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On Similar Curves and Similar Ruled Surfaces in Euclidean 3-space

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Abstract: *In this paper, we introduced a new (more general and comprehensive) definition of similar ruled surfaces with variable transformations in Euclidean 3-space. Some important results on similar striction curves and similar ruled surfaces are obtained. Furthermore, we have been proved that: a family of k-slant helices is a family of similar curves with variable transformations.MSC: 53A04.*
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I. Introduction

From the view of differential geometry, a straight line is a geometric curve with the cur- vature $\kappa(s) = 0$. A plane curve is a family of geometric curves with torsion $\tau(s) = 0$. Helix (circular helix) is a geometric curve with non-vanishing constant curvature κ and non-vanishing constant torsion τ [3]. A curve of constant slope or general helix is defined by the property that the tangent makes a constant angle with a fixed straight line called the axis of the general helix. A necessary and sufficient condition that a curve be a general helix is that the function

$$\sigma_0 = \frac{\tau(s)}{\kappa(s)} \tag{1}$$

is constant along the curve, where κ and τ denote the curvature and the torsion, respectively [13]. Izumiya and Takeuchi [8] introduced the concept of *slant helix* by saying that the normal lines make a constant angle with a fixed straight line. They characterize a slant helix if and only if the *geodesic curvature* of the principal image of the principal normal indicatrix

$$\sigma_1 = \frac{\sigma_0'(s)}{\kappa(s)\Big(1+\sigma_0^2(s)\Big)^{3/2}} \tag{2}$$

is a constant function. Ali [2], defined a new special curve called it a $k-$*slant helix* and proved that the straight lines, plane curves, general helices, slant helices and slant-slant helices are a special subclasses from k-slant helix. He characterized that: the curve is a k-slant helices if and only if the *geodesic curvature* of the spherical image of ψ_k indicatrix of the curve ψ

$$\sigma_k = \frac{\sigma_{k-1}'(s)}{\kappa(s)\sqrt{1+\sigma_0^2(s)}\sqrt{1+\sigma_1^2(s)}\ldots\Big(1+\sigma_{k-1}^2(s)\Big)^{3/2}}, \tag{3}$$

is a constant function, where $\psi_{\kappa+1} = \frac{\psi_k'(s)}{\|\psi_k'(s)\|}$, $\psi_0(s) = \psi(s)$, $\sigma_0(s) = \frac{\tau(s)}{\kappa(s)}$ and $k \in \{0, 1, 2, ...\}$.

The surface pairs especially ruled surface pairs have an important applications in the study of design problems in spatial mechanisms and physics, kinematics and computer aided design (CAD) [11, 12]. So, these surfaces are one of the most important topics of the surface theory. Ruled surfaces are surfaces which are generated by moving a straight line continuously in the space and are one of the most important topics of differential geometry

[14]. Recently, Ali et al [1] studied a family of ruled surfaces generated by a linear combination of Frenet frame (tangent, normal and binormal) vectors with fixed components in Euclidean 3-space at the points $(s, 0)$.
El-Sabbagh and Ali [5] introduced a new definition of the associated curves called it *family of similar curves with variable transformation*. After this, Onder [9] used the definition of similar curves to define a *family of similar ruled surfaces with variable transformation*. In this paper, we introduce a new (more general and comprehensive) definition of a similar ruled surfaces with variable transformation in Euclidean 3-space. Also, we prove that: a family of *k-slant helices* is a family of similar curves with variable transformations, where
$k \in \{0, 1, 2, ...\}$.

II. Basic Concepts for surfaces

Let $\mathbf{E}^3$ be a 3-dimensional Euclidean space provided with the metric given by

$$\langle , \rangle = dx_1^2 + dx_2^2 + dx_3^2,$$

where (x_1, x_2, x_3) is a rectangular coordinate system of $\mathbf{E}^3$. Let $\psi = \psi(s) : I \subset \mathbb{R} \to \mathbf{E}^3$ be an arbitrary curve of arc-length parameter s. Let $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ be the moving Frenet frame along ψ, then the Frenet formulae is given by [13]

$$\begin{bmatrix} \mathbf{T}'(s) \\ \mathbf{N}'(s) \\ \mathbf{B}'(s) \end{bmatrix} = \begin{bmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}(s) \end{bmatrix}, \tag{4}$$

where the functions $\kappa(s)$ and $\tau(s)$ are the curvature and the torsion of the curve $\psi(s)$, respectively.

Let $S : \Psi = \Psi(x_1, x_2) \subset \mathbf{E}^3$ be a regular surface. Then, the standard unit normal vector field $\mathbf{U}$ on the surface Ψ can be defined by:

$$\mathbf{U} = \frac{\Psi_1 \wedge \Psi_2}{\|\Psi_1 \wedge \Psi_2\|}, \tag{5}$$

where $\Psi_1 = \dfrac{\partial \Psi}{\partial x_1}$ and $\Psi_2 = \dfrac{\partial \Psi}{\partial x_2}$, $x_1 = s$ and $x_2 = v$.

Now, we will write some basic concepts and important properties of ruled surfaces as the following:

Definition 2.1. *[15] A ruled surface in $\mathbf{E}^3$ is a surface which contains at least one-parameter family of straight lines. Thus a ruled surface has a parametric representation as*

$$\Psi(s, v) = \psi(s) + v\,\mathbf{L}(s), \tag{6}$$

where $\psi(s)$ is called the base curve and $\mathbf{L}(s)$ is a unit space curve called director curve which represents the direction of straight line $v \mapsto \psi(s) + v\,\mathbf{L}(s)$ which is called ruling. If the direction of $\mathbf{L}$ is constant, then the ruled surface is said to be cylindrical surface, otherwise is said to be non-cylindrical surface.

The distribution parameter of the ruled surface (6) is given by

$$\mathrm{d}(s) = \frac{[\psi'(s), \mathbf{L}(s), \mathbf{L}'(s)]}{\|\mathbf{L}'(s)\|^2} = \frac{[\mathbf{T}, \mathbf{L}, \mathrm{m}]}{\|\mathbf{L}'\|}, \tag{7}$$

where $\mathbf{T} = \psi'(s) = \dfrac{d\psi}{ds}$, $\dfrac{d\mathbf{L}}{ds} = \mathbf{L}'(s) = \|\mathbf{L}'(s)\|\,\mathrm{m}$ and m is a unit vector in the direction $\mathbf{L}'$. Because, the vector $\mathbf{L}$ is unit vector, then $\mathbf{L}'$ is perpendicular to $\mathbf{L}$ and then $\langle \mathbf{L}, \mathrm{m} \rangle = 0$.

The ruled surface is developable if and only if the distribution parameter vanishes and it is minimal if and only if its mean curvature vanishes [6].

The unit normal vector of the ruled surface (6)

$$\mathbf{U}(s,v)=\frac{(\psi'+v\,\mathbf{L}')\wedge\mathbf{L}}{\sqrt{\|\psi'+v\,\mathbf{L}'\|^2-\langle\psi',\mathbf{L}\rangle^2}}. \tag{8}$$

along a ruling $s=s_0$ approaches a limiting direction as v infinitely decreases. This direction is called the asymptotic normal (central tangent) direction and is defined by:

$$\mathbf{a}=\lim_{v\to-\infty}\mathbf{U}(s_0,v)=\frac{\mathbf{L}\wedge\mathbf{L}'}{\|\mathbf{L}'\|}=\mathbf{L}\wedge\mathbf{m}. \tag{9}$$

The point at which the unit normal of Ψ is perpendicular to $\mathbf{a}$ is called the striction point (or central point) $\mathbf{C}$ and the locus of the central points of all rulings is called striction curve. The parametrization of the striction curve on the ruled surface (6) is given by

$$\gamma(s_\gamma)=\psi(s)-\frac{\langle\mathbf{T}(s),\mathbf{m}(s)\rangle}{\|\mathbf{L}'(s)\|}\,\mathbf{L}(s)=\psi(s)+\mathrm{q}(s)\,\mathbf{L}(s), \tag{10}$$

where

$$\mathrm{q}=-\frac{\langle\mathbf{T},\mathbf{m}\rangle}{\|\mathbf{L}'\|} \tag{11}$$

is called strictional distance and s_γ is the arclength of the striction curve. In the case of $\mathrm{q}(s)=0$, the base curve is a striction curve.

The unit vector $\mathbf{m}=\mathbf{a}\wedge\mathbf{L}=\dfrac{\mathbf{L}'}{\|\mathbf{L}'\|}$ is called central normal which is the surface normal along the striction curve. Then the orthonormal system $\{\mathrm{C};\mathbf{L},\mathbf{m},\mathbf{a}\}$ is called Frenet frame of the ruled surfaces Ψ, where $\mathbf{C}$ is the central points of ruling of the ruled surface and $\mathbf{L}$, $\mathbf{m}$ and $\mathbf{a}$ are unit vectors of ruling, central normal and central tangent, respectively.

For the derivatives of the vectors of Frenet frame $\{\mathrm{C};\mathbf{L},\mathbf{m},\mathbf{a}\}$ of ruled surface Ψ with respect to the arclength parameter s_γ of striction curve, we have

$$\begin{bmatrix}\mathbf{L}'(s_\gamma)\\ \mathbf{m}'(s_\gamma)\\ \mathbf{a}'(s_\gamma)\end{bmatrix}=\begin{bmatrix}0 & \bar{\kappa} & 0\\ -\bar{\kappa} & 0 & \bar{\tau}\\ 0 & -\bar{\tau} & 0\end{bmatrix}\begin{bmatrix}\mathbf{L}(s_\gamma)\\ \mathbf{m}(s_\gamma)\\ \mathbf{a}(s_\gamma)\end{bmatrix}, \tag{12}$$

where $\bar{\kappa}=\dfrac{ds_{\mathbf{L}}}{ds_\gamma}$, $\bar{\tau}=\dfrac{ds_{\mathbf{a}}}{ds_\gamma}$, $s_{\mathbf{L}}$ and $s_{\mathbf{a}}$ are arclengths of the spherical curves circumscribed by the bound vectors $\mathbf{L}$ and $\mathbf{a}$, respectively. It is worth noting that when $\bar{\kappa}\neq 0$ and $\bar{\tau}=0$, the ruled surface is called conoid [7].

On the other hand, the unit normal vector field (8) can be written in the form:

$$\mathbf{U}(s,v)=\frac{\mathbf{T}\wedge\mathbf{L}-v\,s'_{\mathbf{L}}\,\mathbf{a}}{\|\mathbf{T}\wedge\mathbf{L}-v\,s'_{\mathbf{L}}\,\mathbf{a}\|}. \tag{13}$$

Therefore, we have

$$\mathbf{U}(s,0)=\frac{\mathbf{T}\wedge\mathbf{L}}{\|\mathbf{T}\wedge\mathbf{L}\|}=\frac{\mathbf{T}\wedge\mathbf{L}}{\sqrt{1-\langle\mathbf{T},\mathbf{L}\rangle^2}}, \tag{14}$$

$$\mathbf{U}'(s,0)=\frac{\kappa\,\mathbf{N}\wedge\mathbf{L}+s'_{\mathbf{L}}\,\mathbf{T}\wedge\mathbf{m}}{\|\mathbf{T}\wedge\mathbf{L}\|}+\frac{\langle\mathbf{T},\mathbf{L}\rangle\left[\kappa\,\langle\mathbf{N},\mathbf{L}\rangle+s'_{\mathbf{L}}\,\langle\mathbf{T},\mathbf{m}\rangle\right]\mathbf{U}(s,0)}{\|\mathbf{T}\wedge\mathbf{L}\|^2}. \tag{15}$$

Then, the geodesic curvature, the normal curvature and the geodesic torsion which associate the curve $\psi(s)$ on the ruled surface Ψ can be written as follows:

$$\begin{cases}\kappa_g=[\mathbf{U}(s,0),\psi'(s),\psi''(s)]=\dfrac{\kappa\,\langle\mathbf{N},\mathbf{L}\rangle}{\|\mathbf{T}\wedge\mathbf{L}\|},\\[2ex] \kappa_n=\langle\mathbf{U}(s,0),\psi''(s)\rangle=-\dfrac{\kappa\,\langle\mathbf{B},\mathbf{L}\rangle}{\|\mathbf{T}\wedge\mathbf{L}\|},\\[2ex] \tau_g=[\mathbf{U}(s,0),\mathbf{U}'(s,0),\psi'(s)]=\dfrac{\kappa\,\langle\mathbf{T},\mathbf{L}\rangle\,\langle\mathbf{B},\mathbf{L}\rangle+s'_{\mathbf{L}}\langle\mathbf{T},\mathbf{a}\rangle}{\|\mathbf{T}\wedge\mathbf{L}\|^2}.\end{cases} \tag{16}$$

Furthermore, we can write the following important definitions (results):

Definition 2.2. *[4] For a curve $\psi(s)$ lying on a ruled surface, the following statements are well-known:*
(1): *The base curve $\psi(s)$ of the ruled surface Ψ is a geodesic curve if and only if the geodesic curvature κ_g vanishes.*
(2): *The base curve $\psi(s)$ of the ruled surface Ψ is an asymptotic line if and only if the normal curvature κ_n vanishes.*
(3): *The base curve $\psi(s)$ of the ruled surface Ψ is a principal line if and only if the geodesic torsion τ_g vanishes.*

On the other hand, let us consider the Darboux frame $\{\mathbf{T},\mathbf{V},\mathbf{U}\}$ instead of the Frenet frame $\{\mathbf{T},\mathbf{N},\mathbf{B}\}$ on the curve ψ, where $\mathbf{U}$ is the normal in the surface restricted to ψ and $\mathbf{V}=\mathbf{U}\wedge\mathbf{T}$. Then we have

$$\begin{bmatrix}\mathbf{T}'(s)\\ \mathbf{V}'(s)\\ \mathbf{U}'(s)\end{bmatrix}=\begin{bmatrix}0 & \kappa_g & \kappa_n\\ -\kappa_g & 0 & \tau_g\\ -\kappa_n & -\tau_g & 0\end{bmatrix}\begin{bmatrix}\mathbf{T}(s)\\ \mathbf{V}(s)\\ \mathbf{U}(s)\end{bmatrix}, \tag{17}$$

where κ_n is the normal curvature of the surface in the direction of the tangent vector $\mathbf{T}$; κ_g and τ_g are the geodesic torsion and the geodesic curvature of ψ, respectively [10]. The geodesic torsion and the geodesic curvature of ψ are given by:

$$\begin{aligned}\kappa_g&=\langle\mathbf{T}',\mathbf{V}\rangle=\kappa\,\langle\mathbf{N},\mathbf{V}\rangle=\kappa\,\cos[\varphi],\\ \kappa_n&=\langle\mathbf{T}',\mathbf{U}\rangle=\kappa\,\langle\mathbf{N},\mathbf{U}\rangle=-\kappa\,\sin[\varphi],\\ \tau_g&=\langle\mathbf{V}',\mathbf{U}\rangle=[\mathbf{U},\mathbf{U}',\mathbf{T}]=\tau+\varphi'.\end{aligned} \tag{18}$$

where ϕ is the angle between the vectors **V** and **N**. That is **V** and **U** are the rotation of **N** and **B** of the curve ψ in the normal plane. Then

$$\mathbf{V}=\cos[\phi]\mathbf{N}+\sin[\phi]\mathbf{B},\qquad \mathbf{U}=-\sin[\phi]\mathbf{N}+\cos[\phi]\mathbf{B}. \tag{19}$$

III. Similar curves and similar ruled surfaces

Recently, a new definition of associated curves was given by El-Sabbagh and Ali [5]. They called these new curves as similar curves with variable transformation and defined it as follows:

Definition 3.1. *[5] Let ψ and $\tilde{\psi}$ be two regular curves in $\mathbf{E}^3$ parameterized by arclengths s_α and s_β with curvatures κ_α and κ_β, torsion τ_α and τ_β and Frenet frames $\{\boldsymbol{T}_\alpha, \boldsymbol{N}_\alpha, \boldsymbol{B}_\alpha\}$ and $\{\boldsymbol{T}_\beta, \boldsymbol{N}_\beta, \boldsymbol{B}_\beta\}$. The curves ψ and $\tilde{\psi}$ are called similar curves with variable transformation $\frac{ds_\beta}{ds_\alpha} = \lambda_\alpha^\beta$ if and only if the tangent vectors are the same for the two curves i.e.,*

$$\boldsymbol{T}_\beta = \boldsymbol{T}_\alpha. \tag{20}$$

The relation "similar" between curves is an equivalence relation, so that all curves satisfying this condition are called a family of similar curves with variable transformation.

Also, new special curves were given by Ali [2] who called them by k-slant helices and defined it as follows:

Definition 3.2. *[2] Let $\psi = \psi(s)$ be a natural representation of a unit speed regular curve in Euclidean 3-space with Frenet apparatus $\{\kappa, \tau, \boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$. A curve ψ is called a k-slant helix if the unit vector*

$$\psi_{\kappa+1} = \frac{\psi_k'(s)}{\|\psi_k'(s)\|} \tag{21}$$

makes a constant angle with a fixed direction, where $\psi_0 = \psi(s)$ and $\psi_1 = \frac{\psi_0'(s)}{\|\psi_0'(s)\|}$.

El-Sabbagh and Ali [5] proved that:
(1): A family of *straight lines* is a family of similar curves with variable transformations. **(2):** A family of *plane curves* is a family of similar curves with variable transformations. **(3):** A family of *general helices* is a family of similar curves with variable transformations. **(4):** A family of *slant helices* is a family of similar curves with variable transformations.
Now, we can introduce a general theorem for a family of similar curves with variable transformation within the following important theorem:

Theorem 3.3. *The family of k-slant helices with fixed angle φ between the axis of k-slant helix and the unit vector $\psi_{\kappa+1}$ forms a family of similar curves with variable transformation, where $k \in \{0,1,2,...\}$.*

Proof: We will use the *Principal of Mathematical Induction* in this proof.
(1): El-Sabbagh and Ali [5] proved that: A family of 0-slant helices (general helices) is a family of similar curves with variable transformations. Then the theorem is true when $k = 0$.
(2): Let a family of k-slant helices with fixed angle φ between the axis of k-slant helix and the unit vector $\psi_{\kappa+1}$ forms a family of similar curves with variable transformations.
Then assume that

$$\tilde{\psi}_{k+1}(s_\beta) = \psi_{k+1}(s_\alpha)$$

where $\psi = \psi_0(s_\alpha)$ and $\tilde{\psi} = \tilde{\psi}_0(s_\beta)$ are two k-slant helices with arclength parameters s_α and s_β, respectively. Differentiating the above equation with respect to s_β, we have

$$\dot{\tilde{\psi}}_{k+1}(s_\beta) = \frac{d\tilde{\psi}_{k+1}(s_\beta)}{ds_\beta} = \frac{d\psi_{k+1}(s_\alpha)}{ds_\alpha}\frac{ds_\alpha}{ds_\beta} = \psi_{k+1}'(s_\alpha)\,\frac{ds_\alpha}{ds_\beta} \tag{22}$$

and

$$\| \dot{\tilde{\psi}}_{k+1}(s_\beta) \| = \| \psi_{k+1}'(s_\alpha) \| \; \frac{ds_\alpha}{ds_\beta}. \tag{23}$$

Here, "prime" and "dot" refer the differentiation with respect to s_α and s_β, respectively. From (22) and (23), we obtain the following:

$$\tilde{\psi}_{k+2}(s_\beta) = \frac{\dot{\tilde{\psi}}_{k+1}(s_\beta)}{\| \dot{\tilde{\psi}}_{k+1}(s_\beta) \|} = \frac{\psi_{k+1}'(s_\alpha)}{\| \psi_{k+1}'(s_\alpha) \|} = \psi_{k+2}(s_\alpha).$$

This proves that the family of $(k+1)$-slant helices forms a family of similar curves with variable transformations.

Hence the proof is complete.

On the other hand, Onder [9] used the definition of similar curves with variable transformations and defined new associated ruled surfaces. He called these new ruled surfaces as similar ruled surfaces with variable transformations and defined it as follows:

Definition 3.4. *[9] Let $\Psi(s_\alpha, v)$ and $\tilde{\Psi}(s_\beta, v)$ be two regular ruled surfaces in $\mathbf{E}^3$ given by the parameterizations*

$$\Psi(s_\alpha, v) = \psi(s_\alpha) + v\,\mathbf{L}_\alpha(s_\alpha), \qquad \tilde{\Psi}(s_\beta, v) = \tilde{\psi}(s_\beta) + v\,\tilde{\mathbf{L}}_\beta(s_\beta), \tag{24}$$

respectively, where ψ and $\tilde{\psi}$ are the striction curves of Ψ and $\tilde{\Psi}$, respectively. Ψ and $\tilde{\Psi}$ are called similar ruled surfaces with variable transformation if the direction vectors are the same for the two surfaces i.e., $\tilde{\mathbf{L}}_\beta = \mathbf{L}_\alpha = \mathbf{L}$.

Remark 3.5. *From Onder definition above of similar ruled surface, we show that Onder definition of the similar ruled surfaces is a special case because he takes the three conditions to define the similar ruled surfaces as the following:*

(1): *The striction curve and the base curve on the ruled surface Ψ are the same.*

(2): *The striction curve and the base curve on the ruled surface $\tilde{\Psi}$ are the same.*

(3): *The rulings on the similar ruled surfaces Ψ and $\tilde{\Psi}$ are the same.*

The first and the second conditions are one condition on the surface Ψ and another con- dition for the surface $\tilde{\Psi}$, respectively, i.e., not a relation between them (similar surfaces Ψ and $\tilde{\Psi}$). So that, the third condition and the fourth condition have weakened his definition.
This remark is what invited us to think in a more general and comprehensive definition of similar ruled surfaces by omitting the first and the second conditions.

The general suggestion definition of similar ruled surfaces can be introduced as the following:

Definition 3.6. *Let $\Psi(s_\alpha, v)$ and $\tilde{\Psi}(s_\beta, v)$ be two regular ruled surfaces in $\mathbf{E}^3$ given by*

$$\Psi(s_\alpha, v) = \psi(s_\alpha) + v\,\mathbf{L}_\alpha(s_\alpha), \qquad \tilde{\Psi}(s_\beta, v) = \tilde{\psi}(s_\beta) + v\,\tilde{\mathbf{L}}_\beta(s_\beta), \tag{25}$$

where ψ and $\tilde{\psi}$ are the base curves of the surfaces Ψ and $\tilde{\Psi}$, respectively and the quantities s_α and s_β are arclength parameters of the base curves $\psi(s_\alpha)$ and $\tilde{\psi}(s_\beta)$, respectively. The ruled surfaces Ψ and $\tilde{\Psi}$ are called similar ruled surfaces with variable transformation if the two curves ψ and $\tilde{\psi}$ are similar curves with variable transformation and the direction vectors for the two surfaces are the same i.e., $\tilde{\mathbf{L}}_\beta = \mathbf{L}_\alpha = \mathbf{L}$.

The relation "similar" between ruled surfaces is an equivalence relation, so that all similar ruled surfaces are called a family of similar ruled surfaces with variable transformation.

Now, we can give some new characterizations and more general theorems of similar ruled surfaces as the following:

Theorem 3.7. *Let $\Psi(s_\alpha, v)$ and $\tilde{\Psi}(s_\beta, v)$ be two regular ruled surfaces in $\mathbf{E}^3$ given by*

$$\Psi(s_\alpha, v) = \psi(s_\alpha) + v\,\mathbf{L}, \qquad \tilde{\Psi}(s_\beta, v) = \tilde{\psi}(s_\beta) + v\,\mathbf{L} \tag{26}$$

with the same rulings. Then the following statements are equivalent:

(1): The surfaces Ψ *and* $\tilde{\Psi}$ *are similar ruled surfaces with variables transformation* $\frac{ds_\beta}{ds_\alpha} = \lambda_\alpha^\beta$.

(2): The base curves on the ruled surfaces Ψ *and* $\tilde{\Psi}$ *are similar curves with variables transformation* $\frac{ds_\beta}{ds_\alpha} = \lambda_\alpha^\beta$.

(3): The tangents of the base curves on the ruled surfaces Ψ *and* $\tilde{\Psi}$ *are the same, i.e.,* $\tilde{\mathbf{T}}_\beta = \mathbf{T}_\alpha = \mathbf{T}$.

(4): The position vector of the base curve on the ruled surface $\tilde{\Psi}$ *takes the form:*

$$\tilde{\psi}(s_\beta) = \int \mathbf{T}_\alpha \lambda_\alpha^\beta \, ds_\alpha. \tag{27}$$

(5): The normals of the base curves on the ruled surfaces Ψ *and* $\tilde{\Psi}$ *are the same, i.e.,* $\tilde{\mathbf{N}}_\beta = \mathbf{N}_\alpha = \mathbf{N}$ *and the following condition is satisfied:*

$$\lambda_\alpha^\beta = \frac{\kappa_\alpha}{\kappa_\beta}. \tag{28}$$

(6): The binormals of the base curves on the ruled surfaces Ψ *and* $\tilde{\Psi}$ *are the same, i.e.,* $\tilde{\mathbf{B}}_\beta = \mathbf{B}_\alpha = \mathbf{B}$.

(7): The position vector of the ruled surfaces $\tilde{\Psi}$ *takes the form:*

$$\tilde{\Psi}(s_\beta, v) = \int \mathbf{T}_\alpha \lambda_\alpha^\beta \, ds_\alpha + v \, \mathbf{L}. \tag{29}$$

Proof. (**1**) ⇔ (**2**). It is clear from the definition 3.4 of similar ruled surfaces.

(**2**) ⇔ (**3**). It is clear from the definition 3.4 of similar curves.

(**2**) ⇔ (**4**). The proof results from the theorem 4.2 in El-Sabbagh and Ali [5].

(**2**) ⇔ (**5**). The proof results from the theorem 4.3 in El-Sabbagh and Ali [5].

(**2**) ⇔ (**6**). The proof results from the theorem 4.4 in El-Sabbagh and Ali [5].

(**4**) ⇔ (**7**).It is clear from the definition 3.6 of similar ruled surfaces and from the definition 3.4 of similar curves.

Therefore the proof is completed.

Theorem 3.8. *Let $\Psi(s_\alpha, v)$ and $\tilde{\Psi}(s_\beta, v)$ be two regular similar ruled surfaces with variable transformation* $\frac{ds_\beta}{ds_\alpha} = \lambda_\alpha^\beta$ *in* $\mathbf{E}^3$ *given by*

$$\Psi(s_\alpha, v) = \psi(s_\alpha) + v\,\mathbf{L}_\alpha(s_\alpha), \qquad \tilde{\Psi}(s_\beta, v) = \tilde{\psi}(s_\beta) + v\,\tilde{\mathbf{L}}_\beta(s_\beta). \tag{30}$$

Then the following relations are satisfied:

(1): $\kappa_\beta = \lambda_\beta^\alpha\,\kappa_\alpha.$

(2): $\tau_\beta = \lambda_\beta^\alpha\,\tau_\alpha.$

(3): $\tilde{\mathbf{U}}_\beta\,(s_\beta, 0) = \mathbf{U}_\alpha\,(s_\alpha, 0)\,.$

(4): $\tilde{\mathbf{V}}_\beta = \mathbf{V}_\alpha.$

(5): $\dot{s}_{\mathbf{L}_\beta} = \lambda_\beta^\alpha\, s'_{\mathbf{L}_\alpha}.$

(6): $\tilde{\mathbf{m}}_\beta = \mathbf{m}_\alpha.$

(7): $\tilde{\mathbf{a}}_\beta = \mathbf{a}_\alpha.$

(8): $\tilde{\kappa}_g = \lambda_\beta^\alpha\,\kappa_g.$

(9): $\tilde{\kappa}_n = \lambda_\beta^\alpha\,\tilde{\kappa}_n.$

(10): $\tilde{\tau}_g = \lambda_\beta^\alpha\,\tau_g.$

(11): $\tilde{\mathbf{q}}_\beta = \lambda_\alpha^\beta\,\mathbf{q}_\alpha.$

(12): $\tilde{\mathbf{d}}_\beta = \lambda_\alpha^\beta\,\mathbf{d}_\alpha.$

(13): $\tilde{\varphi}_\beta = \varphi_\alpha.$

Proof. (1): The proof of this part is resulting directly from (39).

(2): Differentiating the equation $\tilde{\mathbf{B}}_\beta = \mathbf{B}_\alpha$ with respect to s_β one gets

$$-\tau_\beta\,\mathbf{N}_\beta = -\tau_\alpha\,\mathbf{N}_\alpha\,\frac{ds_\alpha}{ds_\beta},$$

which leads to the relation in part (2).

(3): The proof is clear from (14).

(4): $\tilde{\mathbf{V}}_\beta = \tilde{\mathbf{U}}_\beta \wedge \tilde{\mathbf{T}}_\beta = \mathbf{U}_\alpha \wedge \mathbf{T}_\alpha = \mathbf{V}_\alpha$.

(5) and (6): Differentiating the equation $\tilde{\mathbf{L}}_\beta = \mathbf{L}_\alpha$ with respect to s_β one gets

$$\|\dot{\mathbf{L}}(s_\beta)\|\,\mathrm{m}_\beta = \|\mathbf{L}'(s_\alpha)\|\,\mathrm{m}_\alpha \frac{ds_\alpha}{ds_\beta},$$

or

$$\dot{s}_{\mathbf{L}}\,\tilde{\mathrm{m}}_\beta = s'_{\mathbf{L}}\,\mathrm{m}_\alpha \frac{ds_\alpha}{ds_\beta},$$

which leads to the relation in part (5) and part (6).

(7): $\tilde{\mathbf{a}}_\beta = \tilde{\mathbf{L}}_\beta \wedge \tilde{\mathbf{m}}_\beta = \mathbf{L}_\alpha \wedge \mathbf{m}_\alpha = \mathbf{a}_\alpha$.

(8): From equation (16), we have

$$\tilde{\kappa}_g = \frac{\kappa_\beta\,\langle \tilde{\mathbf{N}}_\beta, \tilde{\mathbf{L}}_\beta\rangle}{\|\tilde{\mathbf{T}}_\beta \wedge \tilde{\mathbf{L}}_\beta\|} = \frac{\lambda_\beta^\alpha\,\kappa_\alpha\,\langle \mathbf{N}_\alpha, \mathbf{L}_\alpha\rangle}{\|\mathbf{T}_\alpha \wedge \mathbf{L}_\alpha\|} = \lambda_\beta^\alpha\,\kappa_g.$$

(9): From equation (16), we have

$$\tilde{\kappa}_n = -\frac{\kappa_\beta\,\langle \tilde{\mathbf{B}}_\beta, \tilde{\mathbf{L}}_\beta\rangle}{\|\tilde{\mathbf{T}}_\beta \wedge \tilde{\mathbf{L}}_\beta\|} = -\frac{\lambda_\beta^\alpha\,\kappa_\alpha\,\langle \mathbf{B}_\alpha, \mathbf{L}_\alpha\rangle}{\|\mathbf{T}_\alpha \wedge \mathbf{L}_\alpha\|} = \lambda_\beta^\alpha\,\kappa_n.$$

(10): From equation (16), we have

$$\begin{aligned}\tilde{\tau}_g &= \frac{\kappa_\beta\,\langle \tilde{\mathbf{T}}_\beta, \tilde{\mathbf{L}}_\beta\rangle\,\langle \tilde{\mathbf{B}}_\beta, \tilde{\mathbf{L}}_\beta\rangle + \dot{s}_{\mathbf{L}}\,\langle \tilde{\mathbf{T}}_\beta, \tilde{\mathbf{a}}_\beta\rangle}{\|\tilde{\mathbf{T}}_\beta \wedge \tilde{\mathbf{L}}_\beta\|^2}\\ &= \frac{\lambda_\beta^\alpha\,\kappa_\alpha\,\langle \mathbf{T}_\alpha, \mathbf{L}_\alpha\rangle\,\langle \mathbf{B}_\alpha, \mathbf{L}_\alpha\rangle + \lambda_\beta^\alpha\, s'_{\mathbf{L}}\,\langle \mathbf{T}_\alpha, \mathbf{a}_\alpha\rangle}{\|\mathbf{T}_\alpha \wedge \mathbf{L}_\alpha\|^2} = \lambda_\beta^\alpha\,\tau_g.\end{aligned}$$

(11): From equation (11), we have

$$\mathrm{q}_\beta = -\frac{\langle \mathbf{T}_\beta, \mathbf{m}_\beta\rangle}{\|\dot{\mathbf{L}}_\beta\|} = -\frac{\langle \mathbf{T}_\alpha, \mathbf{m}_\alpha\rangle}{\lambda_\beta^\alpha\,\|\mathbf{L}'_\alpha\|} = \lambda_\alpha^\beta\,\mathrm{q}_\alpha.$$

(12): From equation (11), we have

$$\mathrm{d}_\beta = \frac{[\mathbf{T}_\beta, \mathbf{L}_\beta, \mathbf{m}_\beta]}{\|\dot{\mathbf{L}}_\beta\|} = \frac{[\mathbf{T}_\alpha, \mathbf{L}_\alpha, \mathbf{m}_\alpha]}{\lambda_\beta^\alpha\,\|\mathbf{L}'_\alpha\|} = \lambda_\alpha^\beta\,\mathrm{d}_\alpha.$$

(13): From equation (18), we have

$$\varphi_\beta = -\tan^{-1}\left[\frac{\tilde{\kappa}_n}{\tilde{\kappa}_g}\right] = -\tan^{-1}\left[\frac{\kappa_n}{\kappa_g}\right] = \varphi_\alpha.$$

From the above discussion, it is easy to write the following:

Theorem 3.9. *Let Ψ and $\tilde{\Psi}$ be two regular similar ruled surfaces with variable transformation λ_α^β in $\mathbf{E}^3$ given by (30). Then the followings hold:*

(1): There are many invariant vectors, for example: $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, $\mathbf{L}$, $\mathbf{m}$, $\mathbf{a}$, $\mathbf{U}(s)$, $\mathbf{V}$, $\mathbf{T}\wedge\mathbf{a}$ *and so on.*

(2): There are many invariant scalars, for example: $\dfrac{\tau}{\kappa}$, $\tau\,\mathbf{d}$, $\dfrac{\mathbf{q}}{\mathbf{d}}$, $\int\kappa\,ds$, $\int\dfrac{ds}{\mathbf{d}}$, $\langle\mathbf{N},\mathbf{m}\rangle$, $\dfrac{\kappa_n}{\kappa_g}$, φ *and so on.*

Theorem 3.10. *Let Ψ and $\tilde{\Psi}$ be two regular similar ruled surfaces with variable transformation λ_α^β in $\mathbf{E}^3$ given by (30). Then the quantities:*

$$\sigma_k=\frac{\sigma'_{k-1}(s)}{\kappa(s)\sqrt{1+\sigma_0^2(s)}\sqrt{1+\sigma_1^2(s)}\ldots\left(1+\sigma_{k-1}^2(s)\right)^{3/2}},\quad k\in\{0,1,2,\ldots\}$$

are invariants.

Proof: We will use the *Principal of Mathematical Induction* in this proof.

(1): From the above theorem, we have $\sigma_0=\dfrac{\tau}{\kappa}$ invariant. Then the theorem is true when $k=0$.

(2): Let the quantities σ_0, σ_1,..., σ_k are invariants. Then we have $\tilde{\sigma}_0(s_\beta)=\sigma_0(s_\alpha)$,...,$\tilde{\sigma}_k(s_\beta)=\sigma_k(s_\alpha)$ and

$$\begin{aligned}\tilde{\sigma}_{k+1}(s_\beta)&=\frac{\dot{\tilde{\sigma}}_k(s_\beta)}{\kappa_\beta(s_\beta)\sqrt{1+\tilde{\sigma}_0^2(s_\beta)}\sqrt{1+\tilde{\sigma}_1^2(s_\beta)}\ldots\left(1+\tilde{\sigma}_k^2(s_\beta)\right)^{3/2}}\\&=\frac{\sigma'_k(s_\alpha)\,\lambda_\beta^\alpha}{\kappa_\alpha(s_\alpha)\,\lambda_\beta^\alpha\sqrt{1+\sigma_0^2(s_\alpha)}\sqrt{1+\sigma_1^2(s_\alpha)}\ldots\left(1+\sigma_k^2(s_\alpha)\right)^{3/2}}=\sigma_{k+1}(s_\alpha).\end{aligned}$$

Hence the proof is complete.

From the theorem 3.10 and the parts 8, 9, 10 ,11, 12 and 5 in theorem 3.8, respectively, we can write the following lemma:

Lemma 3.11. *Let Ψ and $\tilde{\Psi}$ be two regular similar ruled surfaces with variable transformation λ_α^β in $\mathbf{E}^3$ given by (30). Then the following relations are satisfied:*

(1): The base curve ψ of surface Ψ is a k-slant helix if and only if the base curve $\tilde{\psi}$ of surface $\tilde{\Psi}$ is a k-slant helix also.

(2): The base curve ψ of surface Ψ is a geodesic curve if and only if the base curve $\tilde{\psi}$ of surface Ψ is a geodesic curve too.

(3): The base curve ψ of surface Ψ is an asymptotic line if and only if the base curve $\tilde{\psi}$ of surface Ψ is an asymptotic line.

(4): The base curve ψ of surface Ψ is a principal line if and only if the base curve $\tilde{\psi}$ of surface Ψ is a principal line.

(5): The striction curve and the base curve are the same on the surface Ψ if and only if The striction curve and the base curve are the same on the surface $\tilde{\Psi}$.

(6): The surface Ψ is developable if and only if the the surface $\tilde{\Psi}$ is developable.

(7): The surface Ψ is cylindrical if and only if the the surface $\tilde{\Psi}$ is cylindrical.

I. Similar striction curves with variable transformation

In this section we show the following important theorem:

Theorem 4.1. *Let $\Psi(s_\alpha, v)$ and $\tilde{\Psi}(s_\beta, v)$ be two regular ruled surfaces in $\mathbf{E}^3$ given by (30) with the Frenet frames $\{\mathbf{C}_\gamma; \mathbf{L}_\gamma, \mathbf{m}_\gamma, \mathbf{a}_\gamma\}$ and $\{\mathbf{C}_{\tilde{\gamma}}; \mathbf{L}_{\tilde{\gamma}}, \mathbf{m}_{\tilde{\gamma}}, \mathbf{a}_{\tilde{\gamma}}\}$, respectively, γ and $\tilde{\gamma}$ are the striction curves on the ruled surfaces Ψ and $\tilde{\Psi}$ with arclength parameters s_γ and $s_{\tilde{\gamma}}$, respectively, $\bar{\kappa}_\gamma = \frac{ds_{\mathbf{L}_\gamma}}{ds_\gamma}$, $\bar{\tau}_\gamma = \frac{ds_{\mathbf{a}_\gamma}}{ds_\gamma}$, $s_{\mathbf{L}_\gamma}$, $s_{\mathbf{a}_\gamma}$ are the arclength of the spherical curves circumscribed by the bound vector $\mathbf{L}_\gamma$ and $\mathbf{a}_\gamma$ of the ruled surface Ψ and $\bar{\kappa}_{\tilde{\gamma}} = \frac{ds_{\mathbf{L}_{\tilde{\gamma}}}}{ds_{\tilde{\gamma}}}$, $\bar{\tau}_{\tilde{\gamma}} = \frac{ds_{\mathbf{a}_{\tilde{\gamma}}}}{ds_{\tilde{\gamma}}}$, $s_{\mathbf{L}_{\tilde{\gamma}}}$, $s_{\mathbf{a}_{\tilde{\gamma}}}$ are the arclength of the spherical curves circumscribed by the bound vector $\mathbf{L}_{\tilde{\gamma}}$ and $\mathbf{a}_{\tilde{\gamma}}$ of the ruled surface $\tilde{\Psi}$. Then the following statements are equivalent:*

(1): The rulings of the ruled surfaces Ψ and $\tilde{\Psi}$ are the same, i.e., $\mathbf{L}_{\tilde{\gamma}} = \mathbf{L}_\gamma = \mathbf{L}$.

(2): The curves $\phi(s_\gamma) = \int \mathbf{L}_\gamma \, ds_\gamma$ and $\tilde{\phi}(s_{\tilde{\gamma}}) = \int \mathbf{L}_{\tilde{\gamma}} \, ds_{\tilde{\gamma}}$ are similar curves with variables transformation $\lambda_\gamma^{\tilde{\gamma}} = \frac{ds_{\tilde{\gamma}}}{ds_\gamma}$.

(3): The position vector of the curve $\tilde{\phi}$ on the ruled surfaces $\tilde{\Psi}$ takes the form:

$$\tilde{\phi} = \int \mathbf{L}_\gamma \, \lambda_\gamma^{\tilde{\gamma}} \, ds_\gamma.$$

(4): The central normals on the ruled surfaces Ψ and $\tilde{\Psi}$ are the same, i.e., $\tilde{\mathbf{m}}_\gamma = \mathbf{m}_{\tilde{\gamma}} = \mathbf{m}$ and the following condition is satisfied:

$$\lambda_\gamma^{\tilde{\gamma}} = \frac{\bar{\kappa}_\gamma}{\bar{\kappa}_{\tilde{\gamma}}}.$$

(5): The central tangents on the ruled surfaces Ψ and $\tilde{\Psi}$ are the same, i.e., $\tilde{\mathbf{a}}_{\tilde{\gamma}} = \mathbf{a}_\gamma = \mathbf{a}$.

Proof. The proof is very easy and we will omit here.

Theorem 4.2. *Let the rulings of the similar ruled surfaces $\Psi(s_\alpha, v)$ and $\tilde{\Psi}(s_\beta, v)$ with variable transformation $\lambda_\gamma^{\tilde{\gamma}} = \frac{ds_{\tilde{\gamma}}}{ds_\gamma}$ in $\mathbf{E}^3$ given by (30) be the same. Then the following relations are satisfied:*

$$\lambda_\gamma^{\tilde{\gamma}} = \frac{\bar{\kappa}_\gamma}{\bar{\kappa}_{\tilde{\gamma}}} = \frac{\bar{\tau}_\gamma}{\bar{\tau}_{\tilde{\gamma}}}.$$

Proof. The proof is similar to the proof of theorem 3.8.

Theorem 4.3. *Let Ψ and $\tilde{\Psi}$ be two regular similar ruled surfaces in $\mathbf{E}^3$ given by (30) and*

$$\begin{cases} \gamma(s_\gamma) = \psi(s_\alpha) - \dfrac{\langle \psi'(s_\alpha), \mathbf{L}'(s_\alpha) \rangle}{\|\mathbf{L}'(s_\alpha)\|^2} \mathbf{L}(s_\alpha), \\ \\ \tilde{\gamma}(s_{\tilde{\gamma}}) = \tilde{\psi}(s_\beta) - \dfrac{\langle \dot{\tilde{\psi}}(s_\beta), \dot{\tilde{\mathbf{L}}}(s_\beta) \rangle}{\|\dot{\tilde{\mathbf{L}}}(s_\beta)\|^2} \tilde{\mathbf{L}}(s_\beta), \end{cases} \tag{31}$$

be the striction curves on the ruled surfaces Ψ and $\tilde{\Psi}$ respectively. Then the following statements are equivalent:

(1): The striction curves on the ruled surfaces Ψ and $\tilde{\Psi}$ are similar curves with variable transformation.

(2): The striction curves and the base curves on the ruled surfaces Ψ and $\tilde{\Psi}$ are the same.

(3): The striction parameters on the ruled surfaces Ψ and $\tilde{\Psi}$ vanish.

(4): The tangent vector $\mathbf{T}$ of the base curves on the ruled surfaces Ψ and $\tilde{\Psi}$ lies in the cental ruling-central tangent plane ($\mathbf{Lm}$-plane).

(5): The tangent vector $\mathbf{T}$ of the base curves are perpendicular to the central normal vector $\mathbf{a}$ on the ruled surfaces Ψ and $\tilde{\Psi}$.

Proof. If the ruled surfaces Ψ and $\tilde{\Psi}$ are similar ruled surfaces with variable transformation, then the curves ψ and $\tilde{\psi}$ are similar curves with variable transformation and $\tilde{\mathbf{L}} = \mathbf{L}$. Therefore, it is easy to show that:

$$\begin{cases} \psi'(s_\alpha) = \mathbf{T}_\alpha = \mathbf{T}, & \mathbf{L}'(s_\alpha) = \mathbf{L}', \\ \dot{\tilde{\psi}}(s_\beta) = \mathbf{T}_\beta = \mathbf{T}, & \dot{\tilde{\mathbf{L}}}(s_\beta) = \dot{s}\,\mathbf{L}'. \end{cases} \tag{32}$$

We refer $s' = \dfrac{ds_\beta}{ds_\alpha}$ and $\dot{s} = \dfrac{ds_\alpha}{ds_\beta}$. Then, from the above equations the striction distances of the two similar surfaces are given by:

$$\begin{cases} \mathbf{q}(s_\alpha) = \dfrac{\langle \psi'(s_\alpha), \mathbf{L}'(s_\alpha)\rangle}{\|\mathbf{L}'(s_\alpha)\|^2} = \dfrac{\langle \mathbf{T}, \mathbf{L}'\rangle}{\|\mathbf{L}'\|^2} = \dfrac{\langle \mathbf{T}, \mathbf{m}\rangle}{\|\mathbf{L}'\|} = \dfrac{\langle \mathbf{T}, \mathbf{m}\rangle}{s'_{\mathbf{L}}}, \\ \tilde{\mathbf{q}}(s_\beta) = \dfrac{\langle \dot{\tilde{\psi}}(s_\beta), \dot{\tilde{\mathbf{L}}}(s_\beta)\rangle}{\|\dot{\tilde{\mathbf{L}}}(s_\beta)\|^2} = s'\,\dfrac{\langle \mathbf{T}, \mathbf{L}'\rangle}{\|\mathbf{L}'\|^2} = s'\,\dfrac{\langle \mathbf{T}, \mathbf{m}\rangle}{s'_{\mathbf{L}}}. \end{cases} \tag{33}$$

Therefore, we get the relation between the strictianal distances as the following:

$$\tilde{\mathbf{q}}(s_\beta) = s'\,\mathbf{q}(s_\alpha) = \lambda_\alpha^\beta\,\mathbf{q}(s_\alpha). \tag{34}$$

Then, the equations in (31) becomes:

$$\gamma = \psi - \mathbf{q}\,\mathbf{L}, \qquad \tilde{\gamma} = \tilde{\psi} - s'\,\mathbf{q}\,\mathbf{L}. \tag{35}$$

Now, by computing the tangent of the striction curves γ and $\tilde{\gamma}$, we obtain:

$$\begin{aligned} \mathbf{T}_\gamma &= \gamma'(s_\gamma) = (\mathbf{T} - \mathbf{q}'\,\mathbf{L} - \mathbf{q}\,s'_{\mathbf{L}}\,\mathbf{m})\,\frac{ds_\alpha}{ds_\gamma}, \\ \mathbf{T}_{\tilde{\gamma}} &= \dot{\tilde{\gamma}}(s_{\tilde{\gamma}}) = \left(\mathbf{T} - (s'\,\mathbf{q})'\,\mathbf{L} - \mathbf{q}\,s'^2_{\mathbf{L}}\,\mathbf{m}\right)\,\frac{ds_\alpha}{ds_{\tilde{\gamma}}}. \end{aligned} \tag{36}$$

$(\mathbf{1}) \Rightarrow (\mathbf{4})$: If the striction curves are similar curves, then $\mathbf{T}_{\tilde{\gamma}} = \mathbf{T}_\gamma$ which leads to the following equation:

$$(s'_\gamma - s'_{\tilde{\gamma}})\;\mathbf{T} = \left[(s'\,\mathbf{q})'\;s'_\gamma - \mathbf{q}\,s'_{\tilde{\gamma}}\right]\;\mathbf{L} + s'_{\mathbf{L}}\,\mathbf{q}\;(s'_{\mathbf{L}}\,s'_\gamma - s'_{\tilde{\gamma}})\;\mathbf{m}. \tag{37}$$

Then, the tangent $\mathbf{T}$ must lie in the $\mathbf{Lm}$-plane.

$(\mathbf{4}) \Rightarrow (\mathbf{5})$: The proof is very clear.

$(\mathbf{5}) \Rightarrow (\mathbf{3})$: The proof is very clear.

$(\mathbf{3}) \Rightarrow (\mathbf{2})$: The proof is very clear.

$(\mathbf{2}) \Rightarrow (\mathbf{1})$: The proof is very clear.

Therefore, the proof of the theorem is completed.

II. Applications

Now we give an example to illustrate the above results by considering two similar ruled surfaces given by

$$\Psi(s_\alpha, v) = \psi(s_\alpha) + v\,\mathbf{B}(s_\alpha), \qquad \tilde{\Psi}(s_\beta, v) = \tilde{\psi}(s_\beta) + v\,\tilde{\mathbf{B}}_\beta(s_\beta), \tag{38}$$

with base curves which are circular helix and spherical general helix generated by the binormal vectors. The natural representation of the circular helix is:

$$\psi(s_\alpha) = \frac{n^2}{m^2}\left(\sin\left[\frac{m\,s}{n}\right], -\cos\left[\frac{m\,s}{n}\right], \frac{m^2\,s}{n}\right), \tag{39}$$

where $s_\alpha = s$ is the arclength of the circular helix, $m = \dfrac{n}{\sqrt{1-n^2}}$ and $n = \cos[\phi]$, and ϕ is the constant angle between the tangent vector and the axis of the circular helix. The curvature and torsion of this circular helix are $\kappa_\alpha(s) = 1$ and $\tau_\alpha(s) = m$, respectively. The Frenet vectors of this curve takes the form:

$$\begin{cases} \mathbf{T}_\alpha(s) = \dfrac{n}{m}\left(\cos\left[\dfrac{m\,s}{n}\right], \sin\left[\dfrac{m\,s}{n}\right], m\right), \\ \mathbf{N}_\alpha(s) = \left(-\sin\left[\dfrac{m\,s}{n}\right], \cos\left[\dfrac{m\,s}{n}\right], 0\right), \\ \mathbf{B}_\alpha(s) = n\left(-\cos\left[\dfrac{m\,s}{n}\right], -\sin\left[\dfrac{m\,s}{n}\right], \dfrac{1}{m}\right). \end{cases} \tag{40}$$

On the other hand, the natural representation of a spherical general helix $\tilde{\psi} = (\tilde{\psi}_1, \tilde{\psi}_2, \tilde{\psi}_3)$ is:

$$\begin{cases} \tilde{\psi}_1 = \dfrac{1}{c^2\,(1+m^2) - m^2}\left(c\sqrt{1-m^2\,\tilde{s}^2}\,\sin[t] - n\,m\,\tilde{s}\,\cos[t]\right), \\ \tilde{\psi}_2 = \dfrac{1}{m^2 - c^2\,(1+m^2)}\left(c\sqrt{1-m^2\,\tilde{s}^2}\,\cos[t] - n\,m\,\tilde{s}\,\sin[t]\right), \\ \tilde{\psi}_3 = n\,\tilde{s}, \end{cases} \tag{41}$$

where c is an arbitrary constant, $t = \dfrac{c}{n}\arcsin[m\,\tilde{s}]$, $s_\beta = \tilde{s}$ is the arclength of the spherical general helix, $m = \dfrac{n}{\sqrt{1-n^2}}$ and $n = \cos[\phi]$, and ϕ is the constant angle between the tangent vector and the axis of a spherical general helix. The curvature and the torsion of this curve is $\kappa_\beta(\tilde{s}) = \dfrac{c}{\sqrt{1-m^2\,\tilde{s}^2}}$ and $\tau_\beta(\tilde{s}) = \dfrac{m\,c}{\sqrt{1-m^2\,\tilde{s}^2}}$.

The two curves $\psi(s)$ and $\tilde{\psi}(\tilde{s})$ are similar curves with variable transformations, then

$$\lambda_\alpha^\beta = \frac{d\tilde{s}}{ds} = \frac{\kappa_\alpha}{\kappa_\beta} = \frac{\sqrt{1-m^2\,\tilde{s}^2}}{c}. \tag{42}$$

Solving the above equation we have:

$$s=\frac{c}{m}\sin^{-1}[m\,\tilde{s}] \qquad \Leftrightarrow \qquad \tilde{s}=\frac{1}{m}\sin\left[\frac{m\,s}{c}\right] \tag{43}$$

The Frenet vectors of the spherical general helix (41) take the form:

$$\begin{cases} \mathbf{T}_\beta(\tilde{s})=\frac{n}{m}\left(\cos\left[\frac{c}{n}\sin^{-1}[m\,\tilde{s}]\right],\sin\left[\frac{c}{n}\sin^{-1}[m\,\tilde{s}]\right],m\right), \\ \mathbf{N}_\beta(\tilde{s})=\left(-\sin\left[\frac{c}{n}\sin^{-1}[m\,\tilde{s}]\right],\cos\left[\frac{c}{n}\sin^{-1}[m\,\tilde{s}]\right],0\right), \\ \mathbf{B}_\beta(\tilde{s})=n\left(-\cos\left[\frac{c}{n}\sin^{-1}[m\,\tilde{s}]\right],-\sin\left[\frac{c}{n}\sin^{-1}[m\,\tilde{s}]\right],\frac{1}{m}\right). \end{cases} \tag{44}$$

It is easy to show that $\mathbf{T}_\beta(\tilde{s})=\mathbf{T}_\alpha(s)$, $\mathbf{N}_\beta(\tilde{s})=\mathbf{N}_\alpha(s)$ and $\mathbf{B}_\beta(\tilde{s})=\mathbf{B}_\alpha(s)$ under the transformation (43).

Now, we will begin to calculate all quantities in theorem 3.8 and check for all statements as follows.

(1): $\kappa_\beta(s)=\frac{c}{\sqrt{1-m^2\,\tilde{s}^2}}$, $\kappa_\alpha=1$ and $\lambda_\beta^\alpha=\frac{c}{\sqrt{1-m^2\,\tilde{s}^2}}$. Therefore the relation $\kappa_\beta=\lambda_\beta^\alpha\,\kappa_\alpha$ is satisfied.

(2): $\tau_\beta=\frac{m\,c}{\sqrt{1-m^2\,\tilde{s}^2}}$ and $\tau_\alpha=m$. Then the relation $\tau_\beta=\lambda_\beta^\alpha\,\tau_\alpha$ is satisfied.

(3): From (14) we have $\mathbf{U}_\alpha=\frac{\mathbf{T}_\alpha\wedge\mathbf{B}_\alpha}{\parallel\mathbf{T}_\alpha\wedge\mathbf{B}_\alpha\parallel}=-\mathbf{N}_\alpha$ and $\mathbf{U}_\beta=\frac{\mathbf{T}_\beta\wedge\mathbf{B}_\beta}{\parallel\mathbf{T}_\beta\wedge\mathbf{B}_\beta\parallel}=-\mathbf{N}_\beta$. Then the relation $\mathbf{U}_\beta=\mathbf{U}_\alpha$ is satisfied.

(4): $\mathbf{V}_\alpha=\mathbf{U}_\alpha\wedge\mathbf{T}_\alpha=-\mathbf{N}_\alpha\wedge\mathbf{T}_\alpha=\mathbf{B}_\alpha$ and $\mathbf{V}_\beta=\mathbf{U}_\beta\wedge\mathbf{T}_\beta=-\mathbf{N}_\beta\wedge\mathbf{T}_\beta=\mathbf{B}_\beta$. Then the relation $\mathbf{V}_\beta=\mathbf{V}_\alpha$ is satisfied.

(5): $s'_{\mathbf{B}_\alpha}=\parallel\mathbf{B}'_\alpha\parallel=\tau_\alpha$ and $\dot{s}_{\mathbf{B}_\beta}=\parallel\dot{\mathbf{B}}_\beta\parallel=\tau_\beta$. Then the relation $\dot{s}_{\mathbf{B}_\beta}=\lambda_\beta^\alpha\,s'_{\mathbf{B}_\alpha}$ is satisfied.

(6): From the definition of the vector $\mathbf{m}$, we have $\mathbf{m}_\alpha=\mathbf{N}_\alpha$ and $\mathbf{m}_\beta=\mathbf{N}_\beta$. Then the relation $\mathbf{m}_\beta=\mathbf{m}_\alpha$ is satisfied.

(7): From (9) we have $\mathbf{a}_\alpha=\mathbf{B}_\alpha\wedge\mathbf{m}_\alpha=\mathbf{B}_\alpha\wedge\mathbf{N}_\alpha=-\mathbf{T}_\alpha$ and $\mathbf{a}_\beta=\mathbf{B}_\beta\wedge\mathbf{m}_\beta=\mathbf{B}_\beta\wedge\mathbf{N}_\beta=-\mathbf{T}_\beta$. Then the relation $\mathbf{a}_\beta=\mathbf{a}_\alpha$ is satisfied.

(8): $\kappa_g=0$ and $\tilde{\kappa}_g=0$. Then the relation $\tilde{\kappa}_g=\lambda_\beta^\alpha\,\kappa_g$ is satisfied.

(9): $\kappa_n=-1$ and $\tilde{\kappa}_n=-\frac{c}{\sqrt{1-m^2\,\tilde{s}^2}}$. Under the transformation (43), the relation $\tilde{\kappa}_n=\lambda_\beta^\alpha\,\kappa_n$ is satisfied.

(10): $\tau_g=m$ and $\tilde{\tau}_g=\frac{m\,c}{\sqrt{1-m^2\,\tilde{s}^2}}$. Under the transformation (43), the relation $\tilde{\tau}_g=\lambda_\beta^\alpha\,\tau_g$ is satisfied.

(11): $\mathrm{q}_\beta=0$ and $\mathrm{q}_\alpha=0$. Then the relation $\mathrm{q}_\beta=\lambda_\alpha^\beta\,\mathrm{q}_\alpha$ is satisfied.

(12): $\mathrm{d}_\alpha = \frac{n^2}{2m^2}\left(\sin\left[\frac{m\,s}{n}\right] - \cos\left[\frac{m\,s}{n}\right]\right)\sin\left[\frac{2\,m\,s}{n}\right]$ and

$$\tilde{\mathrm{d}}_\beta = \frac{n^2\sqrt{1-m^2\,\tilde{s}^2}}{4\,c\,m^2}\Bigg(\cos\left[\frac{c}{n}\sin^{-1}[m\,\tilde{s}]\right] - \sin\left[\frac{c}{n}\sin^{-1}[m\,\tilde{s}]\right] - \cos[\left[\frac{3\,c}{n}\sin^{-1}[m\,\tilde{s}]\right] - \sin\left[\frac{3\,c}{n}\sin^{-1}[m\,\tilde{s}]\right]\Bigg). \tag{45}$$

Under the transformation (43), the relation $\mathrm{d}_\beta = \lambda_\alpha^\beta\,\mathrm{d}_\alpha$ is satisfied.

(13): From the part (8) and part (9|) in this theorem, we have $\varphi_\beta = \varphi_\alpha$.

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