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A Futher Identity and a Recurrence Relation on the Coefficient of a Holomorphic Function Several Complex Variables

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Abstract: *In this paper, we reviewed the work of Adepoju et al [1] and also corrected some mistakes in the paper. We then used this known result to obtain our own results using the well known Legendre duplicating formula.*
Keywords: *Holomorphic function, Osgood's theorem, Verdemonde's identity, Legendre duplicating formula and geometric progression formula.*

I. Review

In this section, we review the work of Adepoju et al [1] and we expatiate more on the proof of the result. Adepoju et al [1] proved the result as follows;

Let $$S_n = \sum_{i+j+k=n} \binom{i+j}{i}\binom{j+k}{j}\binom{k+i}{k} \qquad (1.0)$$

where the summation is taken over all non-negative integers *i, j, k* such that $i+j+k=n$ and S_n is the coefficient of the holomorphic function

$$f(z,w) = (1+z)^i (1+w)^i (2+z+w)^{n-i}, \qquad (1.1)$$

Then

$$S_n - S_{n-1} = \binom{2n}{n} \qquad (1.2)$$

For a fixed *i*, let $i+j+k=n$

$$\therefore S_n = \sum_{i=0}^{n}\sum_{j=0}^{n-i} \binom{i+j}{i}\binom{n-i}{j}\binom{n-j}{i}$$

Now let $$\sigma_{n,i} = \sum_{j=0}^{n-i} \binom{i+j}{i}\binom{n-i}{j}\binom{n-j}{i} \qquad (1.3)$$

$$\therefore S_n = \sum_{i=0}^{n} \sigma_{n,i} \qquad (1.4)$$

We consider a holomorphic function of the form

$$f(z,w) = (1+z)^i (1+w)^i (2+z+w)^{n-i} = (1+z)^i (1+w)^i (1+z+1+w)^{n-i}$$

$$= \cancel{(1+z)^i}\,(1+w)^i (1+z)^{n-\cancel{i}} \left(1+\frac{1+w}{1+z}\right)^{n-i} = (1+w)^i (1+z)^n \left(1+\frac{1+w}{1+z}\right)^{n-i}$$

$$= (1+w)^i (1+z)^n \sum_{j=0}^{n-i} \binom{n-i}{j} \left(\frac{1+w}{1+z}\right)^j = \sum_{j=0}^{n-i} \binom{n-i}{j} (1+w)^{i+j} (1+z)^{n-j}$$

$$f(z,w)=\sum_{j=0}^{n-i}\binom{n-i}{j}\binom{i+j}{i}i^{i+j-i}w^{i}\sum_{m=0}^{n-j}\binom{n-j}{m}i^{n-j-m}z^{m}$$

Since m is a dummy variable, let $m=i$

$$\therefore f(z,w)=\sum_{i=0}^{n-j}\sum_{j=0}^{n-i}\binom{i+j}{i}\binom{n-i}{j}\binom{n-j}{i}w^{i}z^{i}$$

Hence the coefficient of $w^{i}z^{i}$ in $f(w,z)$ is

$$\sum_{j=0}^{n-i}\binom{i+j}{i}\binom{n-i}{j}\binom{n-j}{i}=\sigma_{n,i} \qquad (1.5)$$

From Cauchy formula for double complex variables, (1.5) becomes

$$\sigma_{n,i}=\left(\frac{1}{2\pi i}\right)^{2}\oint_{C_r}\oint_{C_r}\frac{f(z,w)}{w^{i+1}z^{i+1}}dw\,dz=\frac{-1}{4\pi^{2}}\oint_{C_r}\oint_{C_r}\frac{(1+z)^{i}(1+w)^{i}(2+z+w)^{n-i}}{w^{i+1}z^{i+1}}dw\,dz \qquad (1.6)$$

Now, $C_r:|z|=r, \quad 0<r<1$ and $\Gamma=C_r\times C_r$, (10) becomes

$$\sigma_{n,i}=\frac{-1}{4\pi^{2}}\oint_{\Gamma}\frac{(1+z)^{i}(1+w)^{i}(2+z+w)^{n-i}}{w^{i+1}z^{i+1}}dw\,dz$$

From (1.4) we have that $S_n=\sum_{i=0}^{n}\sigma_{n,i}$

$$\therefore S_n=\sum_{i=0}^{n}\frac{-1}{4\pi^{2}}\oint_{\Gamma}\frac{(1+z)^{i}(1+w)^{i}(2+z+w)^{n-i}}{(wz)^{i+1}}dw\,dz$$

$$=-\frac{1}{4\pi^{2}}\oint_{\Gamma}\sum_{i=0}^{n}\frac{(1+z)^{i}(1+w)^{i}}{(wz)^{i}(zw)}\times\frac{(2+z+w)^{n}}{(2+z+w)^{i}}dw\,dz$$

$$\Rightarrow S_n=-\frac{1}{4\pi^{2}}\oint_{\Gamma}\sum_{i=0}^{n}\frac{(2+z+w)^{n}}{wz}\left[\frac{(1+z)(1+w)}{(wz)(2+z+w)}\right]^{i}dw\,dz$$

Using the sum of nth of geometric progression, we have that

$$S_n=-\frac{1}{4\pi^{2}}\oint_{\Gamma}\frac{(2+z+w)^{n}}{wz}\left(\frac{1-\left[\frac{(1+z)(1+w)}{zw(2+z+w)}\right]^{n+1}}{1-\frac{(1+z)(1+w)}{zw(2+z+w)}}\right)dw\,dz$$

$$S_n=-\frac{1}{4\pi^{2}}\oint_{\Gamma}\frac{(2+z+w)^{n}}{wz}\frac{(zw)(2+z+w)}{(zw)^{n+1}(2+z+w)^{n+1}}\left(\frac{(zw)^{n+1}(2+z+w)^{n+1}-(1+z)^{n+1}(1+w)^{n+1}}{zw(2+z+w)-(1+z)(1+w)}\right)dw\,dz$$

$$S_n = -\frac{1}{4\pi^2}\oint\!\!\oint_\Gamma \frac{1}{(zw)^{n+1}}\left(\frac{(zw)^{n+1}(2+z+w)^{n+1}-(1+z)^{n+1}(1+w)^{n+1}}{zw(2+z+w)-(1+z)(1+w)}\right)dw\,dz$$

But $zw(2+z+w)-(1+z)(1+w)=(zw-1)(1+z+w)$

$$\therefore S_n = -\frac{1}{4\pi^2}\oint\!\!\oint_\Gamma \frac{(zw)^{n+1}(2+z+w)^{n+1}-(1+z)^{n+1}(1+w)^{n+1}}{(zw)^{n+1}(zw-1)(1+z+w)}dw\,dz$$

$$S_n = -\frac{1}{4\pi^2}\oint\!\!\oint_\Gamma \left[\frac{(zw)^{n+1}(2+z+w)^{n+1}}{(zw)^{n+1}(zw-1)(1+z+w)}-\frac{(1+z)^{n+1}(1+w)^{n+1}}{(zw)^{n+1}(zw-1)(1+z+w)}\right]dw\,dz$$

$$= -\frac{1}{4\pi^2}\oint\!\!\oint_\Gamma \frac{1}{(zw-1)(1+z+w)}\left[(2+z+w)^{n+1}-\frac{(1+z)^{n+1}(1+w)^{n+1}}{(zw)^{n+1}}\right]dw\,dz \qquad (1.7)$$

$$S_{n-1} = -\frac{1}{4\pi^2}\oint\!\!\oint_\Gamma \frac{1}{(zw-1)(1+z+w)}\left((2+z+w)^{n}-\frac{(1+z)^{n}(1+w)^{n}}{(zw)^{n}}\right)dw\,dz \qquad (1.8)$$

Hence (1.7) – (1.8) becomes

$$S_n - S_{n-1} = -\frac{1}{4\pi^2}\oint\!\!\oint_\Gamma \frac{1}{(zw-1)(1+z+w)}\left(\begin{array}{r}(2+z+w)^{n+1}-(2+z+w)^{n}-\dfrac{(1+z)^{n+1}(1+w)^{n+1}}{(zw)^{n+1}}\\ +\dfrac{(1+z)^{n}(1+w)^{n}}{(zw)^{n}}\end{array}\right)dw\,dz$$

$$= -\frac{1}{4\pi^2}\oint\!\!\oint_\Gamma \frac{1}{(zw-1)(1+z+w)}\left[(2+z+w)^{n}(2+z+w)-\frac{(1+z)^{n}(1+w)^{n}}{(zw)^{n}}\left(\frac{(1+z)(1+w)}{zw}-1\right)\right]dw\,dz$$

$$S_n - S_{n-1} = -\frac{1}{4\pi^2}\oint\!\!\oint_\Gamma \frac{1}{(zw-1)}\left[(2+z+w)^{n}-\frac{(1+z)^{n}(1+w)^{n}}{(zw)^{n+1}}\right]dw\,dz$$

$$= -\frac{1}{4\pi^2}\oint\!\!\oint_\Gamma \frac{-1}{(1-zw)}\left[(2+z+w)^{n}-\frac{(1+z)^{n}(1+w)^{n}}{(zw)^{n+1}}\right]dw\,dz$$

$$= -\frac{1}{4\pi^2}\oint\!\!\oint_\Gamma \left[\frac{(1+z)^{n}(1+w)^{n}}{(zw)^{n+1}(1-zw)}-\frac{(2+z+w)^{n}}{(1-zw)}\right]dw\,dz$$

$$= -\frac{1}{4\pi^2}\oint\!\!\oint_\Gamma \frac{(1+z)^{n}(1+w)^{n}}{(zw)^{n+1}(1-zw)}dw\,dz+\frac{1}{4\pi^2}\oint\!\!\oint_\Gamma \frac{(2+z+w)^{n}}{(1-zw)}dw\,dz$$

The second integral has a singular point at $zw=1$ which lies outside the path $\Gamma = C_r \times C_r;\ 0<r<1$, thereby holomorphic inside the path Γ. Hence by Cauchy's theorem, the second integral becomes zero.

$$S_n - S_{n-1} = -\frac{1}{4\pi^2} \oint\!\!\oint_{\Gamma} \frac{(1+z)^n (1+w)^n (1-zw)^{-1}}{z^{n+1} w^{n+1}} dw\, dz \qquad (1.8)$$

In view of (1.6), the R.H.S of (1.8) is the coefficient of $z^n w^n$ in the expansion of the function $g(z,w) = (1+z)^n (1+w)^n (1-zw)^{-1}$ in the powers of *z* and *w*. Hence

$$g(z,w) = \left[\sum_{k=0}^{n} \binom{n}{k} z^n\right]\left[\sum_{k=0}^{n} \binom{n}{k} w^n\right]\left[\frac{1}{1-zw}\right]$$

$$= \sum_{k=0}^{n}\sum_{k=0}^{n} \binom{n}{k}^2 w^n z^n \left(\sum_{n=0}^{\infty} (wz)^n\right) = \sum_{k=0}^{n}\sum_{k=0}^{n} \binom{n}{k}^2 w^n z^n \left(\sum_{n=0}^{\infty} w^n z^n\right)$$

The coefficient of $z^n w^n$ in the expansion is $\sum_{k=0}^{n} \binom{n}{k}^2$ (1.9)

Comparing (1.8) and (1.9) (that are both coefficients of $z^n w^n$), we have that

$$S_n - S_{n-1} = \sum_{k=0}^{n} \binom{n}{k}^2 \qquad (1.10)$$

We now need to show that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Now, we consider the identity

$$(1+x)^n \left(1+\frac{1}{x}\right)^n = \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^{2n}$$

Applying binomial expansion for the identity, we have

$$\left[\sum_{k=0}^{n} \binom{n}{k} x^n\right]\left[\sum_{k=0}^{n} \binom{n}{k} x^{-n}\right] = \sum_{k=0}^{2n} \binom{2n}{k} \left(\sqrt{x}\right)^{2n-k} \left(\frac{1}{\sqrt{x}}\right)^k$$

$$\Rightarrow \sum_{k=0}^{n}\sum_{k=0}^{n} \binom{n}{k}^2 = \sum_{k=0}^{2n} \binom{2n}{k} (x)^{n-\frac{k}{2}} (x)^{\frac{k}{2}} = \sum_{k=0}^{2n} \binom{2n}{k} x^{n-k}$$

$$\therefore \sum_{k=0}^{n}\sum_{k=0}^{n} \binom{n}{k}^2 = \sum_{k=0}^{2n} \binom{2n}{k} x^{n-k}$$

Finally, we apply Verdemonde's identity $\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$

This can be proved by the usual binomial theorem, hence applying this identity, we obtain

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Therefore, (1.10) becomes

$$S_n - S_{n-1} = \binom{2n}{n}$$

This completes the proof.

II. Our Main Results

In this section, we obtained a recurrence relation and a further identity based on the result of Adepoju [1], which serves as an improvement on the results. Our results and the proof are stated in the following theorems;

Theorem 2.1
Let S_n be defined as in (1.1) and (1.2), then we have the identity

$$S_{n+\frac{1}{2}} - S_{n-\frac{1}{2}} = \frac{2^{4n+2}}{(2n+1)\binom{2n}{n}\pi} \qquad (2.1)$$

Proof of theorem (2.1)

We recall from the identity in (1.2), that is $S_n - S_{n-1} = \binom{2n}{n}$, this can be written as

$$S_n - S_{n-1} = \binom{2n}{n} = {}^{2n}C_n = \frac{(2n)!}{(2n-n)!\text{n}!} = \frac{(2n)!}{[\text{n}!]^2}$$

Now, replacing n by n + ½ , we have

$$S_{n+\frac{1}{2}} - S_{n-\frac{1}{2}} = {}^{2\left(n+\frac{1}{2}\right)}C_{n+\frac{1}{2}} = {}^{2n+1}C_{n+\frac{1}{2}} = \frac{\left[2\left(n+\frac{1}{2}\right)\right]!}{\left[\left(n+\frac{1}{2}\right)!\right]^2} = \frac{[2n+1]!}{\left[\left(n+\frac{1}{2}\right)!\right]^2}$$

Recall from gamma function, $n! = \Gamma(n+1)$ and $\Gamma(n+1) = n\Gamma(n)$, hence we have that

$$S_{n+\frac{1}{2}} - S_{n-\frac{1}{2}} = \frac{\Gamma(2n+1+1)}{\left[\Gamma\left(n+\frac{1}{2}+1\right)\right]^2} = \frac{(2n+1)\Gamma(2n+1)}{\left[\left(n+\frac{1}{2}\right)\Gamma\left(n+\frac{1}{2}\right)\right]^2} = \frac{4\Gamma(2n+1)}{(2n+1)\Gamma^2\left(n+\frac{1}{2}\right)}$$

Applying Legendre duplicating formula, we have,

$$S_{n+\frac{1}{2}} - S_{n-\frac{1}{2}} = \frac{4\Gamma(2n+1)}{(2n+1)\left[\frac{(2n)!\sqrt{\pi}}{2^{2n}\,n!}\right]^2} = \frac{2^{4n+2}}{(2n+1)\pi\frac{\Gamma(2n+1)}{(n!)^2}} = \frac{2^{4n+2}}{(2n+1)\pi\frac{(2n)!}{(n!)^2}}$$

$$\Rightarrow S_{n+\frac{1}{2}} - S_{n-\frac{1}{2}} = \frac{2^{4n+2}}{(2n+1)\pi\frac{(2n)!}{(n!)^2}} = \frac{2^{4n+2}}{(2n+1)\pi\binom{2n}{n}}$$

This completes the proof.

Theorem 2.2
Let S_n be defined as in (1.1) and (1.2), then we have the recurrence relation

$$(n+1)S_{n+1} + 2(2n+1)S_{n-1} = (5n+3)S_n \qquad (2.2)$$

Proof of Theorem 2.2

From the result $S_n - S_{n-1} = \binom{2n}{n}$, we replacing n by n +1to get

$$S_{n+1} - S_n = \begin{pmatrix} 2n+2 \\ n+1 \end{pmatrix} = {}^{2n+2}C_{n+1}$$

$$\Rightarrow S_{n+1} - S_n = {}^{2n+2}C_{n+1} = \frac{(2n+2)!}{(2n+2-n-1)!(n+1)!} = \frac{(2n+2)!}{\left[(n+1)!\right]^2}$$

$$\Rightarrow S_{n+1} - S_n = \left\{\frac{(2n+2)(2n+1)}{(n+1)^2}\right\}\left\{\frac{(2n)!}{(n!)^2}\right\} = \left\{\frac{(2n+2)(2n+1)}{(n+1)^2}\right\}\begin{pmatrix} 2n \\ n \end{pmatrix}$$

$$= \left\{\frac{2(2n+1)}{(n+1)}\right\}\left\{S_n - S_{n-1}\right\}$$

$$\Rightarrow (n+1)S_{n+1} - (n+1)S_n = 2(2n+1)S_n - 2(2n+1)S_{n-1}$$

$$\Rightarrow (n+1)S_{n+1} + 2(2n+1)S_{n-1} = (5n+3)S_n$$

This completes the proof.

III. Conclusion

Base on our results, we conclude that our results are improvement of the work of Adepoju et al [1]. Also for further research, applying the solution of difference equation method to the recurrence relation (2.2), the actual expression of S_n can be found.

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