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An Efficient Treatments For Linear And Nonlinear Heat-Like And Wave-Like Equations With Variable Coefficients

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Abstract: *This paper presents the implementation of the new iterative method proposed by Daftardar-Gejji and Jafari (DGJ method) [V. Daftardar-Gejji, H. Jafari, An iterative method for solving non linear functional equations, J. Math. Anal. Appl. 316 (2006) 753-763] to solve the linear and nonlinear heat-like and wave-like equations with variable coefficients in one and higher dimensional spaces. The solution is obtained in the series form that converge to the exact solution with easily computed components. The DGJ method has many attractive features such as being derivative-free, overcome the difficulty arising in calculating Adomian polynomials to handle the nonlinear terms in Adomian Decomposition Method (ADM). No need to calculate the Lagrange multiplier as in Variational Iteration Method (VIM) and does not require to construct a homotopy and solve the corresponding algebraic equations as in Homotopy Perturbation Method (HPM). Several test examples are given to demonstrate the effectiveness of the proposed method. The software used for the calculations in this study was MATHEMATICA® 8.0.*

Keywords- *Heat-like equations, Iterative method, Variable coefficients, Wave-like equations*

I. Introduction

Linear and nonlinear ordinary or partial differential equations are widely used to model many problems in physics, chemistry, biology, mechanics and engineering. Many analytic and approximate methods have been developed to obtain the solutions for differential equations, especially nonlinear. In physics for example, the heat flow and the wave propagation phenomena are well described by partial differential equations.

Moreover, most physical phenomena of fluid dynamics, quantum mechanics, electricity, plasma physics, propagation of shallow water waves, and many other models are formulated by partial differential equations [1].

Due to these huge applications, there is a demand on the development of accurate and efficient analytic or approximate methods able to deal with the PDEs.

Many fields of science, engineering and physical problems can be described by initial boundary value

Problems (IBVP) with variable coefficients. These linear and nonlinear models were treated numerically and analytically, see [2] with examples and references therein.

Therefore, seeking the solutions of these equations becomes more and more important. Several analytic and approximate methods have been proposed to solve the linear and nonlinear heat-like and wave-like equations with variable coefficients, see [2–4,6]. Some difficulties and drawbacks have appeared, for examples, evaluating the Adomian polynomials to handle the nonlinear terms in ADM [7], calculating the Lagrange multiplier in VIM [3], constructing the homotopy and solve the corresponding algebraic equations in HPM [6].

Daftardar-Gejji and Jafari [8] have proposed an efficient technique for solving linear/nonlinear functional equations in a similar manner namely DGJ method. The DGJ method has been implemented in literature, see [9–12].

Recently, AL-Jawary et al. [13–17] have successfully implemented the DGJ method for solving different linear and nonlinear ordinary and partial differential equations.

In this paper, the applications of the DGJ method for the 1D, 2D and 3D linear and nonlinear heat-like and wave-like equations with variable coefficients will be presented to obtain exact solutions.

The results obtained in this paper are compared with those obtained by other iterative methods such as ADM [2], VIM[3], HAM[4] and HPM[6]. Comparisons show that the DGJ method is effective and convenient to use and overcomes the difficulties arising in others existing techniques.

The outline of this paper is as follows: In section 2 is devoted to introduce the DGJ method and its convergence. In section 3 the heat-like and wave-like equations with variable coefficients are solved by the DGJ method. In section 4 some test examples are solved by the DGJ method for linear and nonlinear heat-like and wave-like equations with variable coefficients in one and higher dimensional spaces and finally in section 5 the conclusion is presented.

II. The DGJ Method

Considering the following general non-linear equation:

$$u = N(u) + f \qquad \ldots(1)$$

where N is a nonlinear operator from a Banach space B → B and f is a known function [8–12]. We are looking for a solution u of equation (1) having the series form:

$$u = \sum_{i=0}^{\infty} u_i \qquad \ldots(2)$$

The nonlinear operator N can be decomposed as

$$N\left(\sum_{i=0}^{\infty} u_i\right) = N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\} \qquad \ldots(3)$$

From equations (2) and (3), equation (1) is equivalent to

$$\sum_{i=0}^{\infty} u_i = f + N(u_0) + \sum_{i=1}^{\infty}\left\{N\left(\sum_{j=0}^{i} u_j\right) - N\left(\sum_{j=0}^{i-1} u_j\right)\right\} \qquad \ldots(4)$$

We define the recurrence relation:

$$G_0 = u_0 = f,$$
$$G_1 = u_1 = N(u_0), \qquad \ldots(5)$$
$$G_m = u_{m+1} = N(u_0 + \ldots + u_m) - N(u_0 + \ldots + u_{m-1}),\ m = 1, 2, \ldots$$

Then

$$(u_1 + \ldots + u_{m+1}) = N(u_1 + \ldots + u_m),\ m = 1, 2, \ldots \quad \ldots(6)$$

and

$$u(x) = f + \sum_{i=1}^{\infty} u_i \qquad \ldots(7)$$

The m-term approximate solution of equation (2) is given by $u = u_0 + u_1 + \ldots + u_{m-1}$.

2.1 Convergence of the DGJ method

The condition for convergence of the series $\sum u_i$ will be presented below. For further details the reader can see [18].

Theorem 2.1.1: [18]

If N is $C^{(\infty)}$ in a neighbourhood of u_0 and $\| N^{(n)}(u_0)\| \le L$, for any n and for some real $L > 0$ and $\|u_i\| \le M < \frac{1}{e}$, i = 1, 2, ..., then the series $\sum_{n=0}^{\infty} G_n$ is absolutely convergent and moreover, $\|G_n\| \le LM^n e^{n-1}(e-1)$, n = 1, 2, ...

Theorem 2.1.2 : [18]

If N is $C^{(\infty)}$ and $\| N^{(n)}(u_0)\| \le M \le e^{-1}$, ∀n, then the series $\sum_{n=0}^{\infty} G_n$ is absolutely convergent.

III. Solution Of Heat-Like And Wave-Like Equations With Variable Coefficients By Using Dgj Method

In this section the DGJ method will be applied to heat-like and wave-like equations with variable coefficients independently.

3.1 Heat-like equations

The heat-like equation with variable coefficients in three-dimensional is given in the form [2]:

$$u_t = f(x, y, z)u_{xx} + g(x, y, z)u_{yy} + h(x, y, z)u_{zz},\ 0 < x < a,\ 0 < y < b,\ 0 < z < c,\ t > 0 \qquad \ldots(8)$$

with initial condition:

$$u(x, y, z, 0) = \phi(x, y, z) \qquad \ldots(9)$$

and the Neumann boundary conditions

$$u_x(0, y, z, t) = f_1(y, z, t),\ u_x(a, y, z, t) = f_2(y, z, t),$$
$$u_y(x, 0, z, t) = g_1(x, z, t),\ u_y(x, b, z, t) = g_2(x, z, t), \qquad \ldots(10)$$
$$u_z(x, y, 0, t) = h_1(x, y, t),\ u_z(x, y, c, t) = h_2(x, y, t).$$

Equation (8) can be written in an operator form as

$L_t u = f(x, y, z)u_{xx} + g(x, y, z)u_{yy} + h(x, y, z)u_{zz}$,
…(11)

where $L_t = \frac{\partial}{\partial t}$. Let us assume the inverse operator L_t^{-1} exists and it can be taken with respect t from 0 to t, i.e.

$$L_t^{-1}(.) = \int_0^t (.)dt \qquad \text{…(12)}$$

Then, by taking the inverse operator L_t^{-1} to both sides of the equation (11) and using the initial condition, leads to

$u(x, y, z, t) = \phi(x, y, z) + L_t^{-1}(f(x, y, z)u_{xx} + g(x, y, z)u_{yy} + h(x, y, z)u_{zz})$
…(13)

By applying the DGJ method for equation (13) the following recurrence relation for the determination of the components $u_{n+1}(x, y, z, t)$ are obtained:

$u_0(x, y, z, t) = \_(x, y, z)$, …(14)

$u_1(x, y, z, t) = N(u_0) = L_t^{-1}(f(x, y, z)(u_0)_{xx} + g(x, y, z)(u_0)_{yy} + h(x, y, z)(u_0)_{zz})$,
…(15)

$u_2(x, y, z, t) = N(u_1 + u_0) - N(u_0) = L_t^{-1}(f(x, y, z)(u_1 + u_0)_{xx} + g(x, y, z)(u_1 + u_0)_{yy} + h(x, y, z)(u_1 + u_0)_{zz}) - u_1$,
…(16)

$u_3(x, y, z, t) = N(u_2 + u_1 + u_0) - N(u_1 + u_0) = L_t^{-1}(f(x, y, z)(u_2 + u_1 + u_0)_{xx} + g(x, y, z)(u_2 + u_1 + u_0)yy + h(x, y, z)(u_2 + u_1 + u_0)_{zz}) - L_t^{-1}(f(x, y, z)(u_1 + u_0)_{xx} + g(x, y, z)(u_1 + u_0)_{yy} + h(x, y, z)(u_1 + u_0)_{zz})$, …(17)

and so on.

Continuing in this manner, the (n + 1)th approximation of the exact solutions for the unknown functions u(x, y, z, t) can be achieved as:

$u_{n+1} = N(u_0 + \ldots + u_n) - N(u_0 + \ldots + u_{n-1}) = L_t^{-1}(f(x, y, z)(u_0 + \cdot \cdot \cdot + un)_{xx} + g(x, y, z)(u_0 + \ldots + u_n)_{yy} + h(x, y, z)(u_0 + \ldots + u_n)_{zz}) - L_t^{-1}(f(x, y, z)(u_0 + \ldots + u_{n-1})_{xx} + g(x, y, z)(u_0 + \ldots + u_{n-1})_{yy} + h(x, y, z)(u_0 + \ldots + u_{n-1})_{zz})$, n = 1, 2, ...
(18)

Based on the DGJ method, we constructed the solution u(x, y, z, t) as:

$$u(x, y, z, t) = \sum_{k=0}^{n} u_k(x, y, z, t),\ n \geq 0 \qquad \text{…(19)}$$

3.2 Wave-like equations

The wave-like equation with variable coefficients in three-dimensional is given in the form [2]:

$u_{tt} = f(x, y, z)u_{xx} + g(x, y, z)u_{yy} + h(x, y, z)u_{zz}$, $0 < x < a$, $0 < y < b$, $0 < z < c$, $t > 0$ …(20)

with initial condition:

$u(x, y, z, 0) = \psi(x, y, z)$ $\quad u_t(x, y, z, 0) = \theta(x, y, z)$, …(21)

and the Neumann boundary conditions

$u_x(0, y, z, t) = f_1(y, z, t)$, $u_x(a, y, z, t) = f_2(y, z, t)$, …(22)
$u_y(x, 0, z, t) = g_1(x, z, t)$, $u_y(x, b, z, t) = g_2(x, z, t)$,
$u_z(x, y, 0, t) = h_1(x, y, t)$, $u_z(x, y, c, t) = h_2(x, y, t)$.

Equation (20) can be written in an operator form as

$\tilde{L}_{tt}u = f(x, y, z)u_{xx} + g(x, y, z)u_{yy} + h(x, y, z)u_{zz}$, …(23)

where $\tilde{L}_{tt}u = \frac{\partial}{\partial t^2}$. Let us assume the inverse operator $\tilde{L}_{tt}^{-1}$ exists and it can be taken with respect t from 0 to t, i.e.

$$\tilde{L}_{tt}^{-1}(.) = \int_0^t \int_0^t (.)dtdt \qquad \text{…(24)}$$

Then, by taking the inverse operator $\tilde{L}_{tt}^{-1}$ to both sides of the equation (23) and using the initial conditions in equation (21), leads to

$u(x, y, z, t) = \psi (x, y, z) + t\theta(x, y, z) + \tilde{L}_{tt}^{-1} (f (x, y, z)u_{xx} + g(x, y, z)u_{yy} + h(x, y, z)u_{zz})$ …(25)

By applying the DGJ method for equation (25) the following recurrence relation for the determination of the components $u_{n+1}(x, y, z, t)$ are obtained:

$u_0(x, y, z, t) = (x, y, z) + t\theta(x, y, z)$, …(26)

$u_1(x, y, z, t) = N(u_0) = \tilde{L}_{tt}^{-1} (f (x, y, z)(u_0)_{xx} + g(x, y, z)(u_0)_{yy} + h(x, y, z)(u_0)_{zz})$, …(27)

$u_2(x, y, z, t) = N(u_1 + u_0) - N(u_0) = \tilde{L}_{tt}^{-1} (f (x, y, z)(u_1 + u_0)_{xx} + g(x, y, z)(u_1 + u_0)_{yy} + h(x, y, z)(u_1 + u_0)_{zz}) - u_1$, (28)

$u_3(x, y, z, t) = N(u_2 + u_1 + u_0) - N(u_1 + u_0) = \tilde{L}_{tt}^{-1} (f (x, y, z)(u_2 + u_1 + u_0)_{xx} + g(x, y, z)(u_2 + u_1 + u_0)_{yy} +$
$h(x, y, z)(u_2 + u_1 + u_0)_{zz}) - \tilde{L}_{tt}^{-1} (f (x, y, z)(u_1 + u_0)_{xx} + g(x, y, z)(u_1 + u_0)_{yy} + h(x, y, z) (u_1 + u_0)_{zz})$, (29)

and so on.

Continuing in this manner, the (n + 1)th approximation of the exact solutions for the unknown functions u(x, y, z, t) can be achieved as:

$u_{n+1} = N(u_0 + \ldots + u_n) - N(u_0 + \ldots + u_{n-1}) = \tilde{L}_{tt}^{-1} (f (x, y, z)(u_0 + \ldots + u_n)_{xx} + g(x, y, z)(u_0 + \ldots + u_n)_{yy} +$
$h(x, y, z)(u_0 + \ldots + u_n)_{zz}) - \tilde{L}_{tt}^{-1} (f (x, y, z)(u_0 + \ldots + u_{n-1})_{xx} + g(x, y, z)(u_0 + \ldots + u_{n-1})_{yy} +$
$h(x, y, z)(u_0 + \ldots + u_{n-1})_{zz})$, n = 1, 2, ...
…(30)

where the solution u(x, y, z, t) is given in equation (19).

It can also be clearly seen that from the DGJ method algorithm for both heat-like and wave-like equations with variable coefficients, the exact solutions are obtained by using the initial conditions only, where the given boundary conditions can be used for justification only.

It is worth to mention that the DGJ method's algorithm above is also valid for nonlinear heat-like and wave-like equations in a straightforward without using linearization, perturbation or restrictive assumption or calculating Adomian polynomials to handle the nonlinear terms as in ADM.

IV. Test Examples

In this section, some test examples will be examined to assess the performance of the DGJ method for 1D, 2D and 3D the linear and nonlinear heat-like and wave-like equations with variable coefficients. To verify the convergence of the method, we applied the method to some test problems for which an analytical solution are available.

4.1 Linear heat-Like Models

To assess the efficiency of DGJ method, three linear heat-like equations with variable coefficients will be solved.

Example 1: Consider the following one-dimensional linear IBVP [2–5]

$u_t = \frac{1}{2}x^2u_{xx}$, $0 < x < 1$, $t > 0$. …(31)

with initial condition:

$u(x, 0) = x^2$,

and boundary conditions:

$u(0, t) = 0$, $u(1, t) = e^t$.

By using the DGJ method, we get the recurrence relation:

$u_0(x, t) = x^2$, …(32)

$u_{n+1} = N(u_0 + \ldots + u_n) - N(u_0 + \ldots + u_{n-1}) = L_t^{-1} \frac{1}{2}x^2(u_0 + \ldots + u_n)_{xx}) - L_t^{-1} \frac{1}{2}x^2(u_0 + \ldots + u_{n-1})_{xx})$, n = 1, 2, ... …(33)

According to the DGJ method, we achieve the following components:

$u_1(x, t) = N(u_0) = L_t^{-1} \frac{1}{2} x^2(u_0)_{xx}) = x^2t$, …(34)

$u_2(x, t) = N(u_1 + u_0) - N(u_0) = L_t^{-1} \frac{1}{2} x^2(u_1 + u_0)_{xx}) - u_1 = x^2 \frac{t^2}{2!}$, …(35)

$u_3(x, t) = N(u_2 + u_1 + u_0) - N(u_1 + u_0) = L_t^{-1} \frac{1}{2} x^2(u^2 + u^1 + u^0)_{xx}) - L_t^{-1} \frac{1}{2} x2(u_1 + u_0)_{xx}) = x^2 \frac{t^3}{3!}$, …(36)

and so on.

Therefore, according to equation (19), we get:

$$u(x,t) = x^2(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+...) \qquad \text{…(37)}$$

This has the closed form

$u(x, t) = x^2e^t$. …(38)

which is the exact solution of the problem and it is the same results obtained by ADM [2], VIM [3], HAM [4] and HPM [5].

Example 2: Consider the following two-dimensional linear IBVP [2, 3, 5]

$u_t = \frac{1}{2}(y^2u_{xx} + x^2u_{yy})$, $0 < x, y < 1, t > 0$. …(39)

with initial condition:

$u(x, y, 0) = y^2$,

and Neumann boundary conditions:

$u_x(0, y, t) = 0$, $u_x(1, y, t) = 2 \sinh t$,

$u_y(x, 0, t) = 0$, $u_y(x, 1, t) = 2 \cosh t$.

Proceeding as before, the recurrence relation

$u_0(x, y, t) = y^2$, …(40)

$u_{n+1} = N(u_0 +…+ u_n) - N(u_0 +…+ u_{n-1}) = \frac{1}{2} L_t^{-1} (y^2(u_0 +…+ u_n)_{xx} + x^2(u_0 +…+ u_n)_{yy}) - \frac{1}{2} L_t^{-1} (y^2(u_0 +… + u_{n-1})_{xx}$ +

$x^2(u_0 + … + u_{n-1})_{yy})$, n = 1, 2, ... …(41)

According to the DGJ method we achieve the following components:

$u_1(x, y, t) = N(u_0) = \frac{1}{2} L_t^{-1} (y^2(u_0)_{xx} + x^2(u_0)_{yy}) = x^2t$, …(42)

$u_2(x, y, t) = N(u_1 + u_0) - N(u_0) = \frac{1}{2} L_t^{-1} (y^2(u_1 + u_0)_{xx} + x^2(u_1 + u_0)_{yy}) - u_1 = y^2 \frac{t^2}{2!}$,…(43)

$u_3(x, y, t) = N(u_2 + u_1 + u_0) - N(u_1 + u_0) = \frac{1}{2} L_t^{-1} (y^2(u_2 + u_1 + u_0)_{xx} + x^2(u_2 + u_1 + u_0)_{yy}) - \frac{1}{2} L_t^{-1} (y^2(u_1 + u_0)_{xx}$ +

$x^2(u_1 + u_0)_{yy}) = x^2 \frac{t^3}{3!}$,…(44)

and so on.

Therefore, according to equation (19) we have:

$u(x, y, t) = x^2(t + \frac{t^3}{3!} ...) + y^2(t + \frac{t^2}{2!} ...)$. …(45)

This has the closed form

$u(x, t) = x^2 \sinh t + y^2 \cosh t$. …(46)

which is the exact solution of the problem and it is the same results obtained by ADM [2], VIM [3] and HPM [5].

Example 3: Consider the following three-dimensional inhomogeneous linear IBVP [2–5]

$u_t = x^4y^4z^4 + \frac{1}{36}(x^2u_{xx} + y^2u_{yy} + z^2u_{zz})$, $0 < x, y, z < 1, t > 0$, …(47)

with initial condition:

$u(x, y, z, 0) = 0$,

and boundary conditions:

$u(0, y, z, t) = 0$ $u(1, y, z, t) = y^4z^4(e^t - 1)$,
$u(x, 0, z, t) = 0$ $u(x, 1, z, t) = x^4z^4(e^t - 1)$,
$u(x, y, 0, t) = 0$ $u(x, y, 1, t) = x^4y^4(e^t - 1)$.
In this example there is inhomogeneous term $x^4y^4z^4$ which after integrating it with respect t from 0 to t, leads to $x^4y^4z^4t$.
Therefore, proceeding as before, the recurrence relation
$u_0(x, y, z, t) = x^4y^4z^4t$, …(48)

$$u_{n+1} = N(u_0 + \dots + u_n) - N(u_0 + \dots + u_{n-1}) = \frac{1}{36} L_t^{-1}(x^2(u_0 + \dots + u_n)_{xx} + y^2(u_0 + \dots + u_n)_{yy} + z^2(u_0 + \dots + u_n)_{zz}) - \frac{1}{2} L_t^{-1}(x^2(u_0 + \dots + u_{n-1})_{xx} + y^2(u_0 + \dots + u_{n-1})_{yy} + z^2(u_0 + \dots + u_{n-1})_{zz}),\ n = 1, 2, \dots$$
…(49)
This gives the following components:

$$u_1(x, y, z, t) = N(u_0) = \frac{1}{36} L_t^{-1}(x^2(u_0)_{xx} + y^2(u_0)_{yy} + z^2(u_0)_{zz}) = x^4y^4z^4 \frac{t^2}{2!},$$
…(50)

$$u_2(x, y, z, t) = N(u_1 + u_0) - N(u_0) = \frac{1}{36} L_t^{-1}(x^2(u_1+ u_0)_{xx}+y^2(u_1+u_0)_{yy}+z^2(u_1+u_0)_{zz}) - u_1 = x^4y^4z^4 \frac{t^3}{3!},$$
…(51)

$$u_3(x, y, z, t) = N(u_2 + u_1 + u_0) - N(u_1 + u_0) = \frac{1}{36} L_t^{-1}(x^2(u_2 + u_1 + u_0)xx + y^2(u_2 + u_1 + u_0)_{yy} + z^2(u_2 + u_1 + u_0)_{zz}) - \frac{1}{36} L_t^{-1}(x^2(u_1 + u_0)_{xx} + y^2(u_1 + u_0)_{yy} + z^2(u_1 + u_0)_{zz}) = x^4y^4z^4 \frac{t^4}{4!},$$
…(52)
and so on.
Therefore, according to equation (19) we have:

$u(x, y, z, t) = x^4y^4z^4(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} \dots)$.…(53)
This has the closed form
$u(x, t) = x^4y^4z^4(e^t - 1)$. …(54)
which is the exact solution of the problem and it is the same results obtained by ADM [2], VIM [3] and HPM [5].

4.2 Nonlinear Heat-Like Models
The DGJ method will be applied for solving three examples of nonlinear heat-like equations with variable coefficients.

Example 4: Consider the following one-dimensional inhomogeneous nonlinear IBVP
$u_t = xuu_{xx} - 2x^3t^4 + 2tx^2,\ 0 < x < 1,\ t > 0$. …(55)
with initial condition:
$u(x, 0) = 0$,
and boundary conditions:
$u(0, t) = 0,\ u(1, t) = t^2$.
In this example there is inhomogeneous term $-2x^3t^4 + 2tx^2$, which after integrating it with respect t from 0 to t, leads to $-\frac{2}{5}x^3t^5 + t^2x^2$.
By using the DGJ method, we get the recurrence relation:

$u_0(x, t) = -\frac{2}{5}x^3t^5 + t^2x^2$, …(56)

$$u_{n+1} = N(u_0 + \dots + u_n) - N(u_0 + \dots + u_{n-1}) = L_t^{-1}(x(u_0 + \dots + u_n)(u_0 + \dots + u_n)_{xx}) - L_t^{-1}(x(u_0 + \dots + u_n)(u_0 + \dots + u_{n-1})_{xx}),\ n = 1, 2, \dots$$
…(57)
According to the DGJ method, we achieve the following components:

$u_1(x, t) = N(u_0) = L_t^{-1}(xu(u_0)_{xx}) = \frac{2}{5}x^3t^5 - \frac{2}{5}t^8x^4 + \dots$,…(58)

$u_2(x, t) = N(u_1 + u_0) - N(u_0) = L_t^{-1}(xu(u_1 + u_0)_{xx}) - u_1 = \frac{2}{5}t^8x^4 + \ldots$, …(59)

and so on.

Considering the first two components u0 and u1 in above, we observe the appearance of the noise terms $-\frac{2}{5}x^3t^5$ in u_0 and $\frac{2}{5}x^3t^5$ in u_1. By canceling the identical terms with opposite signs the remaining term of u_0 justifies the equation. This called noise phenomena, for more details about necessary and sufficient conditions see [1, 19]. Therefore, the exact solution is obtained in the closed form

$u(x, t) = x^2t^2$. …(60)

Example 5: Consider the following two-dimensional inhomogeneous nonlinear IBVP

$ut = xuuxx + yuu_{yy} - 2t^4x^3y^3(y + x) + 2tx^2y^2$, $0 < x, y < 1$, $t > 0$. …(61)

with initial condition:

$u(x, y, 0) = 0$,

and Neumann boundary conditions:

$u_x(0, y, t) = 0$, $u_x(1, y, t) = 2y^2t^2$,

$u_y(x, 0, t) = 0$, $u_y(x, 1, t) = 2x^2t^2$.

Equation (61) contains inhomogeneous term $-2t^4x^3y^3(y+x)+2tx^2y^2$ which after integrating it with respect t from 0 to t, leads to $-\frac{2}{5}t^5x^3y^3(y + x) + t^2x^2y^2$.

Proceeding as before, the recurrence relation

$u_0(x, y, t) = -\frac{2}{5}t^5x^3y^3(y + x) + t^2x^2y^2$, …(62)

$u_{n+1} = N(u_0 + \ldots + u_n) - N(u_0 + \ldots + u_{n-1}) = L_t^{-1}(x(u_0 + \ldots + u_n)(u_0 + \ldots + u_n)_{xx} + y(u_0 + \ldots + u_n)(u_0 + \ldots + u_n)_{yy})$

$L_t^{-1}(x(u_0 + \ldots + u_{n-1})(u_0 + \ldots + u_{n-1})_{xx} + y(u_0 + \ldots + u_{n-1})(u_0 + \ldots + u_{n-1})_{yy})$, $n = 1, 2, \ldots$ …(63)

We achieve the following components:

$u_1(x, y, t) = N(u_0) = L_t^{-1}(xu_0(u_0)_{xx} + yu_0(u_0)_{yy}) = \frac{2}{5}t^5x^3y^3(y + x) - \frac{2}{5}t^8x^6y^4 + \ldots$,…(64)

$u_2(x, y, t) = N(u_1 + u_0) - N(u_0) = L_t^{-1}(x(u_1 + u_0)(u_1 + u_0)_{xx} + y(u_1 + u_0)(u_1 + u_0)_{yy}) - u_1 = \frac{2}{5}t^8x^6y^4 + \ldots$, …(65)

and so on.

Considering the first two components u_0 and u_1 in above, we observe the appearance of the noise terms $-\frac{2}{5}t^5x^3y^3(y + x)$ in u_0 and $\frac{2}{5}t^5x^3y^3(y + x)$ in u_1. By canceling the identical terms with opposite signs the remaining term of u_0 justifies the equation, the exact solution is obtained in the closed form

$u(x, y, t) = t^2x^2y^2$…(66)

Example 6: Consider the following three-dimensional inhomogeneous nonlinear IBVP

$u_t = xuu_{xx} + yuu_{yy} + \frac{1}{6}z^2u_{zz}$, $0 < x, y, z < 1$, $t > 0$…(67)

with initial condition:

$u(x, y, z, 0) = z^3$,

and Neumann boundary conditions:

$u_x(0, y, z, t) = 0$, $u_x(1, y, z, t) = 0$,

$u_y(x, 0, z, t) = 0$, $u_y(x, 1, z, t) = 0$,

$u_z(x, y, 0, t) = 0$, $u_z(x, y, 1, t) = 3e^t$,

Proceeding as before, the recurrence relation

$u_0(x, y, z, t) = z^3$, …(68)

$u_{n+1} = N(u_0 + \ldots + u_n) - N(u_0 + \ldots + u_{n-1}) = L_t^{-1}(x(u_0 + \ldots + u_n)(u_0 + \ldots + u_n)_{xx} + y(u_0 + \ldots + u_n)(u_0 + \ldots + u_n)_{yy} +$

$\frac{1}{6}z^2(u_0 + \ldots + u_n)_{zz}) - L_t^{-1}\ (x(u_0 + \ldots + u_{n-1})(u_0 + \ldots + u_{n-1})_{xx} + y(u_0 + \ldots + u_{n-1})(u_0 + \ldots + u_{n-1})_{yy} +$

$\frac{1}{6}z^2(u_0 + \ldots + u_{n-1})_{zz})$, n = 1, 2, ... …(69)

The following components are achieved:

$u_1(x, y, z, t) = N(u_0) = L_t^{-1}(xu_0(u_0)_{xx} + yu_0(u_0)_{yy} + \frac{1}{6}z^2(u_0)_{zz}) = tz^3$, …(70)

$u_2(x,y,z,t)=N(u_1+u_0) - N(u_0)=L_t^{-1}(x(u_1 + u_0)(u_1 + u_0)_{xx} + y(u_1 + u_0)(u_1 + u_0)_{yy} + \frac{1}{6}z^2(u_1 + u_0)_{zz}) - u_1 = \frac{t^2}{2!}z^3$, …(71)

$u_3(x,y,z,t) = N(u_2 + u_1 + u_0) - N(u_1 + u_0) = L_t^{-1}(x(u_2 + u_1 + u_0)(u_2 + u_1 + u_0)_{xx} + y(u_2 + u_1 + u_0)(u_2 + u_1 + u_0)_{yy} +$

$\frac{1}{6}z^2(u_2 + u_1 + u_0)_{zz}) - L_t^{-1}(x(u_1 + u_0)(u_1 + u_0)_{xx} + y(u_1 + u_0)(u_1 + u_0)_{yy} + \frac{1}{6}z^2(u_1 + u_0)_{zz}) = \frac{t^3}{3!}z^3$, …(72)

and so on.

Therefore, according to equation (19) we have:

$u(x,y,z,t) = z^3(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} \ldots)$…(73)

This has the closed form

$u(x, t) = z^3e^t$ …(74)

which is the exact solution of the problem.

It can also be clearly seen that from solving examples 1-6, the exact solutions are obtained by using the initial conditions only. Also, the obtained solutions can be used to justify the given boundary conditions.

Moreover, the DGJ method is overcomes the difficulty arising in calculating Adomian polynomials to handle the nonlinear terms in ADM.

4.3 Linear wave-like models

To assess the efficiency of DGJ method, three linear wave-like equations with variable coefficients will be solved.

Example 1: Consider the following one-dimensional linear IBVP [2–5]

$u_{tt} = \frac{1}{2}x^2u_{xx}$, $0 < x < 1$, $t > 0$…(75)

with initial conditions:

$u(x, 0) = x$, $u_t(x, 0) = x^2$.

and boundary conditions:

$u(0, t) = 0$, $u(1, t) = 1 + \sinh t$.

By using the same procedure given in equations (26)-(30), we get the recurrence relation:

$u_0(x, t) = x + x^2t$, …(76)

$u_{n+1} = N(u_0 + \ldots + u_n) - N(u_0 + \ldots + u_{n-1}) = \tilde{L}_{tt}^{-1}\frac{1}{2}x^2(u_0 + \ldots + u_n)_{xx}) - \tilde{L}_{tt}^{-1}\frac{1}{2}x^2(u_0 + \ldots + u_{n-1})_{xx})$, n = 1, 2, ... …(77)

The following components are obtained:

$u_1(x, t) = N(u_0) = \tilde{L}_{tt}^{-1}\frac{1}{2}x^2(u_0)_{xx}) = x^2\frac{t^3}{3!}$, …(78)

$u_2(x, t) = N(u_1 + u_0) - N(u_0) = \tilde{L}_{tt}^{-1}\frac{1}{2}x^2(u_1 + u_0)_{xx}) - u_1 = x^2\frac{t^5}{5!}$, …(79)

$u_3(x, t) = N(u_2 + u_1 + u_0) - N(u_1 + u_0) = \tilde{L}_{tt}^{-1}\frac{1}{2}x^2(u_2 + u_1 + u_0)_{xx}) - \tilde{L}_{tt}^{-1}\frac{1}{2}x^2(u_1 + u_0)_{xx}) = x^2\frac{t^7}{7!}$, …(80)

and so on.

Therefore, according to equation (19), we get:

$u(x, t) = x + x^2(t + \frac{t^3}{3!} + \frac{t^5}{5!} + \dots)$, …(81)

This has the closed form
$u(x, t) = x + x^2 \sinh t$ …(82)
which is the exact solution of the problem and it is the same results obtained by ADM [2], VIM [3], HAM [4] and HPM [5].

Example 2: Consider the following two-dimensional linear IBVP [2, 3, 5]
$u_{tt} = \frac{1}{12}(x^2u_{xx} + y^2u_{yy})$, $0 < x, y < 1, t > 0$…(83)
with initial conditions:
$u(x, y, 0) = x^4$, $u_t(x, y, 0) = y^4$,
and Neumann boundary conditions:
$u_x(0, y, t) = 0$, $u_x(1, y, t) = 4 \cosh t$,
$u_y(x, 0, t) = 0$, $u_y(x, 1, t) = 4 \sinh t$.
Proceeding as before, the recurrence relation
$u_0(x, y, t) = x^4 + y^4t$, …(84)

$u_{n+1} = N(u_0 + \dots + u_n) - N(u_0 + \dots + u_{n-1}) = \frac{1}{12}\tilde{L}_{tt}^{-1}(x^2(u_0 + \dots + u_n)_{xx} + y^2(u_0 + \dots + u_n)_{yy}) -$

$\frac{1}{12}\tilde{L}_{tt}^{-1}(x^2(u_0 + \dots + u_{n-1})_{xx} + y^2(u_0 + \dots + u_{n-1})_{yy})$, $n = 1, 2, \dots$ …(85)

The first few components of $u(x, y, t)$ are given by:

$u_1(x, y, t) = N(u_0) = \frac{1}{12}\tilde{L}_{tt}^{-1}(x^2(u_0)_{xx} + y^2(u_0)_{yy}) = x^4\frac{t^2}{2!} + y^4\frac{t^3}{3!}$, …(86)

$u_2(x, y, t) = N(u_1 + u_0) - N(u_0) = \frac{1}{12}\tilde{L}_{tt}^{-1}(x^2(u_1 + u_0)_{xx} + y^2(u_1 + u_0)_{yy}) - u_1 = x^4\frac{t^4}{4!} + y^4\frac{t^5}{5!}$,…(87)

$u_3(x, y, t) = N(u_2 + u_1 + u_0) - N(u_1 + u_0) = \frac{1}{12}\tilde{L}_{tt}^{-1}(x^2(u_2 + u_1 + u_0)_{xx} + y^2(u_2 + u_1 + u_0)_{yy}) - \frac{1}{12}\tilde{L}_{tt}^{-1}(x^2(u_1 + u_0)_{xx} +$

$y^2(u_1 + u_0)_{yy}) = x^4\frac{t^6}{6!} + y^4\frac{t^7}{7!}$,…(88)

and so on.
Therefore, according to equation (19) we have:

$u(x, y, t) = x^4(1 + t + \frac{t^2}{2!} + \frac{t^4}{4!}\dots) + y^4(t + \frac{t^3}{3!} + \frac{t^5}{5!}\dots)$…(89)

This has the closed form
$u(x, t) = x^4 \cosh t + y^4 \sinh t$ …(90)
which is the exact solution of the problem and it is the same results obtained by ADM [2], VIM [3] and HPM [5].

Example 3: Consider the following three-dimensional inhomogeneous linear IBVP [2–5]
$u_{tt} = (x^2 + y^2 + z^2) + \frac{1}{2}(x^2u_{xx} + y^2u_{yy} + z^2u_{zz})$, $0 < x, y, z < 1, t > 0$, …(91)
with initial conditions:
$u(x, y, z, 0) = 0$ $u_t(x, y, z, 0) = x^2 + y^2 - z^2$,
and boundary conditions:
$u(0, y, z, t) = y^2(e^t - 1) + z^2(e^t - 1)$, $u(1, y, z, t) = (1 + y^2)(e^t - 1) + z^2(e^t - 1)$,
$u(x, 0, z, t) = x^2(e^t - 1) + z^2(e^t - 1)$, $u(x, 1, z, t) = (1 + x^2)(e^t - 1) + z^2(e^t - 1)$,
$u(x, y, 0, t) = (x^2 + y^2)(e^t - 1)$, $u(x, y, 1, t) = (x^2 + y^2)(e^t - 1) + (e^t - 1)$.
Proceeding as before, the recurrence relation

$u_0(x, y, z, t) = (x^2 + y^2)(t + \frac{t^2}{2!}) + z^2(-t + \frac{t^2}{2!})$,…(92)

$u_{n+1} = N(u_0 + \ldots + u_n) - N(u_0 + \ldots + u_{n-1}) = \frac{1}{2}\tilde{L}_{tt}^{-1}(x^2(u_0 + \ldots + u_n)_{xx} + y^2(u_0 + \ldots + u_n)_{yy} + z^2(u_0 + \ldots + u_n)_{zz}) -$

$\frac{1}{2}\tilde{L}_{tt}^{-1}(x^2(u_0 + \ldots + u_{n-1})_{xx} + y^2(u_0 + \ldots + u_{n-1})_{yy} + z^2(u_0 + \ldots + u_{n-1})_{zz})$, n = 1, 2, ...(93)

This gives the following components:

$u_1(x, y, z, t) = N(u_0) = \frac{1}{2}\tilde{L}_{tt}^{-1}(x^2(u_0)_{xx} + y^2(u_0)_{yy} + z^2(u_0)_{zz}) = (x^2+y^2)(\frac{t^3}{3!} + \frac{t^4}{4!}) + z^2(-\frac{t^3}{3!} + \frac{t^4}{4!})$,…(94)

$u_2(x, y, z, t) = N(u_1 + u_0) - N(u_0) = \frac{1}{2}\tilde{L}_{tt}^{-1}(x^2(u_1 + u_0)_{xx} + y^2(u_1 + u_0)_{yy} + z^2(u_1 + u_0)_{zz}) - u_1 = (x^2 + y^2)(\frac{t^5}{5!} + \frac{t^6}{6!})$ +

$z^2(-\frac{t^5}{5!} + \frac{t^6}{6!})$,…(95)

$u_3(x, y, z, t) = N(u_2 + u_1 + u_0) - N(u_1 + u_0) = \frac{1}{2}\tilde{L}_{tt}^{-1}(x^2(u_2 + u_1 + u_0)_{xx} + y^2(u_2 + u_1 + u_0)_{yy} + z^2(u_2 + u_1 + u_0)_{zz}) -$

$\frac{1}{2}\tilde{L}_{tt}^{-1}(x^2(u_1 + u_0)xx + y^2(u_1 + u_0)_{yy} + z^2(u_1 + u_0)_{zz}) = (x^2 + y^2)(\frac{t^7}{7!} + \frac{t^8}{8!}) + z^2(-\frac{t^7}{7!} + \frac{t^8}{8!})$, …(96)

and so on.

Therefore, according to equation (19) we have:

$u(x, y, z, t) = (x^2 + y^2)(t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} ...) + z^2(-t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} ...)$,…(97)

This has the closed form

$u(x, t) = (x^2 + y^2)e^t + z^2e^{-t} - (x^2 + y^2 + z^2)$ …(98)

which is the exact solution of the problem and it is the same results obtained by ADM [2], VIM [3] and HPM [5].

4.4 Nonlinear wave-like models

The DGJ method will be applied for solving three examples of nonlinear wave-like equations with variable coefficients.

Example 4: Consider the following one-dimensional inhomogeneous nonlinear IBVP

$u_{tt} = xuu_{xx} - 2x^3t^4 + 2x^2$, $0 < x < 1$, $t > 0$ …(99)

with initial condition:

$u(x, 0) = 0$,

and boundary conditions:

$u(0, t) = 0$, $u(1, t) = t^2$.

In this example there is inhomogeneous term −2x3t4 + 2x2, which after integrating it with respect t from 0 to t twice, leads to $-\frac{t^6x^3}{15} + t^2x^2$.

By using the DGJ method, we get the recurrence relation:

$u_0(x, t) = -\frac{t^6x^3}{15} + t^2x^2$, …(100)

$u_{n+1} = N(u_0 + \ldots + u_n) - N(u_0 + \ldots + u_{n-1}) = \tilde{L}_{tt}^{-1}(x(u_0 + \ldots + u_n)(u_0 + \ldots + u_n)_{xx}) - \tilde{L}_{tt}^{-1}(x(u_0 + \ldots + u_{n-1})$
$(u_0 + \ldots + u_{n-1})_{xx})$, n = 1, 2, ... …(101)

The following components are obtained:

$u_1(x, t) = N(u_0) = \tilde{L}_{tt}^{-1}(xu(u_0)_{xx}) = -\frac{t^6x^3}{15} - \frac{4t^{10}x^4}{675} + \ldots$, …(102)

$u_2(x, t) = N(u_1 + u_0) - N(u_0) = \tilde{L}_{tt}^{-1}(xu(u_1 + u_0)_{xx}) - u_1 = -\frac{4t^{10}x^4}{675} - \frac{37t^14x^5}{61425} + \ldots$, …(103)

and so on.

Considering the first two components u_0 and u_1 in above, we observe the appearance of the noise terms $-\frac{t^6x^3}{15}$ in u_0 and $\frac{t^6x^3}{15}$ in u_1. By canceling the identical terms with opposite signs the remaining term of u_0 justifies the equation, the exact solution is obtained in the closed form

$$u(x, t) = x^2t^2 \qquad \ldots(104)$$

Example 5: Consider the following two-dimensional inhomogeneous nonlinear IBVP

$$u_{tt} = -\frac{x^2}{6} u_{xx} + yuu_{yy},\ 0 < x,\ y < 1,\ t > 0 \ \ldots(105)$$

with initial conditions:

$u(x, y, 0) = x^3$, $u_t(x, y, 0) = 0$,

and Neumann boundary conditions:

$u_x(0, y, t) = 0$, $u_x(1, y, t) = 3 \cos t$,

$u_y(x, 0, t) = 0$, $u_y(x, 1, t) = 0$.

Proceeding as before, the recurrence relation

$$u_0(x, y, t) = x^3, \qquad \ldots(106)$$

$$u_{n+1} = N(u_0 + \ldots + u_n) - N(u_0 + \ldots + u_{n-1}) = \tilde{L}_{tt}^{-1}\left(-\frac{x^2}{6}(u_0 + \ldots + u_n)_{xx} + y(u_0 + \ldots + u_n)(u_0 + \ldots + u_n)_{yy}\right) - \tilde{L}_{tt}^{-1}\left(-\frac{x^2}{6}(u_0 + \ldots + u_{n-1})_{xx} + y(u_0 + \ldots + u_{n-1})\, u_0 + \ldots + u_{n-1})_{yy}\right),\ n = 1, 2, \ldots \qquad \ldots(107)$$

The first few components of u(x, y, t) are given by:

$$u_1(x, y, t) = N(u_0) = \tilde{L}_{tt}^{-1}\left(-\frac{x^2}{6}(u_0)_{xx} + yu_0(u_0)_{yy}\right) = -x^3\ \frac{t^2}{2!},\ldots(108)$$

$$u_2(x, y, t) = N(u_1 + u_0) - N(u_0) = \tilde{L}_{tt}^{-1}\left(-\frac{x^2}{6}(u_1 + u_0)_{xx} + y(u_1 + u_0)(u_1 + u_0)_{yy}\right) - u_1 = x^3\ \frac{t^4}{4!}, \qquad \ldots(109)$$

$$u_3(x, y, t) = N(u_2 + u_1 + u_0) - N(u_1 + u_0) = \tilde{L}_{tt}^{-1}\left(-\frac{x^2}{6}(u_2 + u_1 + u_0)_{xx} + y(u_2 + u_1 + u_0)(u_2 + u_1 + u_0)_{yy}\right) - \tilde{L}_{tt}^{-1}\left(-\frac{x^2}{6}(u_1 + u_0)_{xx} + y(u_1 + u_0)(u_1 + u_0)_{yy}\right) = -x^3\ \frac{t^6}{6!},\ \ldots(110)$$

and so on.

Therefore, according to equation (19) we have:

$$u(x, y, t) = x^3\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} \ldots\right)\ldots(111)$$

This has the closed form

$$u(x, t) = x^3 \cos t \qquad \ldots(112)$$

Example 6: Consider the following three-dimensional inhomogeneous nonlinear lBVP

$$u_{tt} = xuu_{xx} - \frac{y^2}{2} u_{yy} + zuu_{zz},\ 0 < x, y, z < 1,\ t > 0,\ \ldots(113)$$

with initial conditions:

$u(x, y, z, 0) = y^2$, $u_t(x, y, z, 0) = 0$,

and boundary conditions:

$u(0, y, z, t) = 0$, $u(1, y, z, t) = 0$,

$u(x, 0, z, t) = 0$, $u(x, 1, z, t) = \cos t$,

$u(x, y, 0, t) = 0$, $u(x, y, 1, t) = 0$.

Proceeding as before, the recurrence relation

$$u_0(x, y, z, t) = y^2,\ \ldots(114)$$

$$u_{n+1} = N(u_0 + \ldots + u_n) - N(u_0 + \ldots + u_{n-1}) = \tilde{L}_{tt}^{-1}\left(x(u_0 + \ldots + u_n)(u_0 + \ldots + u_n)_{xx} - \frac{y^2}{2}(u_0 + \ldots + u_n)_{yy} + z(u_0 + \ldots + u_n)(u_0 + \ldots + u_n)_{zz}\right) - \tilde{L}_{tt}^{-1}\left(x(u_0 + \ldots + u_{n-1})_{xx} - \frac{y^2}{2}(u_0 + \ldots + u_{n-1})_{yy} + z(u_0 + \ldots + u_{n-1})\right.$$

$(u_0 + \ldots + u_{n-1})_{zz})$, n = 1, 2, ... …(115)

This gives the following components:

$u_1(x, y, z, t) = N(u_0) = \tilde{L}_{tt}^{-1}(x(u_0)(u_0)_{xx} - \frac{y^2}{2}(u_0)_{yy} + z(u_0)(u_0)_{zz}) = -y^2\ \frac{t^2}{2!}$, …(116)

$u_2(x, y, z, t) = N(u_1 + u_0) - N(u_0) = \tilde{L}_{tt}^{-1}(x(u_1 + u_0)(u_1 + u_0)_{xx} - \frac{y^2}{2}(u_1 + u_0)_{yy} + z(u_1 + u_0)(u_1 + u_0)_{zz}) - u_1 = y^2\ \frac{t^4}{4!}$,

…(117)

$u_3(x, y, z, t) = N(u_2 + u_1 + u_0) - N(u_1 + u_0) = \tilde{L}_{tt}^{-1}(x(u_2 + u_1 + u_0)(u_2 + u_1 + u_0)_{xx} - \frac{y^2}{2}(u_2 + u_1 + u_0)_{yy} +$

$z(u_2 + u_1 + u_0)(u_2 + u_1 + u_0)_{zz}) - \frac{1}{2}\tilde{L}_{tt}^{-1}(x^2(u_1 + u_0)_{xx} + y^2(u_1 + u_0)_{yy} + z^2(u_1 + u_0)_{zz}) = -y^2\ \frac{t^6}{6!}$,

…(118)

and so on.

Therefore, according to equation (19) we have:

$u(x, y, z, t) = y^2(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!}\ ...)$, …(119)

This has the closed form

$u(x, t) = y^2 \cos t$ …(120)

V. Conclusion

In this paper, an efficient iterative method namely DGJ method is implemented to obtain the exact solution for solving linear and nonlinear heat-like and wave-like equations with variable coefficients in one and higher dimensional spaces using the initial condition only. In DGJ method, it is possible to derive the exact solution for both linear and nonlinear problems by using few iterations only. The obtained exact solution of applying the DGJ method is in full agreement with the results obtained with those methods available in the literature such as ADM [2], VIM [3], HAM [4] and HPM [5]. The method simple, easy does not required any restrictive assumptions and can be easily comprehended with only a basic knowledge of Calculus. This confirms that the DGJ method is reliable and promising when compared with some existing methods.

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References

[1]. A.M. Wazwaz, Linear And Nonlinear Integral Equations Methods And Applications (Beijing And Springer-Verlag Berlin Heidelberg, 2011.

[2]. A.M.Wazwaz, A. Gorguis, Exact Solutions For Heat-Like And Wave-Like Equations With Variable Coefficients, Applied Mathematics And Computation, 149, 2004, 15-29.

[3]. Da-Hua Shou, Ji-Huan He, Beyond Adomian Method: The Variational Iteration Method For Solving Heatlike And Wave-Like Equations With Variable Coefficients, Physics Letters, A 372, 2008, 233-237.

[4]. A. K. Alomari., M. S. M. Noorani, R. Nazar, Solutions Of Heat-Like And Wave-Like Equations With Variable Coefficients By Means Of The Homotopy Analysis Method, Chineses Physics Letters, 25, 2008, 589-592.

[5]. L. Jin, Homotopy Perturbation Method For Solving Partial Differential Equations With Variable Coefficients, International Journal Of Contemporary Mathematical Sciences, 3, 2008, 1395-1407.

[6]. V.G.Gupta, S. Gupta, Homotopy Perturbation Transform Method For Solving Nonlinear Wave-Like Equations Of Variable Coefficients, Journal Of Information And Computing Science, 8, 2013, 163-172.

[7]. M. Ghoreishi, Adomian Decomposition Method (ADM) For Nonlinear Wave-Like Equations With Variable Coefficient, Applied Mathematical Sciences, 4, 2010, 2431-2444.

[8]. V. Daftardar-Gejji, H. Jafari, An Iterative Method For Solving Nonlinear Functional Equations, Journal Of Mathematical Analysis And Applications, 316, 2006, 753-763.

[9]. S. Bhalekar, V. Daftardar-Gejji, Solving A System Of Nonlinear Functional Equations Using Revised New Iterative Method, World Academy Of Science, Engineering And Technology, 68, 2012, 08-21.

[10]. S. Bhalekar, V. Daftardar-Gejji, New Iterative Method: Application To Partial Differential Equations, Applied Mathematics And Computation, 203, 2008, 778-783.

[11]. V. Daftardar-Gejji, S. Bhalekar, Solving Fractional Boundary Value Problems With Dirichlet Boundary Conditions Using A New Iterative Method, Computers And Mathematics With Applications, 59, 2010, 1801-180.

[12]. M. Yaseen, M. Samraiz, S. Naheed, The DJ Method For Exact Solutions Of Laplace Equation, Results In Physics, 3, 2013, 38-40.

[13]. M.A. AL-Jawary, A Reliable Iterative Method For Solving The Epidemic Model And The Prey And Predator Problems, International Journal Of Basic And Applied Sciences, 3, 2014, 441-450.

[14]. M.A. AL-Jawary, Approximate Solution Of A Model Describing Biological Species Living Together Using A New Iterative Method, International Journal Of Applied Mathematical Research, 3, 2014, 518-528.
[15]. M.A. AL-Jawary, A Reliable Iterative Method For Cauchy Problems, Mathematical Theory And Modeling, 4, 2014, 148-153.
[16]. M.A. AL-Jawary, Exact Solutions To Linear And Nonlinear Wave And Diffusion Equations, International Journal Of Applied Mathematical Research, 4, 2015, 106-118.
[17]. M.A. AL-Jawary, H. R. AL-Qaissy, A Reliable Iterative Method For Solving Volterra Integro-Differential Equations And Some Applications For The Lane-Emden Equations Of The First Kind, Monthly Notices Of The Royal Astronomical Society, 448, 2015, 3093-3104.
[18]. S. Bhalekar, V. Daftardar-Gejji, Convergence Of The New Iterative Method, International Journal Of Differential Equations, 2011, 2011, Article ID 989065, 10 Pages.
[19]. A.M. Wazwaz, Necessary Conditions For The Appearance Of Noise Terms In Decomposition Solution Series, Applied Mathematics And Computation, 81, 1997, 265-274.