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Waiting Time Analysis of A Single Server Queue in an Out-Patient Clinic

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Abstract: *Waiting on a queue is not usually interesting, but reduction in this waiting time usually requires planning and extra investments. Queuing theory was developed to study the queuing phenomenon in the commerce, telephone traffic, transportation, etc [Cooper (1981), Gross and Harris (1985)]. The rising population and health-need due to adverse environmental conditions have led to escalating waiting times and congestion in hospital Emergency Departments (ED). It is universally acknowledged that a hospital should treat its patients, especially those in need of critical care, in timely manner. Incidentally, this is not achieved in practice particularly in government owned health institutions because of high demand and limited resources in these hospitals. In this paper, we develop the equations of steady state probabilities. Example from a out-patient department of a clinic was presented to demonstrate how the various parameters of the model influence the behavior of the system.*

Keywords: *M/M/1 queue, Poisson arrival, Exponential distribution,*

I. Introduction

Waiting on a queue is not usually interesting, but reduction in this waiting time usually requires planning and extra investments. Queueing theory involves the mathematical study of waiting lines. Queuing systems is a system consisting of flow of customers requiring service where there is some restriction in the service that can be provided. Three main elements are commonly identified in any service centre namely; a population of customers, the service facility and the waiting line. We usually investigate queues in order to answer questions like, the mean waiting time in the queue, the mean response time in the system, utilization of service facilities, distribution of number of customers in the queue, etc. Decisions regarding the amount of capacity to provide a service must be made frequently by any service provider for optimality. The application of operations research (queuing model in particular) brings greater versatility, variety and control to the management of healthcare organization (Ginter et al., 1998)..

Single server queue has extensively been studied by researchers and applied. Queuing theory was developed to study the queuing phenomenon in the commerce, telephone traffic, transportation, etc (Cooper (1981),Gross and Harriss (1985)). To improve patient satisfaction, the performance of key processes has to be improved (Torres and Guo, (2004)) . The use of queueing theory in a healthcare setting was rarely used until the pioneering work of Bailey, (1952) appeared. In this paper queueing theory was used to develop an out-patient clinic scheduling system that gave acceptable results for patients (in terms of waiting time) and staff (in terms of utilisation). Homogeneity of patients was assumed as far as their service time distributions were concerned, and also it was assumed that all patients arrived for appointments on time. Adeleke R. A et al (2009) considered application of queuing theory to the waiting time of out-patients in a hospital. The average number of patients and the time each patient waits in the hospital were determined.

We will consider a single server queue where the inter-arrival times and service times are exponentially distributed. We investigate it in order to determine the mean waiting times, number of customers in the system and the distribution of busy period of the server.

II. The M/M/1 Queue

In this section we will analyze the model with exponential inter-arrival times with mean $1/\lambda$ and exponential service times with mean $1/\mu$ and a single server. The exponential distribution allows for a very simple description of the state of the system at any time t. First Come First Served (FCFS) service discipline is assumed. It is also required that $\lambda/_{\mu} = \rho < 1$ to achieve stability. The number ρ is the fraction of time the server is working.

Let $p_n(t)$ denote the probability that there are n $(n = 0, 1, 2, \ldots)$ customers in the system at time t. Based on the lack memory property of exponential distribution we have

$$p_0(t + \Delta t) = (1 - \lambda\Delta t)p_0(t) + \mu\Delta t p_1 + o(\Delta t)$$

$$p_n(t + \Delta t) = \lambda\Delta t p_{n-1} + (1 - (\lambda + \mu)\Delta t)p_n(t) + \mu\Delta t\, p_{n+1}(t) + o(\Delta t) \qquad (1)$$

By letting $\Delta t \to 0$, we obtain a differential equation for the probabilities $p_n(t)$.

$\frac{d}{dt}p_n(t) = -(\lambda_n + \mu_n)p_n + \lambda_{n-1}p_n(t) + \mu_{n+1}p_{n+1}(t)$ (2)

For stationarity, as $t \to \infty$, $p_n(t) \to p_n$. Hence,

$$\mu_{n+1}p_{n+1}(t) = (\lambda_n + \mu_n)p_n - \lambda_{n-1}p_n(t) \quad , \quad n > 1 \qquad (3)$$

$\lambda_0 p_0 = \mu_1 p_1$

From which we readily obtain $\quad p_n = \frac{\lambda_0\lambda_1\lambda_2 \dots \lambda_n}{\mu_1\mu_2\mu_3 \dots \mu_n} p_0$

$\sum_{i=0}^{\infty} p_i = p_0\left(1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0\lambda_1}{\mu_1\mu_2} + \cdots\right) = p_0 s = 1$ (4)

So that if $s < \infty$ $\quad p_n = [(\lambda_0\lambda_1\lambda_2 \dots \lambda_{n-1})/(\mu_1\,\mu_2\mu_3 \dots \mu_n)]s^{-1}$

Since $\lambda_n = \lambda$ and $\mu_n = \mu$, $\quad p_n = \left(\lambda/_{\mu}\right)p_0 = \rho^n p_0$ (5)

And $\quad p_n = \rho^n(1-\rho)$ (6)

2.1 Mean Performance Measure

From the above equations we can now derive the mean number of customers in the system and the mean time spent in the system. Let N denote the number of customers in the system, then the expected number of customers in the system is given by

$\mathbb{E}(N) = \sum_{n=0}^{\infty} np_n = \frac{\rho}{1-\rho}$ (7)

And if T is the time spent in the system, then by applying the well known Little's formula, the expected time spent in the system is given by

$\mathbb{E}(T) = \frac{1/\mu}{1-\rho}$ (8)

If we look at expressions for $\mathbb{E}(N)$ and $\mathbb{E}(T)$, we see that both quantities goes to infinity as ρ tends to unity. The expected waiting time $\mathbb{E}(W)$ is derived by subtracting the mean service time from $\mathbb{E}(T)$. That is $\mathbb{E}(W) = \mathbb{E}(T) - \frac{1}{\mu} = \frac{\rho/\mu}{1-\rho}$ Denote by N_a the number of customers in the system just before the arrival of a customer and let B_k be the service time of the k^{th} customer. Since the exponential service times is memoryless, the random variables B_k are independently and identically distributed with mean $1/\mu$. So that we have $T = \sum_{k=1}^{N_a+1} B_k$.

$P(T > t) = P\left(\sum_{k=1}^{N_a+1} B_k > t\right) = \sum_{n=0}^{\infty} P(\sum_{k=1}^{n+1} B_k > t)P(N_a = n)$ (9)

Poisson Arrival See Time Average (PASTA) states that the fraction of customer finding on arrival n customers in the system is equal to the fraction of time there are n customers in the system, so,

$P(N_a = n) = p_n = \rho^n(1-\rho)$ (10)

Using this fact we have,

$$P(T > t) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{(\mu t)^k}{k!} e^{-\mu t}\rho^n(1-\rho)$$
$$= e^{-\mu(1-\rho)t}, \qquad t \geq 0 \quad (11)$$

Where we have also used the fact that $\sum_{k=n}^{n+1} B_k$ is Erlang-(n+1) distributed. This gives the distribution of sojourn time, which is the waiting time plus the service time.

III. Example Of A Nigerian Hospital's Out-Patient Clinic

The arrival pattern of the patients into the hospital is usually specified by the inter-arrival time, which is the time between successive patient arrivals into the hospital service facility. Each patient arrives singly, and neither balking nor reneging occurs. Although the hospital opens by 8:00am everyday but it was observed that patients usually arrive from 8:30am. A total of 798 patients arrived for service during this two weeks of observation.

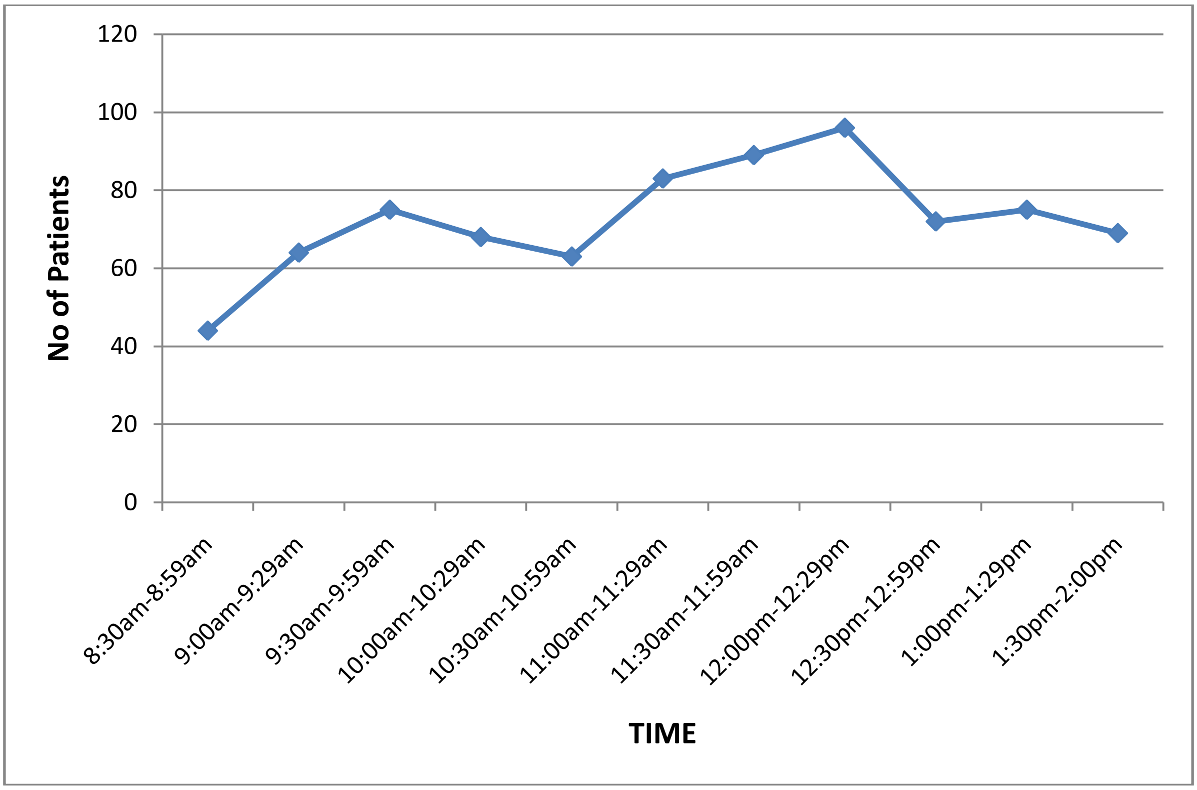

Fig1: distribution of arrival patients

The total number of patients arrival and the mean for the arrivals at different intervals were calculated to be $\lambda = 57$

3.1 Service Distribution

In the hospital .the service rendered to patients is exponentially distributed which eventually satisfy a Poisson process. When service is available, service is made by the doctor immediately a patient arrives therefore, service are rendered to the patients at the same time of arrival, starting from 8:30am to 2:00pm every day. At total of 568 patients were served during the two weeks of observation.
The figure below represents the distribution of the service time.

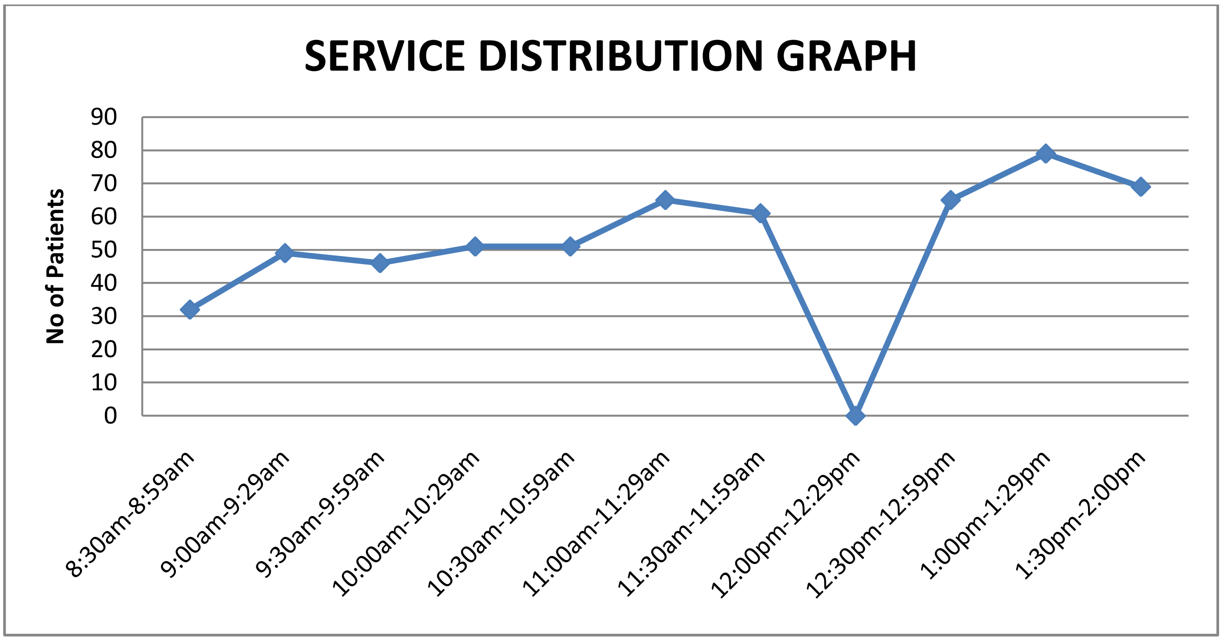

Fig2: distribution of service to patients

The total number of patients served by the doctor and the mean of service rendered at different time intervals was calculated. The mean service rate has its maximum [highest] at 1:00pm-1:29pm . The mean service rate was calculated to be $\mu = 40.57$ patients per day.

3.2 Measure Of Performance Of The System

Since,
(i) The mean arrival rate $\lambda = 57$
(ii) The mean service rate $\mu = 40.57$
(iii) The number of severs $N = 1$

Then,
Therefore, $1/_{\lambda} = 1/_{57} = 0.0175 = (0.0175 * 60)minutes = 1.05patients\ per\ minute$
the average time between service $= 1/_{\boldsymbol{\mu}}$ and $\mu = 40.57$
Therefore, $1/_{\mu} = 00246 = 1.48patients\ per\ minutes$
the potential utilization of service facility ρ
$\rho = \frac{1/_{\lambda}}{1/_{\mu}} = \frac{0.0175}{0.0246} = 0.7114$
Therefore, $T = [0.7114 * 100]\% = 71\%$
Hence the efficiency ratio is 71%

The mean number of patients either waiting on queue or in service $P_i = (x)(y)^i$ where $i = 0,1,2, \ldots$
Since the number of patients in the system has a geometric distribution assume $i = 0$, but $\rho = 0.7114$

Therefore $P_i = (x)(y)^i$ where $x + y = 1$
Hence $x = 1 - y = 1 - 0.7114 = 0.2886$
$\Rightarrow x = 0.2886 \approx 0.29$
$x = (0.29 * 100)\% = 29\%$
$\Rightarrow P_i = (x)(y)^i = x$ Since $i = 0$
Therefore, the successful probability $x = 0.29$ is the idle period
So the number of patients in the system
$$N = \frac{1}{1-y} - 1$$
But $y = 0.7114$
Therefore, $\mathrm{N} = \frac{1}{(1-0.7114)} - 1$
$$\mathrm{N} = \frac{1}{0.2886} - 1 = 3.465 - 1$$
$\mathrm{N} = 2.5 \approx 3$
The mean queue length $[\mathrm{L_Q}]$
$\mathrm{L_q} = 1 - \mathrm{P_i}$

Since the successive probability $\mathrm{x} = 0.29$
From $\mathrm{P_i} = (\mathrm{x})(\mathrm{y})^{\mathrm{i}}$, since $\mathrm{i} = 0$
$\Rightarrow \mathrm{P_i} = \mathrm{x}$ and
$\mathrm{L_q} = 1 - \mathrm{P_i} = 1 - 0.29$
$\mathrm{L_q} = 0.71$

The mean waiting time of a patient on queue $\mathrm{W_q}$
$$\mathrm{W_q} = \frac{\mathrm{L_q}}{\lambda}$$
Where $\mathrm{L_q} = 0.71, \lambda = 57$
Therefore, $\mathrm{W_q} = \frac{\mathrm{L_q}}{\lambda} = \frac{0.71}{57} = 0.0125$
$$\Rightarrow \mathrm{W_q} = 0.0125 = (0.0125 * 60)\mathrm{seconds} = 0.75\mathrm{seconds}$$
The mean time a patient spent in the system $[\mathrm{W_S}]$
$\mathrm{W_S} = \mathrm{W_q} + 1/_{\mu}$

Where W_q is the mean waiting of the patient on queue and $1/\mu$ is the average time spent for service
But $W_q = 0.75$ and $1/\mu = 0.0246$
$\Rightarrow W_S = 0.75 + 0.0246 = 0.7746$
Therefore $W_S = [0.7746 * 60]\text{minutes} = 46.48\text{mins}$

IV. Conclusion

A single server queue has been investigated and its performance measure. We developed the equations of steady state probabilities. Example was presented to demonstrate how the various parameters of the model influence the behavior of the system. There are many practical situations that can be described by exponential distribution; this makes such situations to be well described by this model.

References

[1]. Adeleke, R. A., Ogunwale, O. D., & Halid, O. Y. (2009). Application of Queuing Theory to WaitingTime of Out-Patients in Hospitals". Pacific Journal of Science and Technology , 10(2):270-274.
[2]. Blake, F. I. (1978). An Introduction to Probability Models. USA: John Wiley and Sons Inc.
[3]. Bailey N. T. J. (1952).. A study of queues and appointment systems in hospital out- patient departments, with special reference to waiting-times. Journal of the Royal Statistical Society. Series B (Methodological), 14(2):185–199,
[4]. Churchman, G. W., Ackoff, R. C., & Arnoff., F. C. (1957). Introduction to Operation Research. New York, NY: John Willey and Sons:.
[5]. Gross, D., & Harris, C. M. (1985). Fundamentals of Queuing Theory. New York.: 2nd edition John Wiley.
[6]. Ginter P. M., Swayre, L. M and Duncan, W. J (1998). Stategic Management of Health care Organisations, Cambridge Blackwell Systems
[7]. Lieberman, & Hillier. (1986). Introduction to Operations Research. USA: John Wiley and Sons Inc.
[8]. Torres, J.E. and Guo, K.L. (2004) : Quality improvement techniques to improve patient satisfaction. International Journal of Health Care Quality Assurance, 17 (6), 334-338.
[9]. Taha, H. A. (1987). Operation Research: An Introduction. New York, NY: Fourth Edition. Macmillan Publishing .