IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 11, Issue 3 Ver. IV (May - Jun. 2015), PP 67-71
www.iosrjournals.org

On r- Riemann-Liouville Fractional Calculus Operators and k-Wright function

NEETI GHIYA
(Department Of Mathematics, R. V. College of Engineering, Bangalore, India)

Abstract: *In this paper some certain results of r-Riemann-Liouville fractional integration and differentiation of k-Wright function are established. A new transform Elzaki transform of k-Wright function and r-Riemann-Liouville fractional integral are also obtained. Corollaries of the main theorems have also been derived.*
Keywords: *r-Riemann-Liouville fractional integral and differential operators, k-Wright function, Elzaki transform*

I. Introduction

Diaz and Pariguan [1] introduced k-Pochhammer symbol $(x)_{n,k}$ and k-Gamma function $\Gamma_k(z)$ in the following form

$$(x)_{n,k} = x(x + k)(x + 2k) \dots (x + (n-1)k), \tag{1}$$

where $x \in C,\ k \in R$ and $n \in N$.

$$\Gamma_k(z) = \int_0^\infty e^{-\frac{t^k}{k}} t^{z-1}\, dt, \text{ where } k \in R,\ z \in C,\ Re(z) > 0, \tag{2}$$

and $\Gamma_k(x + k) = x\,\Gamma_k(x)$, (3)

$$\Gamma_k(\eta) = (k)^{\frac{\eta}{k}-1}\Gamma\left(\frac{\eta}{k}\right). \tag{4}$$

Let f be a sufficiently well behaved function with support in R^+ and let λ be a real number such that λ> 0. The r- Riemann-Liouville fractional integral of order λ is given by Mubeem and Habibullah [3]

$$\left(I_{r,a}^{\lambda} f\right)(x) = \frac{1}{r\Gamma_r(\lambda)} \int_a^x (x-t)^{\frac{\lambda}{r}-1} f(t)\, dt, \tag{5}$$

and

$$\left(I_r^{\lambda} f\right)(x) = \frac{1}{r\Gamma_r(\lambda)} \int_0^x (x-t)^{\frac{\lambda}{r}-1} f(t)\, dt, \tag{6}$$

where $r \in R,\ \lambda > 0,\ \Gamma_r(\lambda)$ is k-Gamma function. (6) is the special case of (5).

Let λ be a real number such that $0 < \lambda \le 1$ The r-Riemann-Liouville fractional derivative was introduced as (cf.[8])

$$\left(D_r^{\lambda} f\right)(x) = \left(\frac{d}{dx}\right)^s \left(I_r^{s-\lambda} f\right)(x) \qquad \lambda \in R,\ 0 < \lambda \le 1 \tag{7}$$

r-Riemann-Liouville fractional integral and derivative operators are generalization of Riemann-Liouville fractional integral and derivative operators. If we take r = 1, then (6) and (7) reduce to Riemann-Liouville fractional integral operator and Riemann-Liouville fractional derivative respectively (cf.[6]), defined as

$$\left(I_{0+}^{\lambda} f\right)(x) = \frac{1}{\Gamma(\lambda)} \int_0^x (x-t)^{\lambda-1} f(t)\, dt, \tag{8}$$

$$\left(D_{0+}^{\lambda} f\right)(x) = \left(\frac{d}{dx}\right)^s \left(I_{0+}^{s-\lambda} f\right)(x). \tag{9}$$

Let $k \in R,\ \alpha, \beta, \eta \in C,\ Re(\alpha) > 0,\ Re(\beta) > 0$.The k-Wright function was defined by Romero L. G. and Cerutti R. A.[4] as

$$W_{k,\alpha,\beta}^{\eta}(z) = \sum_{n=0}^{\infty} \frac{(\eta)_{n,k}}{\Gamma_k(\alpha n+\beta)} \frac{z^n}{(n!)^2}, \tag{10}$$

where $(\eta)_{n,k}$ is k-Pochhammer symbol and $\Gamma_k(\alpha n + \beta)$ is k-Gamma function.

Taking $k \to 1$ and η = 1 k-Wright function reduces to the Wright function is defined as

$$W_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\,\Gamma(\alpha n+\beta)};\ \alpha > -1;\ \beta \in C, \tag{11}$$

where Γ(z) is the Euler Gamma function.

A new integral transform called Elzaki transform introduced by[7] defined for functions of exponential order, is proclaimed consider functions in the set A defined by

$$A=\left\{f(t)\,|\,\exists\, M, k_1, k_2>0\,|\, f(t) < M\, e^{\frac{|t|}{k_j}},\ \text{if } t \in (-1)^j X \in [0,\infty)\right\} \tag{12}$$

Elzaki transform defined as

$$E[f(t)]= u^2 \int_0^\infty e^{-t}\, f(u\,t)\, dt = T(u)\,, \qquad u \in (k_1, k_2) \tag{13}$$

II. Main Result

Lemma1. If $\beta \in \mathbb{C}$, $\mathrm{Re}(\beta) > 0$, $\lambda > 0$, $r \in R$, $k \in R$ and $Re\left(\beta + \frac{\lambda}{r} k\right) > 0$,

then $$\left(I_r^{\lambda}\, t^{\frac{\beta}{k}-1}\right)(x) = \left(\frac{k}{r}\right)^{\frac{\lambda}{r}} x^{\frac{\lambda}{r}+\frac{\beta}{k}-1} \frac{\Gamma_k(\beta)}{\Gamma_k\left(\beta+\frac{\lambda}{r}k\right)} \tag{14}$$

Proof. $\left(I_r^{\lambda}\, t^{\frac{\beta}{k}-1}\right)(x) = \frac{1}{r\Gamma_r(\lambda)} \int_0^x (x-t)^{\frac{\lambda}{r}-1}\, t^{\frac{\beta}{k}-1} dt$

taking t = xv then dt = x dv

$$= \frac{1}{r\Gamma_r(\lambda)}\, x^{\frac{\lambda}{r}+\frac{\beta}{k}-1} \int_0^1 (1-v)^{\frac{\lambda}{r}-1}\, v^{\frac{\beta}{k}-1} dv$$

$$= \frac{1}{r\Gamma_r(\lambda)}\, x^{\frac{\lambda}{r}+\frac{\beta}{k}-1} \frac{\Gamma\left(\frac{\beta}{k}\right)\Gamma\left(\frac{\lambda}{r}\right)}{\Gamma\left(\frac{\beta}{k}+\frac{\lambda}{r}\right)}$$

$$= \frac{1}{(r)^{\frac{\lambda}{r}}\Gamma\left(\frac{\lambda}{r}\right)}\, x^{\frac{\lambda}{r}+\frac{\beta}{k}-1} \frac{\Gamma\left(\frac{\beta}{k}\right)\Gamma\left(\frac{\lambda}{r}\right)}{\Gamma\left(\frac{\beta}{k}+\frac{\lambda}{r}\right)}$$

$$= \frac{1}{(r)^{\frac{\lambda}{r}}}\, x^{\frac{\lambda}{r}+\frac{\beta}{k}-1} \frac{\Gamma_k(\beta)\, k^{1-\frac{\beta}{k}}}{\Gamma_k\left(\beta+\frac{\lambda}{r}k\right) k^{1-\frac{\lambda}{r}-\frac{\beta}{k}}}$$

$$= \left(\frac{k}{r}\right)^{\frac{\lambda}{r}} x^{\frac{\lambda}{r}+\frac{\beta}{k}-1} \frac{\Gamma_k(\beta)}{\Gamma_k\left(\beta+\frac{\lambda}{r}k\right)}.$$

Theorem 1. If α, β, η be complex numbers that $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, $\mathrm{Re}(\eta) > 0$, $k > 0$, $r \in \mathbb{R}$, $\lambda > 0$, $w \in \mathbb{C}$ then,

$$\left(I_r^{\lambda}\, t^{\frac{\beta}{k}-1} W^{\eta}_{k,\alpha,\beta}\left(w t^{\frac{\alpha}{k}}\right)\right)(x) = \left(\frac{k}{r}\right)^{\frac{\lambda}{r}} x^{\frac{\lambda}{r}+\frac{\beta}{k}-1}\, W^{\eta}_{k,\alpha,\beta+\frac{\lambda}{r}k}\left(w x^{\frac{\alpha}{k}}\right) \tag{15}$$

Proof. By virtue of (6) and (10), we have

$$\left(I_r^{\lambda}\, t^{\frac{\beta}{k}-1} W^{\eta}_{k,\alpha,\beta}\left(w t^{\frac{\alpha}{k}}\right)\right)(x) = \frac{1}{r\Gamma_r(\lambda)} \int_0^x (x-t)^{\frac{\lambda}{r}-1}\, t^{\frac{\beta}{k}-1} W^{\eta}_{k,\alpha,\beta}\left(w t^{\frac{\alpha}{k}}\right) dt$$

$$= \frac{1}{r\Gamma_r(\lambda)} \int_0^x (x-t)^{\frac{\lambda}{r}-1}\, t^{\frac{\beta}{k}-1} \sum_{n=0}^{\infty} \frac{(\eta)_{n,k}}{\Gamma_k(\alpha n+\beta)} \frac{w^n t^{\frac{\alpha n}{k}}}{(n!)^2}\, dt,$$

interchanging the order of integration and summation, we get

$$= \sum_{n=0}^{\infty} \frac{(\eta)_{n,k}}{\Gamma_k(\alpha n+\beta)} \frac{w^n}{(n!)^2} \frac{1}{r\Gamma_r(\lambda)} \int_0^x (x-t)^{\frac{\lambda}{r}-1}\, t^{\frac{\alpha n}{k}+\frac{\beta}{k}-1} dt,$$

solving integral with the help of Lemma 1, it gives

$$= \sum_{n=0}^{\infty} \frac{(\eta)_{n,k}}{\Gamma_k(\alpha n+\beta)} \frac{w^n}{(n!)^2} \left(\frac{k}{r}\right)^{\frac{\lambda}{r}} x^{\frac{\lambda}{r}+\frac{\alpha n}{k}+\frac{\beta}{k}-1} \frac{\Gamma_k(\alpha n+\beta)}{\Gamma_k\left(\alpha n+\beta+\frac{\lambda}{r}k\right)}$$

$$= \left(\frac{k}{r}\right)^{\frac{\lambda}{r}} x^{\frac{\lambda}{r}+\frac{\beta}{k}-1} \sum_{n=0}^{\infty} \frac{(\eta)_{n,k}}{\Gamma_k\left(\alpha n+\beta+\frac{\lambda}{r}k\right)} \frac{w^n x^{\frac{\alpha n}{k}}}{(n!)^2}$$

$$= \left(\frac{k}{r}\right)^{\frac{\lambda}{r}} x^{\frac{\lambda}{r}+\frac{\beta}{k}-1}\, W^{\eta}_{k,\alpha,\beta+\frac{\lambda}{r}k}\left(w x^{\frac{\alpha}{k}}\right).$$

Corollary1.1. If the conditions of the theorem 1 are satisfied with $k \to 1$ and $\eta = 1$, then

$$\lim_{k\to 1}\left(I_r^{\lambda}\, t^{\frac{\beta}{k}-1} W^{1}_{k,\alpha,\beta}\left(w t^{\frac{\alpha}{k}}\right)\right)(x) = \left(I_r^{\lambda}\, t^{\beta-1} W_{\alpha,\beta}(w t^{\alpha})\right)(x)$$

$$= \left(\frac{1}{r}\right)^{\frac{\lambda}{r}} x^{\frac{\lambda}{r}+\beta-1}\, W_{\alpha,\beta+\frac{\lambda}{r}}(w\, x^{\alpha}). \tag{16}$$

Corollary 1.2. If the conditions of the theorem 1 are satisfied with r = 1, then

$$\left(I_{+}^{\lambda}\, t^{\frac{\beta}{k}-1} W_{k,\alpha,\beta}^{\eta}\left(w t^{\frac{\alpha}{k}}\right)\right)(x) = (k)^{\lambda}\, x^{\lambda+\frac{\beta}{k}-1}\, W_{k,\alpha,\beta+\lambda k}^{\eta}\left(w x^{\frac{\alpha}{k}}\right). \tag{17}$$

Lemma2: If $\beta \in C$, $Re(\beta) > 0$, $\lambda \in R$, $0 < \lambda \leq 1$, $r \in R$, $k \in R$ and $Re\left(\beta + \left(\frac{s-\lambda}{r} - s\right) k\right) 0$,

then, $$\left(D_r^{\lambda}\, t^{\frac{\beta}{k}-1}\right)(x) = \left(\frac{k}{r}\right)^{\frac{s-\lambda}{r}} k^{-s} x^{\frac{s-\lambda}{r}+\frac{\beta}{k}-s-1} \frac{\Gamma_k(\beta)}{\Gamma_k\left(\beta + \left(\frac{s-\lambda}{r} - s\right) k\right)}. \tag{18}$$

Proof. $\left(D_r^{\lambda}\, t^{\frac{\beta}{k}-1}\right)(x) = \left(\frac{d}{dx}\right)^{s}\left(I_r^{s-\lambda}\, t^{\frac{\beta}{k}-1}\right)(x)$

$$= \left(\frac{d}{dx}\right)^{s} \frac{1}{r\Gamma_r(s-\lambda)} \int_0^x (x-t)^{\frac{s-\lambda}{r}-1}\, t^{\frac{\beta}{k}-1} dt$$

Taking t = xv then dt = x dv

$$= \left(\frac{d}{dx}\right)^{s} \frac{1}{r\Gamma_r(s-\lambda)}\, x^{\frac{s-\lambda}{r}+\frac{\beta}{k}-1} \int_0^1 (1-v)^{\frac{s-\lambda}{r}-1}\, v^{\frac{\beta}{k}-1} dv$$

$$= \left(\frac{d}{dx}\right)^{s} \frac{1}{r\Gamma_r(s-\lambda)}\, x^{\frac{s-\lambda}{r}+\frac{\beta}{k}-1} \frac{\Gamma\left(\frac{\beta}{k}\right)\Gamma\left(\frac{s-\lambda}{r}\right)}{\Gamma\left(\frac{\beta}{k}+\frac{s-\lambda}{r}\right)}$$

$$= \left(\frac{d}{dx}\right)^{s} \frac{1}{(r)^{\frac{s-\lambda}{r}}\Gamma\left(\frac{s-\lambda}{r}\right)}\, x^{\frac{s-\lambda}{r}+\frac{\beta}{k}-1} \frac{\Gamma\left(\frac{\beta}{k}\right)\Gamma\left(\frac{s-\lambda}{r}\right)}{\Gamma\left(\frac{\beta}{k}+\frac{s-\lambda}{r}\right)}$$

$$= \frac{1}{(r)^{\frac{s-\lambda}{r}}} \frac{\Gamma\left(\frac{\beta}{k}\right)}{\Gamma\left(\frac{\beta}{k}+\frac{s-\lambda}{r}\right)} \left(\frac{d}{dx}\right)^{s} x^{\frac{s-\lambda}{r}+\frac{\beta}{k}-1}$$

$$= \frac{1}{(r)^{\frac{s-\lambda}{r}}} \frac{\Gamma\left(\frac{\beta}{k}\right)}{\Gamma\left(\frac{\beta}{k}+\frac{s-\lambda}{r}\right)} \frac{\Gamma\left(\frac{\beta}{k}+\frac{s-\lambda}{r}\right)}{\Gamma\left(\frac{\beta}{k}+\frac{s-\lambda}{r}-s\right)} x^{\frac{s-\lambda}{r}+\frac{\beta}{k}-1-s}$$

$$= \frac{1}{(r)^{\frac{s-\lambda}{r}}} \frac{\Gamma_k(\beta)\, k^{1-\frac{\beta}{k}}}{\Gamma_k\left(\beta+\frac{s-\lambda}{r}k - sk\right) k^{1-\frac{s-\lambda}{r}-\frac{\beta}{k}+s}}\, x^{\frac{s-\lambda}{r}+\frac{\beta}{k}-s-1}$$

$$= \left(\frac{k}{r}\right)^{\frac{s-\lambda}{r}} k^{-s} x^{\frac{s-\lambda}{r}+\frac{\beta}{k}-s-1} \frac{\Gamma_k(\beta)}{\Gamma_k\left(\beta+\left(\frac{s-\lambda}{r}-s\right)k\right)}$$

Theorem 2. If α, β, η be complex numbers that $Re(\alpha) > 0$, $Re(\beta) > 0$, $Re(\eta) > 0$, $k > 0$, $r \in R$, $\lambda \in R$, $0 < \lambda \leq 1$, $w \in C$ then,

$$\left(D_r^{\lambda}\, t^{\frac{\beta}{k}-1} W_{k,\alpha,\beta}^{\eta}\left(w t^{\frac{\alpha}{k}}\right)\right)(x) = \left(\frac{k}{r}\right)^{\frac{s-\lambda}{r}} k^{-s} x^{\frac{s-\lambda}{r}+\frac{\beta}{k}-s-1}\, W_{k,\alpha,\beta+\left(\frac{s-\lambda}{r}-s\right)k}^{\eta}\left(w x^{\frac{\alpha}{k}}\right) \tag{19}$$

Proof. By virtue of (7) and (10).we have

$$\left(D_r^{\lambda}\, t^{\frac{\beta}{k}-1} W_{k,\alpha,\beta}^{\eta}\left(w t^{\frac{\alpha}{k}}\right)\right)(x) = \left(\frac{d}{dx}\right)^{s}\left(I_r^{s-\lambda}\, t^{\frac{\beta}{k}-1} W_{k,\alpha,\beta}^{\eta}\left(w t^{\frac{\alpha}{k}}\right)\right)(x)$$

$$= \left(\frac{d}{dx}\right)^{s} \frac{1}{r\Gamma_r(s-\lambda)} \int_0^x (x-t)^{\frac{s-\lambda}{r}-1}\, t^{\frac{\beta}{k}-1} W_{k,\alpha,\beta}^{\eta}\left(w t^{\frac{\alpha}{k}}\right) dt$$

$$= \left(\frac{d}{dx}\right)^{s} \frac{1}{r\Gamma_r(s-\lambda)} \int_0^x (x-t)^{\frac{s-\lambda}{r}-1}\, t^{\frac{\beta}{k}-1} \sum_{n=0}^{\infty} \frac{(\eta)_{n,k}}{\Gamma_k(\alpha n+\beta)} \frac{w^n t^{\frac{\alpha n}{k}}}{(n!)^2} dt,$$

interchanging the order of integration and summation, we get

$$= \sum_{n=0}^{\infty} \frac{(\eta)_{n,k}}{\Gamma_k(\alpha n+\beta)} \frac{w^n}{(n!)^2} \frac{1}{r\Gamma_r(s-\lambda)} \left(\frac{d}{dx}\right)^{s} \int_0^x (x-t)^{\frac{s-\lambda}{r}-1}\, t^{\frac{\alpha n}{k}+\frac{\beta}{k}-1} dt,$$

solving integral with the help of Lemma 2., it gives

$$= \sum_{n=0}^{\infty} \frac{(\eta)_{n,k}}{\Gamma_k(\alpha n+\beta)} \frac{w^n}{(n!)^2} \left(\frac{k}{r}\right)^{\frac{s-\lambda}{r}} k^{-s} x^{\frac{s-\lambda}{r}+\frac{\alpha n}{k}+\frac{\beta}{k}-s-1} \frac{\Gamma_k(\alpha n+\beta)}{\Gamma_k\left(\alpha n+\beta+\left(\frac{s-\lambda}{r}-s\right)k\right)}$$

$$= \left(\frac{k}{r}\right)^{\frac{s-\lambda}{r}} k^{-s} x^{\frac{s-\lambda}{r}+\frac{\beta}{k}-s-1} \sum_{n=0}^{\infty} \frac{(\eta)_{n,k}}{\Gamma_k\left(\alpha n+\beta+\left(\frac{s-\lambda}{r}-s\right)k\right)} \frac{w^n}{(n!)^2} x^{\frac{\alpha n}{k}}$$

$$= \left(\frac{k}{r}\right)^{\frac{s-\lambda}{r}} k^{-s} x^{\frac{s-\lambda}{r}+\frac{\beta}{k}-s-1}\, W_{k,\alpha,\beta+\left(\frac{s-\lambda}{r}-s\right)k}^{\eta}\left(w x^{\frac{\alpha}{k}}\right).$$

Corollary 2.1. If the conditions of the theorem 2 are satisfied with $k \to 1$ and $\eta = 1$, then

$$\lim_{k\to 1}\left(D_r^{\lambda}\, t^{\frac{\beta}{k}-1}W_{k,\alpha,\beta}^{1}\left(wt^{\frac{\alpha}{k}}\right)\right)(x) = \left(\frac{1}{r}\right)^{\frac{s-\lambda}{r}} x^{\frac{s-\lambda}{r}+\beta-1}\, W_{\alpha,\ \beta+\frac{s-\lambda}{r}-s}(w x^{\alpha}). \tag{20}$$

Corollary 2.2. If the conditions of the theorem 2 are satisfied with r = 1, then

$$\left(D_+^{\lambda}\, t^{\frac{\beta}{k}-1}W_{k,\alpha,\beta}^{\eta}\left(wt^{\frac{\alpha}{k}}\right)\right)(x) = (k)^{-\lambda}\, x^{-\lambda+\frac{\beta}{k}-1}\, W_{k,\alpha,\beta-\lambda k}^{\eta}\left(wx^{\frac{\alpha}{k}}\right). \tag{21}$$

Theorem 3. If $k \in \mathbb{R},\ \alpha, \beta, \eta \in \mathbb{C}$, $\mathrm{Re}(\alpha) > 0, \mathrm{Re}(\beta) > 0, \mathrm{Re}(\eta) > 0$, then Elzaki transform of k-Wright function

$$E\{W_{k,\alpha,\beta}^{\eta}(z)\} = u^2\, E_{k,\alpha,\beta}^{\eta}(u), \tag{22}$$

where $E_{k,\alpha,\beta}^{\eta}(u)$ is the k-Mittag Leffler function (cf.[2]).

Proof. $E\{W_{k,\alpha,\beta}^{\eta}(z)\} = u^2 \int_0^{\infty} e^{-z}\, W_{k,\alpha,\beta}^{\eta}(uz)\,dz$

$$= u^2 \int_0^{\infty} e^{-z} \sum_{n=0}^{\infty} \frac{(\eta)_{\alpha,k}}{\Gamma_k(\alpha n+\beta)} \frac{(uz)^n}{(n!)^2}\,dz,$$

changing the order of integration and series

$$= \sum_{n=0}^{\infty} \frac{(\eta)_{\alpha,k}}{\Gamma_k(\alpha n+\beta)} \frac{z^n}{(n!)^2}\, u^{n+2} \int_0^{\infty} e^{-z} z^n\, dz,$$

solving integral using gamma function, we have

$$= u^2 \sum_{n=0}^{\infty} \frac{(\eta)_{\alpha,k}}{\Gamma_k(\alpha n+\beta)} \frac{u^n}{n!}$$

$$= u^2\, E_{k,\alpha,\beta}^{\eta}(u).$$

Corollary 3.1. If the conditions of the theorem 3 are satisfied with $k\to 1$ and $\eta = 1$, then the Elzaki transform of the Wright function is given by

$$\lim_{k\to 1} E\{W_{k,\alpha,\beta}^{1}(z)\} = E\{W_{\alpha,\beta}(z)\} = u^2 E_{\alpha,\beta}(u), \tag{23}$$

where $E_{\alpha,\beta}(u)$ is known as Mittag-Leffler function (cf.[8]).

Theorem 4. Elzaki transform of r-fractional integral operator is

$$(E\, I_r^{\lambda} f)(u) = \left(\frac{u}{r}\right)^{\frac{\lambda}{r}} (E\, f)(u) \tag{24}$$

Proof. Using (13) taking into account (6), we get

$$(E\, I_r^{\lambda} f)(u) = u^2 \int_0^{\infty} e^{-t} (I_r^{\lambda} f)(ut)\,dt$$

$$= u^2 \int_0^{\infty} e^{-t} \frac{1}{r\,\Gamma_r(\lambda)} \int_0^{ut} (ut - x)^{\frac{\lambda}{r}-1} f(x)\,dx\,dt,$$

changing the order of integration, we obtain

$$u^2 \int_0^{\infty} f(x) \frac{1}{r\,\Gamma_r(\lambda)} \int_{\frac{x}{u}}^{\infty} e^{-t}(ut - x)^{\frac{\lambda}{r}-1}\,dt\,dx,$$

by taking $ut - x = w$

$$u \int_0^{\infty} e^{-\frac{x}{u}} f(x) \frac{1}{r\,\Gamma_r(\lambda)} \int_0^{\infty} e^{-\frac{w}{u}} (w)^{\frac{\lambda}{r}-1}\,dw\,dx,$$

on taking $\frac{w}{u} = v$, we get

$$= u^2\ (u)^{\frac{\lambda}{r}-1} \int_0^{\infty} e^{-\frac{x}{u}} f(x) \frac{1}{r\,\Gamma_r(\lambda)} \int_x^{\infty} e^{-v}\,(v)^{\frac{\lambda}{r}-1}\,dv\,dx$$

$$= u^2\ (u)^{\frac{\lambda}{r}-1} \int_0^{\infty} e^{-\frac{x}{u}} f(x) \frac{1}{(r)^{\frac{\lambda}{r}}\Gamma\left(\frac{\lambda}{r}\right)} \Gamma\left(\frac{\lambda}{r}\right) dx$$

$$= \left(\frac{u}{r}\right)^{\frac{\lambda}{r}} u \int_0^{\infty} e^{-\frac{x}{u}} f(x)\,dx$$

$$= \left(\frac{u}{r}\right)^{\frac{\lambda}{r}} (Ef)(u).$$

Corollary 4.1.Let the conditions of theorem 4 are satisfied. If we take r = 1, then the following result holds

$$(E\, I_+^{\lambda} f)(u) = (u)^{\lambda}\ (E\, f)(u). \tag{25}$$

References

[1] Diaz R. and Pariguan E. On hypergeometric functions and k- Pochhammer symbol, Div. Math. *15(2)*, 2007, 179-192.

[2] Dorrego G.A. and Cerutti R. A. , The k-Mittag Leffler function, Int. J. Contemp. Math. Sci. , *7(15)*, 2012, 705-716.

[3] Mubeem, S. and Habibullah, G. M., k-fractional integral and application, Int. J. Contemp. Math. Sci. , *7(2)*, 2012, 89-94.

[4] Romero L. G. and Cerutti R. A. , Fractional calculus of a k-wright type function, Int. J. Contemp. Math. Sci. , *7(31)*, 2012, 1547-1557.

[5] Romero, L. G. , Luque L. L. Dorrego G. A. and Cerutti R. A. , On the k-Riemann-Liouville fractional derivative, Int. J. Contemp. Math. Sci. , 8(1), 2013, 41-51.
[6] Samko S., Kilbas A. and Marichev O., Fractional integrals and derivatives, (Gordon and Breach, Switzerland, 1993).
[7] Tarig, M. E., The new integral transform "Elzaki Transform", Global J. of Pure & Appl. Math., *7(1)*, 2011, 57-64.
[8] Wiman, Ueber den fundamental satz in der theorie der funktionen $E_{\alpha}(x)$, Acta math., *29*, 1905, 191-201.