IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 11, Issue 3 Ver. V (May - Jun. 2015), PP 88-94
www.iosrjournals.org

Operations on Zadeh's Z-numbers

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Abstract: *Zadeh defined a Z-number associated with an uncertain variable X as an ordered pair of fuzzy numbers,(A,B). The first component represents the value of the variable while the second component gives a measure of certainty. Computations with Z-numbers is a topic which is both interesting and useful. In this paper we explain the computational technique and illustrate with examples how sum and product of Z-valuations can be computed.*

Keyword: *Fuzzy event probability ,Restriction, Zadeh's Z-numbers ,Z-valuation*

I. Introduction

Zadeh [1] defined a Z-number associated with an uncertain variable X as an ordered pair of fuzzy numbers,(A,B) where A represents a value of the variable and B represents an idea of certainty .Zadeh refers to the ordered triple,(X,A,B) as a Z-valuation which is equal to the assignment statement X is(A,B). Computations with Z-numbers is a topic which is both interesting and useful. R.R. Yager [2]and Shankar Pal[3] have studied this fascinating toic.

Problems involving computation with Z-numbers are easy to state but far from easy to solve. In this paper we explain the computational technique and illustrate with examples how sum and product of Z-valuations can be computed.

II. Preliminary Definitions

Definition 1: Fuzzy Set

A fuzzy set A defined on a universe X may be given as:
A = $\{(x, \mu_A(x))/x \in X\}$ where μ_A: X→[0,1] is the membership function of A. The membership value $\mu_A(x)$ describes the degree of belonging of $x \in X$ in A .

Definition 2: Restrictions

A restriction may be viewed as generalized constraint. Suppose A is a fuzzy set. The statement "R(X) : X is A" is referred to as a possibilistic restriction .Here A plays the role of possibility distribution of X. The statement " R(X) : X is A" is to be understood as" The Possibility (X=u) is $\mu_A(u)$ "where μ_A is the membership function of A and u is a generic value of X.
When X is a random variable, the probability distribution of X plays the role of probabilistic restriction on X ([9],[4]). A probabilistic restriction is expressed as:

R(X) : X isp f, where f is the p.d.f. of X.

That statement "R(X) : X isp f " is to be understood as" Prob(u≤x≤u+du) = f(u)du"

Definition 4 : Fuzzy event probability[4]

Let X be a random variable taking real values and A a fuzzy set defined on the real line. The fuzzy event probability of " X is A" is
FEP(x∈A) = $\int \mu_A(u)$p(u)du

Definition 3: Z-number

Zadeh defines Z-number as follows[1]
A Z-number is an ordered pair of fuzzy numbers Z = (A,B), associated with a real-valued uncertain variable X, with the first component A, a restriction on the values which X can take and the second component B, a measure of reliability of the first component.

Definition 4: Z-valuation

A Z-valuation is an ordered triple (X,A,B) where A and B are fuzzy numbers.
A Z-valuation is equivalent to an assignment statement "X is (A,B)", where X is an uncertain variable' A is a restriction on the values which X can take and B is referred to as certainty.
Z-valuation (X,A,B) may be viewed as a restriction on X defined by :- FEP(X is A) is B

More explicitly, Possibility(FEP(xεA) = u) = $\mu_B(u)$
The Z-valuation (X,A,B) may be denoted as X isz (A,B)

Extension Principle
A more general version was described in [5]. In this version, we have

$$Y = f(X)$$

R(X): g(X) is A (constraint on u is μ_A (g(u)))
R(Y): $\mu_Y(v) = sup_u \mu_A$(g(u))
subject to

$$v = f(u)$$

For a function with two arguments, the extension principle reads:

$$Z = f(X, Y)$$

R(X): g(X) is A (constraint on v is $\mu_A(g(u))$)
R(Y): h(Y) is B (constraint on v is v_B(h(u)))
R(Z): $\mu_Z(w) = sup_{u,v}(\mu_X(g(u)) \wedge \mu_Y(h(v)))$, $\wedge$ = min
subject to $w = f(u, v)$

III. Operations On Z-Numbers

In his 2011 paper Zadeh had outlined the procedure for operations on Z-numbers as follows:-
Let X, Y be random variables. Given the Z-valuations (X,A,B) and (Y,E,F) and an operation $*$ the problem is to determine the Z-valuation of X $*$ Y . The procedure is as follows:
Let p(x) and q(x) be the underlying probability density functions of X and Y respectively. The known information may be summarized as

$$\mu_B(\int A(x)p(x)dx = u) = B(u)$$

$$\mu_F(\int E(x)q(x)\,dx = u) = F(u)$$

Notice that p,q are not known.
Let us denote the Z-valuation of W by (W,G,H), and the underlying probability density function of W by p_W.
The membership function of G is determined by application of Extension Principle:

$$G(w) = Sup_{u,v}(A(u) \wedge E(v))$$

where the supremum is taken over all u,v such that $w = u * v$.
The next step is to determine the possibility distribution of p_W.Note that

$$p_W(t) = p_W(t) = \int_{t=u*v} p(u)\,q(v)\,du$$

We shall denote the above equation by $p_W = p * q$. By applying the extension principle we have

$$\mu_{p_W}(p_W) = Sup_{\{p,q/p_W = p*q\}}(B(\int A(u)\,p(u)\,du) \wedge F(\int E(v)\,q(v)\,dv))$$

H-the membership function of fuzzy set H is determined to be

$$H(t) = Sup_{p_W}(\mu_{p_W}(p_W))$$

where the supremum is taken over the set $\{p_W \mid t = \int p_W(s)\,G(s)\,ds\}$
In other words we are solving a variational problem here. For each t, we have to locate the probalility density function p_W which optimizes the value of H(t) subject to the condition $t = \int p_W(s)\,G(s)\,ds$. If no other conditions are placed on p_W this problem will be almost impossible to solve. So usually assumptions regarding the nature of p and q must be made to solve the problem.
Note: In case of discrete random variables, in the appropriate places integration is replaced by summation.

IV. Computation Of Sum And Product

We have two Z-valuations on two independent random variables X and Y as X isz (A_X, B_X) and Y isz (A_Y, B_Y).
Consider now a random variable W = X+Y. To find the Z-valuation of W = X+Y

Example 1:
Let X, Y be discrete random variables where the membership functions of A_X and A_Y are given by

$$A_X(x) = \frac{0.3}{0} + \frac{0.6}{1} + \frac{0.8}{2} + \frac{0.6}{3} \qquad A_Y(y) = \frac{0.7}{0} + \frac{0.2}{1} + \frac{0.4}{2} + \frac{0.5}{3}$$

The possibility distributions B_X and B_Y are given by

$$B_X(s) = \frac{10s-5}{2} \quad \text{if } 0.5 \le s \le 0.7 \qquad B_Y(t) = \frac{20t-7}{4} \quad \text{if } 0.35 \le t \le 0.55$$

Each of these generates possibility distribution G_i over the space of respective probability distribution *P* as

$$G_1(p_X) = B_X(\int A_X(u)p_X(u)du) \text{ and } G_2(p_Y) = B_Y(\int A_Y(u)p_Y(u)du)$$

where p_X and p_Y are the probability distributions of X and Y.

Further assume that the p_X, p_Y are known to be :

$p_X(0) = a, p_X(1) = \frac{1}{2} - a, p_X(2) = \frac{1}{2} - a, p_X(3) = a; p_y(0) = \frac{1}{2} - b, p_y(1) = b, p_y(2) = \frac{1}{2} - b, p_Y(3) = b$

Let us denote the Z-valuation of W = X+Y by W isz (A_W, B_W). Our aim is to find the Z-valuation of W = X+Y. That is to find the possibility distributions $A_W(w)$, and $B_W(r)$.

Step (i): To calculate A_W :-

Now $A_W(w) = Sup_v(A_X(v) \wedge A_Y(w-v))$

Thus $A_W(w) = \frac{0.3}{0} + \frac{0.6}{1} + \frac{0.7}{2} + \frac{0.6}{3} + \frac{0.5}{4} + \frac{0.5}{5} + \frac{0.5}{6}$

Step(ii): To calculate the probability distribution of W:

The probability distribution of W is given by $Prob(w = k) = \sum_0^6 p_a(u)p_b(k-u)$

Since the probability distribution depends on two parameters a, and b let us denote it by $q_{(a,b)}$.The distribution $q_{(a,b)}$ is given by the following table

k	0	1	2	3	4	5	6
$q_{(a,b)}(k$	$\frac{a}{2} - a$	$2ab - \frac{a}{2} - \frac{b}{2} + \frac{1}{4}$	$\frac{1}{4} - ab$	$\frac{1}{4}$	$\frac{1}{4} - \frac{a}{2} + ab$	$\frac{a}{2} + \frac{b}{2} - 2ab$	ab

Notice that $q_{(a,b)} = q_{(l,m)} \Leftrightarrow q_{(a,b)}(k) = q_{(l,m)}(k)$ $for\ k = 0,1 \ldots 6.$

S0 $q_{(a,b)}(0) + q_{(a,b)}(6) = q_{(l,m)}(0) + q_{(l,m)}(6)$

That is $\frac{a}{2} - ab + ab = \frac{l}{2} - lm + lm \Rightarrow \frac{a}{2} = \frac{l}{2}$ which gives $a = l$

Also $q_{(a,b)}(6) = q_{(l,m)}(6) \Rightarrow ab = lm \Rightarrow b = m\ (since\ a = l)$

Thus the distribution $q_{(a,b)}$ is indeed a two parameter family. Hence $p_W = p_X \circ p_Y$ implies $p_W(k) = q_{(a,b)}(k)$.

Step (iii): To find the G_W :

To find $B_W(r)$ first we need to find $G_W(p_W)$ the possibility distribution over p_W associated with the random variable W. But using extension principle we have

$G_W(p_W) = sup_{p_X,p_Y}(B_X(s) \wedge B_Y(t))$ where $s = \sum_{x=0}^3 A_X(x)p_X(x)$ and $t = \sum_{y=0}^3 A_Y(y)p_Y(y)$ (1)

subject to $p_W = p_X \circ p_Y$ (2)

But we have already seen that $p_X = p_a$ and $p_Y = p_b \Leftrightarrow p_W = p_X \circ p_Y = q_{(a,b)}$.

Hence the condition $p_W = p_X \circ p_Y$ implies that $p_W = q_{(a,b)}$ if and only if $p_X = p_a$ and $p_Y = p_b$.

So in equation 1, the supremum is taken over a single case $p_X = p_a$ and $p_Y = p_b$.

So $G_W(p_W) = \min(B_X(s), B_Y(t))$ (3)

After calculation we get s = 0.7-0.5a and t= 0.55-0.4b

So $G_W(p_W) = \min(\frac{10s-5}{2}, \frac{20t-7}{4})$ (4)

Step (iv): To find B_W :

Next to calculate the membership function of B_W using the formula

$B_W(D) = sup_{p_W} G_W(p_W)$ subject to $D = \sum A_W(w)p_W(w)$. (5)

Here $D = \sum_0^6 A_W(w)p_W(w) = \frac{3}{5} - \frac{3a}{20} - \frac{b}{20} + \frac{ab}{5}$

$= \frac{113}{160} - \frac{s}{4} - \frac{23t}{40} + st$ (6)

So using equations 4 and 5 we get

$B_W(D) = sup_{s,t}[\min(\frac{10s-5}{2}, \frac{20t-7}{4})]$

Where the supremum is to be calculated over the points (s,t) satisfying

$0.5 \leq s \leq 0.7$ (7)

$0.35 \leq t \leq 0.55$ (8)

$$D = \frac{113}{160} - \frac{s}{4} - \frac{23t}{40} + st \qquad (9)$$

It is to be understood that, for a given D, if no point (s,t) satisfying (7),(8) and (9) exists then $B_W(D) = 0$.
Now for a given D, if there exist points (s,t) satisfying (7),(8) and (9), then the calculation proceeds as follows
Once D is given, it is treated as a constant. Then (9) defines a curve C, a rectangular hyperbola

$$\left(s - \frac{23}{40}\right)\left(t - \frac{1}{4}\right) = \frac{16D-9}{16} \quad \text{in ST plane.}$$

(7) and (8) define a rectangle PQRS given by P(0.5,0.35) , Q(0.7,0.35), R(0.7,0.55) and S(0.5,0.55).
Let G be the area consisting of points which lies inside or on rectangle PQRS.

$$\therefore B_W(D) = sup_{C\cap G}[\min\left(\frac{10s-5}{2}\right), \left(\frac{20t-7}{4}\right)]$$

In the triangular region T_1 : PRS $\frac{10s-5}{2} \leq \frac{20t-7}{4}$ and in the region T_2 : PQR $\frac{20t-7}{4} \leq \frac{10s-5}{2}$.

$$\therefore B_W(D) = Max\left\{sup_{C\cap T_1}\left[\min\left(\frac{10s-5}{2}\right), \left(\frac{20t-7}{4}\right)\right], sup_{C\cap T_2}\left[\min\left(\frac{10s-5}{2}\right), \left(\frac{20t-7}{4}\right)\right]\right\}$$

$$= Max[sup_{C\cap T_1}\left(\frac{10s-5}{2}\right), sup_{C\cap T_2}\left(\frac{20t-7}{4}\right)]$$

It is clear that supremum of $\left(\frac{10s-5}{2}\right)$ occurs at a point in $C \cap T_1$ with maximum possible value of s. Obviously this point lies on the line PR where $\frac{10s-5}{2} = \frac{20t-7}{4}$

If the supremum occurs at s_0, then $sup_{C\cap T_1}\left(\frac{10s-5}{2}\right) = \frac{10s_0-5}{2}$ (10)

where $\frac{10s_0-5}{2} = \frac{20t_0-7}{4}$ (11)

and (s_0, t_0) satisfies (9)and (11).
Arguing similarly we get,

$$sup_{C\cap T_2}\left(\frac{20t-7}{4}\right) = \frac{20t_0-7}{4} \qquad (12)$$

where (s_0, t_0) satisfies (9) and (11).

That is $sup_{C\cap T_1}\left(\frac{10s-5}{2}\right)$ *and* $sup_{C\cap T_2}\left(\frac{20t-7}{4}\right)$ occurs at the same point on the common boundary line PR and both the values are equal. Thus $B_W(D) = sup_{C\cap T_1}\left(\frac{10s-5}{2}\right) = \frac{10s_0-5}{2}$ where s_0 satisfies (9) and (11). From (11), $t_0 = \frac{20s_0-3}{20}$.

Substituting in (3) we get the following quadratic equation in s : $D = s_0^2 - \frac{39}{40}s_0 + \frac{317}{400}$.

Solving this we get $s_0 = \frac{39 \pm \sqrt{6400D-3551}}{80}$

Since $\frac{39-\sqrt{6400D-3551}}{80} \leq \frac{39}{80} = .4875$ lies outside the rectangle PQRS it can be neglected.

$B_W(D) = \frac{\sqrt{6400D-3551}-1}{16}$ when $0.555 \leq D \leq 0.6$

Example 2:
Consider now a random variable W =XY. The problem is to compute the Z-valuation of W.
That is to find the possibility distribution over *P*, G_W, associated with the random variable W.
Let X and Y be discrete random variables.
A_X and A_Y are the membership functions of X and Y given by

$$A_X(x) = \frac{0.7}{1} + \frac{0.5}{2} + \frac{0.3}{3}$$

$$A_Y(y) = \frac{0.6}{1} + \frac{0.3}{2} + \frac{0.4}{4}$$

The possibility distributions B_X and B_Y are given by

$$B_X(s) = \frac{6-10s}{3} \quad \text{if } 0.3 \leq s \leq 0.6$$

$$B_Y(t) = \frac{5-10t}{2} \quad \text{if } 0.3 \leq t \leq 0.5$$

Each of these generates possibility distribution G_i over the space of probability distribution *P* as

$G_1(p) = B_X(\int A_X(u)p_X(u)du)$ and

$G_2(p) = B_Y(\int A_Y(u)p_Y(u)du)$ where p_X and p_Y are the probability distributions of X and Y given by

X	1	2	3
p_X	a	a	$1-2a$

Y	1	2	**4**
p_Y	b	$1-2b$	b

Let us denote the Z-valuation of W = XY by W isz (A_W, B_W). Our aim is to find the Z-valuation of W = XY. That is to find the possibility distributions $A_W(w)$, and $B_W(r)$.

Step (i): To calculate A_W :-

Now $A_W(w) = Sup_v(A_X(v) \wedge A_Y(w/v))$

Thus $A_W(w) = \frac{0.6}{1} + \frac{0.5}{2} + \frac{0.3}{3} + \frac{0.4}{4} + \frac{0.3}{6} + \frac{0.4}{8} + \frac{0.3}{12}$

Step(ii): To calculate the probability distribution of W:

The probability distribution of W is given by

$$Prob(w = k) = \sum_0^6 p_a(u) p_b(k/u)$$

Since the probability distribution depends on two parameters a, and b let us denote it by $q_{(a,b)}$.

The distribution $q_{(a,b)}$ is given by the following table

k	**1**	**2**	**3**	**4**	**6**	**8**	**12**
$q_{(a,b)}(k$	ab	$a-ab$	$b-2ab$	$a-ab$	$1-2a-2b+4ab$	ab	$b-2ab$

Notice that if $q_{(a,b)} = q_{(l,m)}$

then $q_{(a,b)}(1) + q_{(a,b)}(2) = q_{(l,m)}(1) + q_{(l,m)}(2)$

That is $ab + a - ab = lm + l - lm$

$which \;\Rightarrow a = l$

Also $q_{(a,b)}(1) = q_{(l,m)}(1) \Rightarrow ab = lm$

$\Rightarrow b = m \; (since \; a = l)$

Thus the distribution $q_{(a,b)}$ is indeed a two parameter family.

Hence $p_W = p_X \circ p_Y$ implies $p_W(k) = q_{(a,b)}(k)$.

Step (iii): To find the G_W :

To find $B_W(r)$ first we need to find $G_W(p_W)$ the possibility distribution over p_W associated with the random variable W

But using extension principle we have

$G_W(p_W) = sup_{p_X, p_Y}(B_X(s) \wedge B_Y(t))$ (1)

where $s = \sum_{x=0}^3 A_X(x) p_X(x)$ and $t = \sum_{y=0}^3 A_Y(y) p_Y(y)$

subject to $p_W = p_X \circ p_Y$ (2)

But we have already seen that $p_X = p_a \; and \; p_Y = p_b \Leftrightarrow p_W = p_X \circ p_Y = q_{(a,b)}$.

Hence the condition $p_W = p_X \circ p_Y$ implies that $p_W = q_{(a,b)}$ if and only if $p_X = p_a$ and $p_Y = p_b$.

So in equation 1, the supremum is taken over a single case $p_X = p_a$ and $p_Y = p_b$.

So $G_W(p_W) = \min(B_X(s), B_Y(t)$ (3)

After calculation we get

s = 0.6a+0.3

t= 0.4b+0.3

So $G_W(p_W) = \min(\frac{6-10s}{3}, \frac{5-10t}{2})$ (4)

Step (iv): To find B_W :

Next to calculate the membership function of B_W using the formula

$B_W(D) = sup_{p_W} G_W(p_W)$

Subject to $D = \sum A_W(w)p_W(w)$. (5)

Here $D = \sum_0^6 A_W(w)p_W(w)$

$= \frac{3}{10} + \frac{3}{10}a + \frac{1}{10}ab$

$= \frac{3}{16} + \frac{3}{8}s - \frac{1}{8}t + \frac{5}{12}st$ (6)

So using equations 4 and 5 we get

$B_W(D) = sup_{s,t}[\min\left(\frac{6-10s}{3}, \frac{5-10t}{2}\right)]$

Where the supremum is to be calculated over the points (s,t) satisfying

$0.3 \le s \le 0.6$ (7)

$0.3 \le t \le 0.5$ (8)

$D = \frac{3}{16} + \frac{3}{8}s - \frac{1}{8}t + \frac{5}{12}st$ (9)

It is to be understood that, for a given D, if no point (s,t) satisfying (7),(8) and (9) exists then $B_W(D) = 0$.
Now for a given D, if there exist points (s,t) satisfying (7),(8) and (9), then the calculation proceeds as follows
Once D is given, it is treated as a constant. Then (9) defines a curve C, a rectangular hyperbola

$\left(s - \frac{3}{10}\right)\left(t + \frac{9}{10}\right) = \frac{60D-18}{25}$ in ST plane.

(7) and (8) define a rectangle PQRS given by P(0.3,0.3) , Q(0.6,0.3), R(0.6,0.5) and S(0.3,0.5).
Let G be the area consisting of points which lies inside or on rectangle PQRS.

$\therefore B_W(D) = sup_{C\cap G}[\min\left(\frac{6-10s}{3}\right), \left(\frac{5-10t}{2}\right)]$

It can be verified that, in the triangle region T_1 having boundary PRS

$\frac{6-10s}{3} \le \frac{5-10t}{2}$

And in the region T_2, with the boundary PQR

$\frac{5-10t}{2} \le \frac{6-10s}{3}$

$\therefore B_W(D) = Max\left\{sup_{C\cap T_1}\left[\min\left(\frac{6-10s}{3}\right), \left(\frac{5-10t}{2}\right)\right], sup_{C\cap T_2}\left[\min\left(\frac{6-10s}{3}\right), \left(\frac{5-10t}{2}\right)\right]\right\}$

$= Max[sup_{C\cap T_1}\left(\frac{6-10s}{3}\right), sup_{C\cap T_2}\left(\frac{5-10t}{2}\right)]$

It is clear that supremum of $\left(\frac{6-10s}{3}\right)$ occurs at a point in $C \cap T_1$ with minimum possible value of s. Obviously this point lies on the line PR where $\frac{6-10s}{3} = \frac{5-10t}{2}$

If the supremum occurs at s_0, then $sup_{C\cap T_1}\left(\frac{6-10s}{3}\right) = \frac{6-10s_0}{3}$ (10)

Where $\frac{6-10s_0}{3} = \frac{5-10t_0}{2}$ (11)

and (s_0, t_0) satisfies (9).

Arguing similarly we get,

$sup_{C\cap T_2}\left(\frac{5-10t}{2}\right) = \frac{5-10t_0}{2}$ (12)

where (s_0, t_0) satisfies (9) and (11).

That is $sup_{C\cap T_1}\left(\frac{6-10s}{3}\right)$ and $sup_{C\cap T_2}\left(\frac{5-10t}{2}\right)$ occurs at the same point on the common boundary line PR and both the values are equal.

Thus $B_W(D) = sup_{C\cap T_1}\left(\frac{6-10s}{3}\right) = \frac{6-10s_0}{3}$ where s_0 satisfies (9) and (11).

From (11), $t_0 = \frac{20s+3}{30}$

Substituting in (3) we get the following quadratic equation in s

$D = \frac{5}{18}s^2 + \frac{1}{3}s + \frac{7}{40}$.

Solving this we get $s = \frac{-6\pm\sqrt{360D-27}}{10}$

Since $\frac{-6-\sqrt{360D-27}}{10} < 0$,lies outside the rectangle PQRS it can be neglected.

Now $B_W(D) = \frac{12-\sqrt{360D-27}}{3}$ when $0.3 \leq D \leq 0.475$

V. Conclusion

The two examples given in the above section help us to understand the process of computation with Zadeh's Z-numbers. However it is to be noted even in the simplest cases we end up with difficult optimization problems. In these examples it was possible to obtain exact solutions. But in many other cases where the underlying pdf is Poisson, normal, exponential etc it may not be possible to obtain exact solutions. Hence it is important to evolve a numerical method to deal with computation with z-numbers.

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