D_n}{D_n - 1} & j = 0 \\ 0 & j = 1,2,3 \ldots\ldots n. \end{cases}$$

Szegő polynomials are completely characterized by the reflection coefficients $\{a_n\}$, and given by the Favard's theorem [2,11].

Theorem 1.1 [11] Given an arbitrary sequence of complex numbers $\{a_n\}_0^\infty$, with $|a_n| < 1, n \geq 1$, associated with this sequence there exist a unique measure μ on the unit circle such that the polynomials $\{\phi_n\}$ generated by the recurrence relation (2,3) are the respective Szegő polynomials.

The zeros of $\phi_n(z)$ are in the unit disk and if

$$\Psi_n(z) = \int_T \frac{z+w}{z-w} [\phi_n(z) - \phi_n(w)] d\mu(w), \ n \geq 0,$$

then

$$\lim_{n \to \infty} \frac{\Psi^*_n(z)}{\phi^*_n(z)} = F(z) = \int_T \frac{w+z}{w-z} d\mu(w), \tag{6}$$

uniformly on compact subset of D. The function F(z) is called the Carathéodory function of the measure μ. The function F(z) maps the open unit disk $D=\{z\in C:|z|<1\}$ on the closed right half planes $\Re e(z)\geq 0$ and normalized by the condition that $\Re e(f(z))\in(0,\infty)$ and has the following series expansion in terms of the moments

$$F(z)=\mu_0+\Sigma_{k=1}^{\infty}\mu_k z^k. \tag{7}$$

The current paper deals with a generalized class of Szegő polynomials which follows the work of A. Sri Ranga [12] as a particular case, if we take $\beta=b+1$ and $\gamma=b+\overline{b}+1$.

II. Three Term Recurrence Relation For Szegő Polynomials

In this section we will characterize the Szegő polynomials by three term recurrence relations.

Theorem 2.1 A sequence of polynomials $\{\phi_n\}$ generates the sequence of Szegő polynomials polynomials with respect to a distribution function on the unit circle if and only if these polynomials satisfy a three term recurrence relation

$$\phi_{n+1}(z)=(z+\beta_{n+1})\phi_n(z)-\alpha_{n+1}z\phi_{n-1}(z), \qquad n\geq 1, \tag{8}$$

with $\beta_{n+1}\neq 0$, $\alpha_{n+1}\neq 0$ and satisfy

$$0<\frac{\alpha_{n+1}}{\beta_{n+1}}=1-|\phi_n(0)|^2, \qquad \forall\ n\geq 1. \tag{9}$$

Proof. First, let us consider $\{\phi_n(z)\}$ be the sequence of Szegő polynomials then by using the three term recurrence relation (2,3 & 4), we have

$$\phi_{n+1}(z)=Z\phi_n(z)+a_{n+1}\phi^*_n(z)$$

$$\phi(z)=a_n\phi^*_n(z)+(1-|a_n|^2 z\phi_{n-1}(z)$$

eliminating $\phi^*_n(z)$ form above, we can get

$$\phi_{n+1}(z)=Z\phi_n(z)+\frac{a_{n+1}}{a_n}[\phi_n(z)-(1-|a_n|^2)z\phi_{n-1}(z)]$$

$$\phi_{n+1}(z)=(z+\frac{a_{n+1}}{a_n})\phi_n(z)-(1-|a_n|^2)\frac{a_{n+1}}{a_n}z\phi_{n-1}(z)$$

$$\phi_{n+1}(z)=(z+\beta_{n+1})\phi_n(z)-\alpha_{n+1}z\phi_{n-1}(z),$$

where

$$\beta_{n+1}=\frac{a_{n+1}}{a_n},\ \alpha_{n+1}=(1-|a_n|^2)\frac{a_{n+1}}{a_n}.$$

Hence

$$\beta_{n+1}\neq 0,\ \alpha_{n+1}\neq 0<\frac{a_{n+1}}{\beta_{n+1}}=(1-|a_n|^2)$$

A converse result is already proved by A. Sri Ranga in [12], in which he established the existence of the μ measure under the condition of Theorem 2.1.

□

The coefficient of the three term recurrence relations satisfy $\beta_1=-\frac{\mu_{-1}}{\mu_0}$,

$$\alpha_{n+1}=\frac{\int_T z\phi_n(z)d\mu(z)}{\int_T z\phi_{n-1}(z)d\mu(z)} \text{ and } \frac{\alpha_{n+1}}{\beta_{n+1}}=\frac{\int_T z^{-n}\phi_n(z)d\mu(z)}{\int_T z^{-(n-1)}\phi_{n-1}(z)d\mu(z)},\ n\geq 1.$$

Again the value of the normalizing factor is given by $k_0^{-2}=\int_T |\phi_0(z)|^2 d\mu(z)=\mu_0$,

$$k_n^{-2}=\int_T |\phi_n(z)|^2 d\mu(z)=\int_T z^{-n}\phi_n(z)d\mu(z)=\mu_0\,\frac{\alpha_2\alpha_3\cdots\alpha_{n+1}}{\beta_2\beta_3\cdots\beta_{n+1}},\ n\geq 1.$$

and

$$\int_T z\phi_n(z)d\mu(z)=\mu_0\beta_1\alpha_2\alpha_3\cdots\alpha_{n+1}\ \ n\geq 0.$$

III. Szegő Polynomials From Hypergeometric Functions

For $a,b,c\in C$ and $c\neq 0,-1,-2,\cdots$, the hypergeometric function ${}_2F_1(a,b;c;z)$ is defined by the series expansion

$${}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$$

for $|z|<1$ and by analytic continuation for other values of $z\in C$.
A representation of the hypergeometric function for $\Re e(z)<\frac{1}{2}$ is given by the Pfaff's transformation

$${}_2F_1(a,b;c;z)=(1-z)^{-a}{}_2F_1\left(a,c-b;c;\frac{z}{z-1}\right). \tag{10}$$

Another useful transformation is given by Pfaff is

$${}_2F_1(-n,b;c;x)=\frac{(c-b)_n}{(c)_n}{}_2F_1(-n,b;b+1-n-c;1-x). \tag{11}$$

If $|z|<1$ and if $Re(c)>Re(b)>0$, Then ${}_2F_1(a,b;c;z)$ has the integral representation

$$F(a,b;c;z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt. \tag{12}$$

For more details of hypergeometric series, we refer to the book of Andrews, Askey and Roy [1].
Let us consider the following contiguous relations obtained by Gauss in [1]

$${}_2F_1\,(a,b;c;z)=\left(1+\frac{a-b+1}{c}z\right){}_2F_1\,(a+1,b;c+1;z)$$
$$\frac{-(a+1)\,(c-b+1)}{c(c+1)}z\,{}_2F_1\,(a+2,b;c+2;z) \tag{13}$$
$$(c-a){}_2F_1(a-1,b;c;z)=(c-2a-(b-a)z){}_2F_1(a,b;c;z)+a(1-z)\,{}_2F_1(a+1,b;c;z) \tag{14}$$

If we define the monic polynomials

$$R_n(\beta,\gamma;z)=\frac{(\gamma)_n}{(\beta)_n}{}_2F_1[-n,\beta,\gamma;1-z],\ n\geq 0. \tag{15}$$

Replacing $a=-n$, $b=\beta$, $c=\gamma$, and $z\equiv z-1$ in contiguous relation (14), we get

$$(\gamma+n){}_2F_1(-n-1,\beta;\gamma;1-z)=[\,\gamma+2n-(\beta+n)(1-z)]{}_2F_1(-n,\beta;\gamma;1-z)-nz{}_2F_1(-n+1,\beta;\gamma;1-z).$$

Then by using (15), we can get

$$(\gamma+n)\frac{(\beta)_{n+1}}{(\gamma)_{n+1}}R_{n+1}=[n+\gamma-\beta+(\beta+n)z]\frac{(\beta)_n}{(\gamma)_n}R_n-nz\frac{(\beta)_{n-1}}{(\gamma)_{n-1}}R_{n-1}.$$

Thus, we can find that $R_n(\beta,\gamma;z)$ satisfies the following recurrence relation

$$R_{n+1}(\beta,\gamma;z)=(z+\beta_{(n+1,R)})R_n(\beta,\gamma;z)-\alpha_{(n+1,R)}\ zR_{n-1}(\beta,\gamma;z), \tag{16}$$

with $\beta_{(n+1,R)}$ and $\alpha_{(n+1,R)}$ denote the coefficient of three term recurrence relation

$$R_0(\beta,\gamma;z)=1, \quad R_1(\beta,\gamma;z)=z+\beta_{(1,R)},$$

$$\beta_{(n,R)}=\frac{n+\gamma-\beta-1}{\beta+n-1} \quad \text{and} \quad \alpha_{(n+1,R)}=\frac{n(n+\gamma-1)}{(\beta+n)(\beta+n-1)}, \quad n\geq 1.$$

Under the condition $\beta\neq 0,-1,-2,\cdots$ and $\gamma-\beta\neq 0,-1,-2,\cdots$ such that $\frac{\alpha_{(n+1,R)}}{\beta_{(n+1,R)}}$ is real and positive. Then by Theorem 2.1 $\{R_n(\beta,\gamma;z)\}_{n=0}^{\infty}$ form a class of Szegő polynomials if and only if this sequence of polynomials satisfy the following relation

$$1-|R_n(\beta,\gamma;0)|^2 = \frac{\alpha_{(n+1,R)}}{\beta_{(n+1,R)}},$$

$$1-\left|\frac{(\gamma-\beta)_n}{(\beta)_n}\right|^2 = \frac{n(n+\gamma-1)}{(\beta+n-1)(\gamma-\beta+n)},$$

$$(n+\beta-1)(n+\gamma-\beta)|(\gamma-\beta)_n|^2=(\beta-1)(\gamma-\beta)|(\beta)_n|^2. \tag{17}$$

Theorem 3.1 Let $\{R_n(\beta,\gamma;z)\}_{n=0}^{\infty}$ are monic Szegő polynomials defined by

$$\int_T \overline{z}^{\,j} R_n(\beta,\gamma;z)d\mu(\beta,\gamma,z)=(K_{(n,R)})^{-2}\delta_{nj}, \quad 0\leq j\leq n. \tag{18}$$

with respect to the positive measure $\mu(\beta,\gamma,z)$ on the unit circle. The coefficient $K_{(n,R)}=||R_n(\beta,\gamma;z)||^{-1}$ and $a_{(n,R)}=R_n(\beta,\gamma;0)$ associated with these polynomials satisfy

$$K_{(n,R)}=\sqrt{\frac{(\beta)_n(\gamma-\beta+1)_n}{n!(\gamma)_n}}, \quad a_{(n,R)}=\frac{(\gamma-\beta)_n}{(\beta)_n}, \quad n\geq 0. \tag{19}$$

Moreover,

$$L_0(\beta,\gamma,z)-\frac{Q_n(\beta,\gamma,z)}{R_n(\beta,\gamma,z)}=\frac{(\gamma)_n n!}{(\gamma-\beta+1)_n(\beta)_n}z^n+O(z^{n+1}), \tag{20}$$

$$L_\infty(\beta,\gamma,z)-\frac{Q_n(\beta,\gamma,z)}{R_n(\beta,\gamma,z)}=\frac{(\gamma-\beta)(\gamma)_n n!}{(\beta)_n(\beta)_{n+1}}\frac{1}{z^{n+1}}+O(\frac{1}{z^{n+2}}),\ n\geq 0 \tag{20}$$

where $L_0(\beta,\gamma;z)=\sum_{j=0}^{\infty}\mu_{(j,R)}\ z^j$ and $L_\infty(\beta,\gamma;z)=-\sum_{j=1}^{\infty}\mu_{(-j,R)}\ z^{-j}$, with moments

$$\mu_{(0,R)} = \int_T d\mu(\beta,\gamma,z)=1$$

$$\overline{\mu_{(-j,R)}} = \mu_{(j,R)}=\int_T z^{-j}d\mu(\beta,\gamma,z)=\frac{(-\beta+1)_j}{(\gamma-\beta+1)_j}, \quad j\geq 0.$$

Here $Q_n(\beta,\gamma,z)$ is the associated Szegő polynomials of $R_n(\beta,\gamma;z)$.

Proof: We only need to show that $\mu_{(j,R)} = \dfrac{(-\beta+1)_j}{(\gamma-\beta+1)_j}$, $j \geq 0$, as the remaining results follow directly from Section 2.

The contiguous relation (13) can be rewritten as

$$\frac{{}_2F_1(a+1,b,c+1;z)}{{}_2F_1(a,b,c;z)} = \cfrac{1}{1+\dfrac{a-b+1}{c}z - \dfrac{(a+1)(c-b+1)}{c(c+1)} z \dfrac{{}_2F_1(a+2,b,c+2;z)}{{}_2F_1(a+1,b,c+1;z)}}.$$

Replacing b = -β+1 and c = γ-β, we get

$$\frac{{}_2F_1(a+1,b,c+1;z)}{{}_2F_1(a,b,c;z)} = \cfrac{1}{1+\dfrac{a-b+1}{c}z - \dfrac{(a+1)(c-b+1)}{c(c+1)} z \dfrac{{}_2F_1(a+2,b,c+2;z)}{{}_2F_1(a+1,b,c+1;z)}}.$$

If we write $K_n(a,\alpha,\beta,z) = \dfrac{{}_2F_1(a+n+1,-\beta+1,\gamma-\beta+n+1;z)}{{}_2F_1(a+n,-\beta+1,\gamma-\beta+n;z)}$, n=0,1,2… then

$$K_0(a,\alpha,\beta,z)= \frac{1|}{|1+g_1z} - \frac{f_2z|}{|1+g_2z} - \ldots\ldots - \frac{f_{n-1}z|}{|1+g_{n-1}z} - \frac{f_nz|}{|1+g_nz - f_{n+1}K_n(a,\alpha,\beta,z)},$$

where

$$g_n = \frac{a+\beta+n-1}{\gamma-\beta+n-1},\quad f_{n+1} = \frac{(a+n)(\gamma+n-1)}{(\gamma-\beta+n-1)(\gamma-\beta+n)},\ n \geq 1.$$

If we restrict ourself to the case in which a = 0, then

$$K_0(a,\alpha,\beta,z) = {}_2F_1(1,-\beta+1,\gamma-\beta+1;z)$$
$$= \frac{1|}{|1+g_1z} - \frac{f_2z|}{|1+g_2z} - \ldots\ldots - \frac{f_{n-1}z|}{|1+g_{n-1}z} - \frac{f_nz|}{|1+g_nz - f_{n+1}K_n(0,\alpha,\beta,z)},$$

where

$$g_n = \frac{\beta+n-1}{\gamma-\beta+n-1},\quad f_{n+1} = \frac{n(\gamma+n-1)}{(\gamma-\beta+n-1)(\gamma-\beta+n)},\ n \geq 1.$$

Equivalently, we can also write

$$K_0(0,\alpha,\beta,z)= \frac{\beta_1|}{|z+\beta_1} - \frac{\alpha_2z|}{|z+\beta_2} - \ldots\ldots - \frac{\alpha_nz|}{|z+\beta_n} - \frac{\alpha_{n+1}zK_n(0,\alpha,\beta;z)|}{|\beta_{n+1}},$$

where

$$\beta_n = \frac{1}{g_n} = \beta_{(n,R)},\quad \alpha_{n+1} = \frac{f_{n+1}}{g_ng_{n+1}} = \alpha_{(n+1,R)},\quad n \geq 1.$$

Using the theory of continued fraction, we obseved that

$$K_0(0,\alpha,\beta;z) - \frac{Q_n(\beta,\gamma,z)}{R_n(\beta,\gamma,z)} = \frac{\beta_{(n+1,R)}Q_n(\beta,\gamma,z) - \alpha_{(n+1,R)}zK_n(0,\alpha,\beta,;z)Q_{n-1}(\beta,\gamma,z)}{\beta_{(n+1,R)}R_n(\beta,\gamma,z) - \alpha_{(n+1,R)}zK_n(0,\alpha,\beta,;z)R_{n-1}(\beta,\gamma,z)} - \frac{Q_n(\beta,\gamma,z)}{R_n(\beta,\gamma,z)},$$

$$K_0(0,\alpha,\beta;z) - \frac{Q_n(\beta,\gamma,z)}{R_n(\beta,\gamma,z)} = \frac{\beta_{(1,R)}\alpha_{(2,R)}\ldots\alpha_{(n,R)}\alpha_{(n+1,R)}z^nK_n(0,\alpha,\beta;z)}{\beta_{(n+1,R)}R_n(\beta,\gamma,z) - \alpha_{(n+1,R)}zK_n(0,\alpha,\beta,;z)R_{n-1}(\beta,\gamma,z)},$$

$$= \frac{(\gamma)_n n!}{(\gamma-\beta)_n(\gamma-\beta+1)_n} z^n + O(z^{n+1}).$$

Hence $L_0(\beta,\gamma;z) = {}_2F_1(1,-\beta+1,\gamma-\beta+1;z)$ and the theorem follows. □

Again using $\mu_{(-j,R)} = \overline{\mu_{(j,R)}}$, and $L_\infty(\beta,\gamma,z) = -\sum_{j=1}^{\infty} \mu_{(-j,R)} z^{-j}$, we get

$$L_\infty(\beta,\gamma,z)=\frac{(\overline{\beta}-1)z^{-1}}{(\overline{\gamma}-\overline{\beta}-1)}{}_2F_1(1,-\overline{\beta}+2,\overline{\gamma}-\overline{\beta}+2;z^{-1}).$$

The following asymptotic result also hold.

Theorem 3.2

$$\lim_{n\to\infty}K_{(n,R)}=\sqrt{\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta+1)}}\quad\text{and}\quad\lim_{n\to\infty}n^{2\beta-\gamma}a_{(n,R)}=\frac{\Gamma(\beta)}{\Gamma(\gamma-\beta)}\tag{22}$$

IV. More On The Measure And Related Functions

The following theorem gives exact expression for the measure $\mu_R(\beta,\gamma;z)$ in terms of the weight function $w_R(\beta,\gamma;\theta)$ for the function $R(\beta,\gamma;z)$

Theorem 4.1 The measure $\mu_R(\beta,\gamma;z)$ can be given by $d\mu_R(\beta,\gamma;e^{i\theta})=w_R(\beta,\gamma;\theta)d\theta$, where

$$w_R(\beta,\gamma;\theta)=\tau_R\ \mathbf{e}^{\left[-i(\pi-\theta)\frac{2\beta-\gamma-1}{2}\right]}\left[\sin\frac{\theta}{2}\right]^{\gamma-1},\quad 0\le\theta\le2\pi.$$

The constant

$$\tau_R=\frac{2^{\gamma-1}\Gamma(\beta)\Gamma(\gamma-\beta+1)}{2\pi\Gamma(\gamma)}\quad\text{is such that}\quad \mu_{(0,R)}=1.$$

Proof. Since

$$\mu_{(j,R)}=\int_0^{2\pi}e^{-ij\theta}w_R(\beta,\gamma:\theta)d\theta=\overline{\int_0^{2\pi}e^{ij\theta}w_R(\beta,\gamma:\theta)d\theta}=\overline{\mu_{(-j,R)}},\backslash$$

we have only to prove that

$$\mu_{(j,R)}=\tau_R\int_0^{2\pi}e^{-ij\theta}e^{[-i(\pi-\theta)\frac{2\beta-\gamma-1}{2}]}[\mathrm{Sin}\frac{\theta}{2}]^{\gamma-1}d\theta=\frac{(-\beta+1)_j}{(\gamma-\beta+1)_j}.$$

With $2i\mathrm{Sin}(\frac{\theta}{2})=e^{i\frac{\theta}{2}}-e^{-i\frac{\theta}{2}}$,

$$\mu_{(j,R)}=\tau_R e^{-i(\frac{2\beta-\gamma-1}{2})\pi}\int_0^{2\pi}e^{-ij\theta}e^{i(\frac{2\beta-\gamma-1}{2})\theta}[\frac{e^{i\frac{\theta}{2}}-e^{-i\frac{\theta}{2}}}{2i}]^{\gamma-1}d\theta,\quad j\ge0.$$

$$\mu_{(j,R)}=\tilde{\tau}_R\int_0^{2\pi}e^{-i(j+\gamma-\beta)\theta}[e^{i\theta}-1]^{\gamma-1}d\theta.$$

Where,

$$\tilde{\tau}_R=\tau_R e^{-i(\frac{2\beta-\gamma-1}{2})\pi}(2i)^{-(\gamma-1)}.$$

We can write this in the following form

$$\mu_{(j,R)}=\frac{\tilde{\tau}_R}{i}\int_0^{2\pi}e^{-i(j+\gamma-\beta+1)\theta}[e^{i\theta}-1]^{\gamma-1}ie^{i\theta}d\theta,\quad j\ge0.$$

then be integration by parts we establish the relation

$$\mu_{(j,R)}=\frac{j+\gamma-\beta+1}{\gamma}\mu_{(j,R)}-\frac{j+\gamma-\beta+1}{\gamma}\mu_{(j+1,R)}$$

$$\mu_{(j+1,R)}=\left(\frac{-\beta+j+1}{\gamma-\beta+j+1}\right)\mu_{(j,R)},\quad j\ge0.$$

Therefore, the proof will be complete if we can prove $\mu_{(0,R)}=1$, equivalently, if we can show that

$\mu_{(j,R)}=\frac{(-\beta+1)_j}{(\gamma-\beta+1)}$ or $\mu_{(-j,R)}=\frac{(-\bar{\beta}+1)_j}{(\bar{\gamma}-\bar{\beta}+1)}$ for some particular value of j.

Using moment expression, we have

$$\mu_{(-j,R)}=\overline{\mu_{(j,R)}}=\overline{\tilde{\tau}_R\int_0^{2\pi}e^{-i(j+\gamma-\beta)\theta}[e^{i\theta}-1]^{\gamma-1}d\theta},\quad j\geq 0.$$

$$=\overline{\tilde{\tau}_R}\int_0^{2\pi}e^{i(j+\bar{\gamma}-\bar{\beta})\theta}\left[e^{i\theta}-1\right]^{\bar{\gamma}-1}d\theta,$$

$$=\overline{\tilde{\tau}_R}\int_0^{2\pi}e^{i(j+1-\bar{\beta})\theta}\left[1-e^{i\theta}\right]^{\bar{\gamma}-1}d\theta,$$

Where

$$\overline{\tilde{\tau}_R}=\overline{\tau_R\frac{e^{-i\left(\frac{2\beta-\gamma-1}{2}\right)\pi}}{2^{\gamma-1}(-1)^{\frac{\gamma-1}{2}}}}=\overline{\tau_R}\frac{e^{i\left(\frac{2\bar{\beta}-\bar{\gamma}-1}{2}\right)\pi}}{2^{\bar{\gamma}-1}(-1)^{\frac{\bar{\gamma}-1}{2}}}=\frac{(-1)^{\bar{\beta}-\bar{\gamma}}\overline{\tau_R}}{2^{\bar{\gamma}-1}}.$$

Thus,

$$\mu_{(-j,R)}=\frac{\overline{\tau_R}(-1)^{\bar{\beta}-\bar{\gamma}}}{2^{\bar{\gamma}-1}}\int_0^{2\pi}e^{i(j+1-\bar{\beta})\theta}\left[1-e^{i\theta}\right]^{\bar{\gamma}-1}d\theta,\quad j\geq 0.$$

With $z=e^{i\theta}$, one can write

$$\mu_{(-j,R)}=\frac{\overline{\tau_R}(-1)^{\bar{\beta}-\bar{\gamma}}}{2^{\bar{\gamma}-1}}\int_{c:|z|=1}z^{j+1-\bar{\beta}}\left[1-z\right]^{\bar{\gamma}-1}\frac{dz}{iz}$$

$$=\frac{i\overline{\tau_R}}{2^{\bar{\gamma}-1}}\int_{c:|z|=1}z^{j+1-\bar{\gamma}}(-z)^{\bar{\gamma}-\bar{\beta}-1}[1-z]^{\bar{\gamma}-1}dz$$

where the branch cut in $(-z)^{\bar{\gamma}-\bar{\beta}}$ and $(1-z)^{\bar{\gamma}-1}$ are along the postive real axis such that

$$(-z)^{\bar{\gamma}-\bar{\beta}}=|z|^{\bar{\gamma}-\bar{\beta}}\text{ if }\ 0<\arg(z)<2\pi$$

$$(1-z)^{\bar{\gamma}-1}=|1-z|^{\bar{\gamma}-1}\text{ if }\ -\pi<\arg(z)<\pi$$

Hence we choose a j such that Re $(j-\bar{\beta}+1)>0$ and evaluate the integral by contour integration using[4,8],

$$\mu_{(-j,R)}=\frac{i\overline{\tau_R}}{2^{\bar{\gamma}-1}}2i\mathrm{Sin}(\bar{\beta}-1)\pi\int_0^1 t^{j-\bar{\beta}}(1-t)^{\bar{\gamma}-1}dt.$$

Again, if we take $\mu_{(-j,R)}=\frac{(-\bar{\beta}+1)_j}{(\bar{\gamma}-\bar{\beta}+1)_j}$ and using the definition of gamma function and beta function and the Eular's reflection formula, we obtain,

$$\tau_R=\frac{2^{\gamma-1}\Gamma(\beta)\Gamma(\gamma-\beta+1)}{2\pi\Gamma(\gamma)}\text{ is such that }\ \mu_{(0,R)}=1.$$

This complete the proof of the theorem.

□

Theorem 4.2 The polynomials $R_n(\beta,\gamma;z)$ and their reciprocals $R_n^*(\beta,\gamma;z)$ can be given by

$$R_n(\beta,\gamma;z) = \frac{\Gamma(\gamma+n)}{\Gamma(\beta+n)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}[1-t(1-z)]^n dt$$

$$R_n^*(\beta,\gamma;z) = \frac{\Gamma(\overline{\gamma}+n)}{\Gamma(\overline{\beta}+n)\Gamma(\overline{\gamma}-\overline{\beta})}\int_0^1 t^{\overline{\gamma}-\overline{\beta}-1}(1-t)^{\overline{\beta}-1}[1-t(1-z)]^n dt.$$

We now give an expression for the associated Szegő function

$$D(\beta,\gamma;z)=\exp\left[\frac{1}{4\pi}\int_0^{2\pi}\frac{e^{i\theta}+z}{e^{i\theta}-z}\log(w(\beta,\gamma;\theta))d\theta\right]$$

in the following result.

Theorem 4.3 The associated Szegő function for the polynomial $R_n(\beta,\gamma;z)$ is given by

$$D(\beta,\gamma;z)=\sqrt{\frac{\Gamma(\beta)\Gamma(\gamma-\beta+1)}{\Gamma(\gamma)}}(1-z)^{\overline{\gamma}-\overline{\beta}}.$$

Proof. Using the result in [3], we have the following result

$$K_{(n,R)}R_n^*(\beta,\gamma;z)\to[D(\beta,\gamma;z)]^{-1} \text{ as } n\to\infty$$

uniformly on the compact subset of D. With the substitution u=nt in Theorem 4.2, we have,

$$K_{(n,R)}R_n^*(\beta,\gamma;z)=K_{(n,R)}\frac{\Gamma(\overline{\gamma}+n)}{\Gamma(\overline{\beta}+n)\Gamma(\overline{\gamma}-\overline{\beta})}\int_0^n\frac{\mu^{\overline{\gamma}-\overline{\beta}-1}}{n^{\overline{\gamma}-\overline{\beta}-1}}(1-\frac{\mu}{n})^{\overline{\beta}-1}(1-\frac{\mu}{n}(1-z))^n\frac{d\mu}{n}$$

$$=K_{(n,R)}\frac{(\overline{\gamma})_n\,\Gamma(\overline{\gamma})}{(\overline{\beta})_n\Gamma(\overline{\beta})\Gamma(\overline{\gamma}-\overline{\beta})n^{\overline{\gamma}-\overline{\beta}}}\int_0^n\mu^{\overline{\gamma}-\overline{\beta}-1}(1-\frac{\mu}{n})^{\overline{\beta}-1}(1-\frac{\mu}{n}(1-z))^n d\mu$$

$$\lim_{n\to\infty}K_{(n,R)}R_n^*(\beta,\gamma;z)=\lim_{n\to\infty}K_{(n,R)}\frac{\Gamma(\overline{\beta})\ \ \Gamma(\overline{\gamma})}{\Gamma(\overline{\gamma})\ \Gamma(\overline{\beta})(\overline{\gamma}-\overline{\beta})}\int_0^\infty\mu^{\overline{\gamma}-\overline{\beta}-1}e^{-(1-z)\mu}d\mu.$$

$$[D(\beta,\gamma;z)]^{-1}=\lim_{n\to\infty}K_{(n,R)}\frac{\Gamma(\overline{\gamma}-\overline{\beta})}{\Gamma(\overline{\gamma}-\overline{\beta})(1-z)^{\overline{\gamma}-\overline{\beta}}}$$

$$D(\beta,\gamma;z)=\sqrt{\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta+1)}}(1-z)^{\overline{\gamma}-\overline{\beta}}.$$

Hence the Theorem. □

Using (6) and then Theorem (3.1), following result will obtain for associated Carathéodory function for z in the compact subject of D,

$$\lim_{n\to\infty}\frac{2zQ_n^*(\beta,\gamma;z)}{\phi_n^*(\beta,\gamma;z)} = 1-F(z)$$

$$1-\lim_{n\to\infty}\frac{2zQ_n^*(\beta,\gamma;z)}{\phi_n^*(\beta,\gamma;z)} = F(Z)$$

$$=\mu_{(0,R)}+2\Sigma_{k=1}^{\infty}\mu_{(k,R)}z^k$$

$$=-\mu_{(0,R)}+2\Sigma_{k=0}^{\infty}\mu_{(k,R)}z^k$$

$$=-1+L_0(\beta,\gamma;z)$$

$$= -1+z\,{}_2F_1(1,-\beta+1;\gamma-\beta+1;z)$$

Also using (15), $R_n(\beta,\gamma;z)$ have the following hypergeometric representation

$$R_n(\beta,\gamma,z) = \frac{(\gamma-\beta)_n}{(\beta)_n}\,{}_2F_1[-n,\beta;\beta+1-n-\gamma;1-z]. \qquad (23)$$

Finally, we find the generating function for $R_n(\beta,\gamma;z)$.

$$\begin{aligned}
G(\beta,\gamma;z;t) \quad &= (1-t)^{-(\gamma-\beta)}(1-tz)^{-\beta} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\gamma-\beta)_n}{n!}t^n\frac{(\beta)_k}{k!}(tz)^k \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(\gamma-\beta)_{n-k}(\beta)_k}{(n-k)!k!}t^n z^n \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^k(\gamma-\beta)_n(\beta)_k}{(n-k)!k!(1-\gamma+\beta-n)_k}t^n z^n \\
&= \sum_{n=0}^{\infty}\frac{(\gamma-\beta)_n}{n!}\sum_{k=0}^{n}\frac{(-1)^k k!}{(n-k)!}\frac{(\beta)_k}{(1-\gamma+\beta-n)_k}\frac{(zt)^k}{k!} \\
&= \sum_{n=0}^{\infty}\frac{(\gamma-\beta)_n}{n!}\,{}_2F_1(-n,\beta;1+\beta-\gamma-n;z)t^n \\
&= \sum_{n=0}^{\infty}\frac{(\beta)_n}{n!}R_n(\beta,\gamma;z)t^n.
\end{aligned}$$

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