IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 11, Issue 3 Ver. IV (May - Jun. 2015), PP 22-34
www.iosrjournals.org

Geo/Geo/1/k Interdependent Queueing Model with Controllable Arrival Rates and Feedback

G. Rani[1] and A. Srinivasan[2]
[1] *Associate Professor, PG and Research Department of Mathematics, Seethalakshmi Ramaswami College, Tiruchirappalli, 620002. Tamil Nadu, India.*
[2] *Associate Professor, PG and Research Department of Mathematics, Bishop Heber College (Autonomous), Tiruchirappalli, 620017. Tamil Nadu, India.*

Abstract: *In this paper the Geo/Geo/1/k queuing model with controllable arrival rates, single server with identical service rates and feedback is considered. The steady state solution and system characteristics are derived for this model. The analytical results are numerically illustrated, the effect of the nodal parameters on the system characteristics are studied and relevant conclusion is presented.*
Keywords: *Single server, Controllable arrival rates, Bivariate Bernoulli feedback, Finite capacity ,System characteristics.*
AMS 2000 Subject Classification Primary 60K25; *Secondary 68M20.90B22*

I. Introduction

Queueing theory presents the concrete framework for design and analysis of practical applications.One of the important characteristics of a queueing system in the service process,entities in the system may be served individually or in batches.A large number of results in queueing theory is based on research on behavioural problems. Many practical queueing systems especially those with feedback have been widely applied to real life situations, such as the problem involving hospital emergency wards handling critical patients and unsatisfied customers in public telephone booths of coin box type etc. In day today life, one encounters numerous examples of queueing situations where all arriving customers require the main service and only some requires the secondary service provided by the server. "A queueing model in which the arrivals and services are correlated is known as interdependent queueing model. Takacs [6] considered a queue with feedback customers which has applications in real life formulation of queue with feedback mechanism. Kalyanaraman and Renganathan [2] have studied vacation queueing models with instantaneous Bernoulli feedback. In most of the research works, the authors have considered that the arrival and service patterns are independent. But in many real life situations, the arrival and service patterns are interdependent.Kalyanaraman and Sumathy [3] have studied a feedback queue with multiple servers and batch service. Recently Rani and Srinivasan [5] have studied a multiserver loss and delay interdependent queueing model with controllable arrival rates, no passing and feedback.Rani and Srinivasan [4] have analyzed Geo/Geo/c/k Interdependent Queueing Model with Controllable Arrival Rates and Feedback..Goswami and Gupta [1] have obtained the distribution of the number of customer served during a busy period in a discrete time Geo/Geo/1 queue. Thiagarajan and Srinivasan [7] have analysed Geo/Geo/c/∞ interdependent queueing model with controllable arrived rates and obtained the steady state probabilities and the system characteristics when the joint distribution of inter-arrival and service time is a bivariate geometric distribution. Although the literature of queueing theory deals largely with continuous time models, the developments in the practical world of computers and communication are being more and more discrete time in nature. With the rapid growth in the field of computer communications that has been considerable interest in the discrete models. The discrete time models have got some interesting applications in the study of time sharing computer systems, where messages from a collection of terminals are assigned by a variety of different multiplexing methods of a central computer. In this Paper an Geo/Geo/1/k Interdependent queueing model with controllable arrival rates and feedback is considered. In section II, the description of the queueing model is given stating the relevant postulates. In section III the steady state equations are derived and the steady state probabilities are obtained using recursive approach. In section IV the analytical expressions for various characteristics of the queueing model are obtained. In section V the analytical results are numerically illustrated and the effect of the nodal parameters on the system characteristics are studied and relevant conclusion is presented.

II. Description Of The Model

Consider a single server finite capacity queueing system with controllable arrival rate and feedback. Customers arrive at the service station one by one according to a bivariate Geometric stream with arrival rates $(\lambda_0-\varepsilon)$, $(\lambda_1-\varepsilon)$ (>0). There is a single server who provides service to all the arriving customers. Service times are independent and identically distributed Bernoulli random variables with service rate (μ-ε). After the completion of each service, the customer can either join at the end of the queue with probability p or customers can leave the system with probability q with p + q = 1. The customer both newly arrived and those who opted for feedback are served in the order in which they join the tail of the original queue. It is assumed that there is no difference between regular arrival and feedback arrival. The customers are served according to the first come first served rule with following assumptions.

The arrival process $\{X_1(t)\}$ and the service completion process $\{X_2(t)\}$ of the system are correlated and follow a bivariate bernoulli distribution is given by

$$P\left\{X_1(t) = x_1, X_2(t) = x_2\right\}$$
$$= \sum_{j=0}^{\min(x_1,x_2)} \left[(\lambda_i - \varepsilon)\right]^{x_1-j} \left[(\mu-\varepsilon)t\right]^{x_2-j} \left[1-(\lambda_i-\varepsilon)t\right]^{1-(x_1-j)} \left[1-(\mu-\varepsilon)t\right]^{1-(x_2-j)}$$

$x_1, x_2 = 0, 1$; $\lambda_i, \mu > 0$, $i = 0,1$; $0 \le \varepsilon < \min(\lambda_i, \mu)$; $n = 0,1,2,...,r-1, r, r+1,...,R-1, R, R+1,..., k-1, k$
with parameters λ_0 (λ_1), μ and ε as mean faster (slower) rate of arrivals, mean service rate and co-variance between arrival and service processes respectively.

Postulates of the model are

1. Probability that there is no arrival and no service completion during any interval of time t, when the system is in faster rate of arrivals either with feedback or without feedback, is
$$\left[\left(1-(\lambda_0-\varepsilon)t\right)\right]\left[1-\left\{p(\mu-\varepsilon)+q(\mu-\varepsilon)\right\}t\right]$$
2. Probability that there is no arrival and one service completion during any interval of time t, when the system is in faster rate of arrivals either with feedback or without feedback, is
$$\left[\left(1-(\lambda_0-\varepsilon)t\right)\right]\left[\left\{P(\mu-\varepsilon)+q(\mu-\varepsilon)\right\}t\right]$$
3. Probability that there is one arrival and no services completion during any interval of time t, when the system is in faster rate of arrivals either with feedback or without feedback, is
$$\left[(\lambda_0-\varepsilon)t\right]\left[1-\left\{p(\mu-\varepsilon)+q(\mu-\varepsilon)\right\}t\right]$$
4. Probability that there is one arrival and one service completion during any interval of time t, when the system is in faster rate of arrivals either with feedback or without feedback, is
$$\left[(\lambda_0-\varepsilon)t\right]\left[\left\{p(\mu-\varepsilon)t+q(\mu-\varepsilon)t\right\}\right].$$
5. Probability that there is no arrival and no service completion during any interval of time t, when the system is in slower rate of arrivals either with feedback or without feedback, is
$$\left[\left(1-(\lambda_1-\varepsilon)t\right)\right]\left[1-\left\{p(\mu-\varepsilon)+q(\mu-\varepsilon)\right\}t\right].$$
6. Probability that there is no arrival and one service completion during any interval of time t, when the system is in slower rate of arrivals either with feedback or without feedback, is
$$\left[\left(1-(\lambda_1-\varepsilon)t\right)\right]\left[\left\{p(\mu-\varepsilon)+q(\mu-\varepsilon)\right\}t\right].$$
7. Probability that there is one arrival and no service completion during any interval of time t, when the system is in slower rate of arrivals either with feedback or without feedback, is
$$\left[(\lambda_1-\varepsilon)t\right]\left[1-\left\{p(\mu-\varepsilon)+q(\mu-\varepsilon)\right\}t\right].$$
8. Probability that there is one arrival and one service completion during any interval of time t, when the system is in slower rate of arrivals either with feedback or without feedback, is
$$\left[(\lambda_1-\varepsilon)t\right]\left[\left\{p(\mu-\varepsilon)+q(\mu-\varepsilon)\right\}t\right].$$

III. Steady State Equations

We observe that only $P_n(0)$ exists when n = 0,1,2,..., r-1,r; both $P_n(0)$ and $P_n(1)$ exist when n = r+1, r+2, r+3,..., R-1,...,R-1; only $P_n(1)$ exists when n=R, R+1, R+2, ...,k, further $P_n(0) = P_n(1) = 0$ if n > k.

Let $\rho_0 = \dfrac{1}{\lambda_0-\varepsilon}, \rho_1 = \dfrac{1}{\lambda_1-\varepsilon}, q_0 = \dfrac{1}{q(\mu-\varepsilon)}, p_0 = \dfrac{1}{p(\mu-\varepsilon)},$

$\bar{\rho}_0 = 1-\rho_0, \bar{\rho}_1 = 1-\rho_1, \bar{q}_0 = 1-q_0$ and $\bar{p}_0 = 1-p_0$

Then the stationary equations which are written through the matrix of densities are given by

$$\left[\rho_0\bar{q}_0 + \rho_0\bar{p}_0\right]P_0(0) = \bar{\rho}_0 q_0 P_1(0) \quad \text{... (1)}$$

$$\left[\rho_0\bar{q}_0 + \bar{\rho}_0 q_0 + \rho_0\bar{p}_0\right]P_1(0) = \left[\rho_0\bar{p}_0 + \rho_0\bar{q}_0\right]P_0(0) + \bar{\rho}_0 q_0 P_2(0) \quad \text{... (2)}$$

$$\left[\bar{\rho}_0 q_0 + \rho_0\bar{q}_0 + \rho_0\bar{p}_0\right]P_n(0) = \left[\rho_0\bar{q}_0 + \rho_0\bar{p}_0\right]P_{n-1}(0) + \rho_0\bar{q}_0 P_{n+1}(0)$$

$n = 2, 3,4,..., r–1$ ... (3)

$$\left[\bar{\rho}_0 q_0 + \rho_0\bar{q}_0 + \rho_0\bar{p}_0\right]P_r(0) = \left[\rho_0\bar{q}_0 + \rho_0\bar{p}_0\right]P_{r-1}(0) + \bar{\rho}_0 q_0 P_{r+1}(0) + \bar{\rho}_1 q_0 P_{r+1}(1) \quad \text{... (4)}$$

$$\left[\bar{\rho}_0 q_0 + \rho_0\bar{q}_0 + \rho_0\bar{p}_0\right]P_n(0) = \left[\rho_0\bar{q}_0 + \rho_0\bar{p}_0\right]P_{n-1}(0) + \bar{\rho}_0 q_0 P_{n+1}(0)$$

$n = r+1, r+2,..., R–2$ ... (5)

$$\left[\rho_0\bar{q}_0 + \bar{\rho}_0 q_0 + \rho_0\bar{p}_0\right]P_{R-1}(0) = \left[\rho_0\bar{q}_0 + \rho_0\bar{p}_0\right]P_{R-2}(0) \quad \text{... (6)}$$

$$\left[\rho_1\bar{q}_0 + \bar{\rho}_1 q_0 + \rho_1\bar{p}_0\right]P_{r+1}(1) = \bar{\rho}_1 q_0 P_{r+2}(1) \quad \text{... (7)}$$

$$\left[\rho_1\bar{q}_0 + \bar{\rho}_1 q_0 + \rho_1\bar{p}_0\right]P_n(1) = \left[\rho_1\bar{q}_0 + \rho_1\bar{p}_0\right]P_{n-1}(1) + \bar{\rho}_1 q_0 P_{n+1}(1)$$

$n = r+1, r+2, r+3,..., R–1$ ... (8)

$$\left[\bar{\rho}_1 q_0 + \rho_1\bar{q}_0 + \rho_1\bar{p}_0\right]P_R(1) = \left[\rho_0\bar{q}_0 + \rho_0\bar{p}_0\right]P_{R-1}(0) + \left[\rho_1\bar{q}_0 + \rho_1\bar{p}_0\right]P_{R-1}(1) + \bar{\rho}_1 q_0 P_{R+1}(1) \quad \text{... (9)}$$

$$\left[\bar{\rho}_1 q_0 + \rho_1\bar{q}_0 + \rho_1\bar{p}_0\right]P_n(1) = \left[\rho_1\bar{q}_0 + \rho_1\bar{p}_0\right]P_{n-1}(1) + \bar{\rho}_1 q_0 P_{n+1}(1)$$

$n = R+1, R+2, R+3,...,k–1$ ... (10)

$$\left[\bar{\rho}_1 q_0 + \rho_1\bar{q}_0 + \rho_1\bar{p}_0\right]P_k(1) = \left[\rho_1\bar{q}_0 + \rho_0\bar{p}_0\right]P_{k-1}(1) \quad \text{... (11)}$$

Let $\rho(0) = \dfrac{\lambda_0 - \varepsilon}{q(\mu-\varepsilon)}$ and $\rho(1) = \dfrac{\lambda_1 - \varepsilon}{q(\mu-\varepsilon)}$

where $\rho(0) = \left(\dfrac{q_0}{\rho_0}\right)$ is faster rate of arrival intensity and $\rho(1) = \left(\dfrac{q_0}{\rho_1}\right)$ is slower rate of arrivals intensity

From equation (1) we have

$$\left[\rho_0\bar{q}_0 + \rho_0\bar{p}_0\right]\bar{P}_0(0) = \bar{\rho}_0 q_0 P_1(0)$$

$$P_1(0) = \left[\frac{\rho_0\bar{q}_0}{\bar{\rho}_0 q_0} + \frac{\rho_0\bar{p}_0}{\bar{\rho}_0 q_0}\right]P_0(0)$$

$$P_1(0) = [R+S]P_0(0) \quad \text{... (12)}$$

where $R = \dfrac{\rho_0\bar{q}_0}{\bar{\rho}_0 q_0}$ and $S = \dfrac{\rho_0\bar{p}_0}{\bar{\rho}_0 q_0}$

Using the Result (12) in (2) we get

$$P_2(0)=\left[\frac{\bar{\rho}_0 q_0}{\bar{\rho}_0 q_0}+\frac{\rho_0 \bar{q}_0}{\bar{\rho}_0 q_0}+\frac{\rho_0 \bar{p}_0}{\bar{\rho}_0 q_0}\right]P_1(0)-\left[\frac{\rho_0 \bar{p}_0}{\bar{\rho}_0 q_0}+\frac{\rho_0 \bar{q}_0}{\bar{\rho}_0 q_0}\right]P_0(0)$$

$$=\big[[1+R+S][R+S]-[R+S]\big]P_0(0)$$

$$P_2(0)=[R+S]^2 P_0(0) \qquad \text{... (13)}$$

From the equations (12), (13) and (3) we get

$$P_{n+1}(0)=\left[\frac{\bar{\rho}_0 q_0}{\bar{\rho}_0 q_0}+\frac{\rho_0 \bar{q}_0}{\bar{\rho}_0 q_0}+\frac{\rho_0 \bar{p}_0}{\bar{\rho}_0 q_0}\right]P_n(0)-\left[\frac{\rho_0 \bar{q}_0}{\bar{\rho}_0 q_0}+\frac{\rho_0 \bar{p}_0}{\bar{\rho}_0 q_0}\right]P_{n-1}(0)$$

$$P_{n+1}(0)=[1+R+S]P_n(0)-[R+S]P_{n-1}(0) \qquad \text{... (14)}$$

Using the result (12), (13) and (14) we recursively derive

$$P_3(0)=(R+S)^3 P_0(0)$$

$$P_4(0)=(R+S)^4 P_0(0)$$

$$\vdots$$

$$P_r(0)=(R+S)^r P_0(0)$$

and hence $P_n(0) = (R+S)^n P_0(0)$, $n = 3, 4, 5,\ldots,r-1,\ r$ ... (15)

From equation (4) we get

$$P_{r+1}(0)=\left[\frac{\bar{\rho}_0 q_0}{\bar{\rho}_0 q_0}+\frac{\rho_0 \bar{q}_0}{\bar{\rho}_0 q_0}+\frac{\rho_0 \bar{p}_0}{\bar{\rho}_0 q_0}\right]P_r(0)-\left[\frac{\rho_0 \bar{q}_0}{\bar{\rho}_0 q_0}+\frac{\rho_0 \bar{p}_0}{\bar{\rho}_0 q_0}\right]P_{r-1}(0)-\frac{\bar{\rho}_1}{\bar{\rho}_0}P_{r+1}(1)$$

$$=[1+R+S]P_r(0)-[R+S]P_{r-1}(0)-\frac{\bar{\rho}_1}{\bar{\rho}_0}P_{r+1}(1)$$

$$P_{r+1}(0)=[R+S]^{r+1}P_0(0)-\frac{\bar{\rho}_1}{\bar{\rho}_0}P_{r+1}(1) \qquad \text{... (16)}$$

Using the results (13), (15) and (16) in (5) we recursively derive

$$P_{n+1}(0)=\left[\frac{\bar{\rho}_0 q_0}{\bar{\rho}_0 q_0}+\frac{\rho_0 \bar{q}_0}{\bar{\rho}_0 q_0}+\frac{\rho_0 \bar{p}_0}{\bar{\rho}_0 q_0}\right]P_n(0)-\left[\frac{\rho_0 \bar{q}_0}{\bar{\rho}_0 q_0}+\frac{\rho_0 \bar{p}_0}{\bar{\rho}_0 q_0}\right]P_{n-1}(0)$$

$$P_{n+1}(0)=[1+R+S]P_n(0)-[R+S]P_{n-1}(0) \qquad \text{... (17)}$$

From the equation (17) we get recursively

$$P_{r+2}(0)=(R+S)^{r+2}P_0(0)-[1+R+S]\frac{\bar{\rho}_1}{\bar{\rho}_0}P_{r+1}(1)$$

$$P_{r+3}(0)=[R+S]^{r+3}P_0(0)-\left[1+(R+S)+(R+S)^2\right]\frac{\bar{\rho}_1}{\bar{\rho}_0}P_{r+1}(1)$$

$$\vdots$$

$$P_{n-1}(0)=[R+S]^{n-1}P_0(0)-\Big[1+(R+S)+(R+S)^2+(R+S)^3+\ldots$$

$$+(R+S)^{n-r-2}\Big]\frac{\bar{\rho}_1}{\bar{\rho}_0}P_{r+1}(1)$$

$$P_n(0)=[R+S]^n P_0(0)-\Big[1+(R+S)+(R+S)^2+(R+S)^3+\ldots$$

$$+\left(R+S\right)^{n-r-1}\Bigg]\frac{\bar{\rho}_1}{\bar{\rho}_0}P_{r+1}\left(1\right)$$

and hence

$$P_n\left(0\right)=\left[R+S\right]^n P_0\left(0\right)-\left[\frac{1-\left(R+S\right)^{n-r}}{1-\left(R+S\right)}\right]\frac{\bar{\rho}_1}{\bar{\rho}_0}P_{r+1}\left(1\right)$$

$$n = r+1, r+2, r+3,...,R-1 \qquad ...\ (18)$$

Using the result (18) in (6) we get

$$\left[1+\frac{\rho_0\bar{q}_0}{\bar{\rho}_0 q_0}+\frac{\rho_0\bar{p}_0}{\bar{\rho}_0 q_0}\right]P_{R-1}\left(0\right)=\left[\frac{\rho_0\bar{q}_0}{\bar{\rho}_0 q_0}+\frac{\rho_0\bar{p}_0}{\bar{\rho}_0 q_0}\right]P_{R-2}\left(0\right)$$

$$\left[1+R+S\right]\left\{\left(R+S\right)^{R-1}P_0\left(0\right)-\frac{\left[1-\left(R+S\right)^{R-r-1}\right]}{1-\left(R+S\right)}\frac{\bar{\rho}_1}{\bar{\rho}_0}P_{r+1}\left(1\right)\right\}=\left[R+S\right]\left\{\left(R+S\right)^{R-2}P_0\left(0\right)-\frac{\left[1-\left(R+S\right)^{R-r-2}\right]}{1-\left(R+S\right)}\frac{\bar{\rho}_1}{\bar{\rho}_0}P_{r+1}\left(1\right)\right\}$$

$$\left[R+S\right]^R P_0\left(0\right)=\frac{1-\left(R+S\right)^{R-r}}{1-\left(R+S\right)}\frac{\bar{\rho}_1}{\bar{\rho}_0}P_{r+1}\left(1\right)$$

$$P_{r+1}\left(1\right)=\frac{\left(R+S\right)^{R+r}\left(1-\left(R+S\right)\right)}{\left(R+S\right)^r-\left(R+S\right)^R}\frac{\bar{\rho}_0}{\bar{\rho}_1}P_0\left(0\right) \qquad ...\ (19)$$

From equation (7) we get

$$P_{r+2}\left(1\right)=\left[\frac{\rho_1\bar{q}_0}{\bar{\rho}_0 q_0}+\frac{\bar{\rho}_1 q_0}{\bar{\rho}_1 q_0}+\frac{\rho_1\bar{p}_0}{\bar{\rho}_1 q_0}\right]P_{r+1}\left(1\right)$$

$$P_{r+2}\left(1\right)=\left[1+T+U\right]P_{r+1}\left(1\right) \qquad ...\ (20)$$

where $T=\frac{\rho_1\bar{q}_0}{\bar{\rho}_1 q_0}$ and $U=\frac{\rho_1\bar{p}_0}{\bar{\rho}_1 q_0}$

Using the results (19) and (20) in (8) we get

$$P_{n+1}\left(1\right)=\left[\frac{\bar{\rho}_1 q_0}{\bar{\rho}_1 q_0}+\frac{\rho_1\bar{q}_0}{\bar{\rho}_1 q_0}+\frac{\rho_1\bar{p}_0}{\bar{\rho}_1 q_0}\right]P_n\left(1\right)-\left[\frac{\rho_1\bar{q}_0}{\bar{\rho}_1 q_0}+\frac{\rho_1\bar{p}_0}{\bar{\rho}_1 q_0}\right]P_{n-1}\left(1\right)$$

$$P_{n+1}\left(1\right)=\left[1+T+U\right]P_n\left(1\right)-\left[T+U\right]P_{n-1}\left(1\right) \qquad ...\ (21)$$

From equation (21) we recursively derive

$$P_{r+3}\left(1\right)=\left[1+\left(T+U\right)+\left(T+U\right)^2\right]P_{r+1}\left(1\right)$$

$$P_{r+4}\left(1\right)=\left[1+\left(T+U\right)+\left(T+U\right)^2+\left(T+U\right)^3\right]P_{r+1}\left(1\right)$$

$$\vdots$$

$$P_{R-2}(1)=\left[1+(T+U)+(T+U)^2+(T+U)^3+\ldots+(T+U)^{R-r-3}\right]P_{r+1}(1)$$

$$P_{R-1}(1)=\left[1+(T+U)+(T+U)^2+(T+U)^3+\ldots+(T+U)^{R-r-2}\right]P_{r+1}(1)$$

$$P_{n}(1)=\left[1+(T+U)+(T+U)^2+\ldots+(T+U)^{n-r-1}\right]P_{r+1}(1)$$

$$P_{n}(1)=\left[\frac{1-(T+U)^{n-r}}{1-(T+U)}\right]P_{r+1}(1)$$

n = r+1, r+2,... R-2, R-1, R ... (22)

Using the result (22) in (9) we get

$$P_{R+1}(1)=(1+T+U)P_R(1)-(T+U)P_{R-1}(1)-\left(\frac{\rho_0\bar{q}_0}{\bar{\rho}_1 q_0}+\frac{\rho_0\bar{p}_0}{\bar{\rho}_1 q_0}\right)P_{R-1}(0)$$

$$=\frac{(1+T+U)\left[1-(T+U)^{R-r}\right]}{1-(T+U)}P_{r+1}(1)-\frac{(T+U)\left[1-(T+U)^{R-r-1}\right]}{1-(T+U)}P_{r+1}(1)$$

$$-\left(\alpha'+\beta'\right)P_{R-1}(0)$$

$$P_{R+1}(1)=\frac{1-(T+U)^{R-r+1}}{1-(T+U)}P_{r+1}(1)-\left(\alpha'+\beta'\right)P_{R-1}(0) \quad \ldots (23)$$

where $\alpha'=\frac{\rho_0\bar{q}_0}{\bar{\rho}_1 q_0}$ and $\beta'=\frac{\rho_0\bar{p}_0}{\bar{\rho}_1 q_0}$

From the equations (19), (23) and (10) we get

$$P_{n+1}(1)=\left[\frac{\bar{\rho}_1 q_0}{\bar{\rho}_1 q_0}+\frac{\rho_1\bar{q}_0}{\bar{\rho}_1 q_0}+\frac{\rho_1\bar{p}_0}{\bar{\rho}_1 q_0}\right]P_n(1)-\left[\frac{\rho_1\bar{q}_0}{\bar{\rho}_1 q_0}+\frac{\rho_1\bar{p}_0}{\bar{\rho}_1 q_0}\right]P_{n-1}(1)$$

$$P_{n-1}(1)=\left[1+(T+U)\right]P_n(1)-\left[T+U\right]P_{n-1}(1) \quad \ldots (24)$$

Using the results (22), (23) and (24) we recursively derive

$$P_{R+2}(1)=(1+T+U)P_{R+1}(1)-(T+U)P_R(1)$$

$$=(1+T+U)\left[\frac{\left[1-(T+U)^{R-r+1}\right]}{1-(T+U)}P_{r+1}(1)-\left(\alpha'+\beta'\right)P_{R-1}(0)\right]$$

$$-\frac{(T+U)\left[1-(T+U)^{R-r}\right]}{1-(T+U)}P_{r+1}(1)$$

$$=\frac{\left[1-(T+U)^{R-r+2}\right]}{1-(T+U)}P_{r+1}(1)-(1+T+U)\left(\alpha'+\beta'\right)P_{R-1}(0)$$

$$P_{R+3}(1)=\frac{\left[1-(T+U)^{R-r+3}\right]}{1-(T+U)}P_{r+1}(1)-\left(\alpha'+\beta'\right)\left[1+(T+U)+(T+U)^2\right]P_{R-1}(0)$$

$$P_{R+4}(1)=\frac{\left[1-(T+U)^{R-r+4}\right]}{1-(T+U)}P_{r+1}(1)-\left(\alpha^{'}+\beta^{'}\right)\left[1+(T+U)+(T+U)^{2}+(T+U)^{3}\right]P_{R-1}(0)\vdots$$

$$P_{k-1}(1)=\frac{\left[1-(T+U)^{k-r-1}\right]}{1-(T+U)}P_{r+1}(1)-\left(\alpha^{'}+\beta^{'}\right)\frac{\left[1-(T+U)^{k-R-1}\right]}{1-(T+U)}P_{R-1}(0)$$

and hence

$$P_{n}(1)=\frac{\left[1-(T+U)^{n-r}\right]}{1-(T+U)}P_{r+1}(1)-\left(\alpha^{'}+\beta^{'}\right)\frac{\left[1-(T+U)^{n-R}\right]}{1-(T+U)}P_{R-1}(0) \qquad \text{... (25)}$$

From the equations (18), (19) and (25) we get

$$P_{n}(1)=\left[\frac{1-(T+U)^{n-r}}{1-(T+U)}-\left(\frac{\alpha^{1}+\beta^{1}}{R+S}\right)\frac{\left[1-(T+U)^{n-R}\right]}{1-(T+U)}\frac{\bar{\rho}_1}{\bar{\rho}_0}\right]P_{r+1}(1)$$

$$n = R+1, R+2,...,k-1, k \qquad \text{... (26)}$$

where $P_{r+1}(1)$ is given by equation (19).

IV. Characteristics Of The Model

The following system characteristics are considered and their analytical results are derived in this section.

1. The probability P(0) that the system is in faster rate of arrivals either with feedback or without feedback.
2. The probability P(1) that the system is in slower rate of arrivals either with feedback or without feedback.
3. The probability $P_0(0)$ that the system is empty
4. Expected number of customers in the system L_{S_0}, when the system is in faster rate of arrivals either with feedback or without feedback.
5. Expected number of customers in the system L_{S_1}, when the system is in slower rate of arrivals either with feedback or without feedback.
6. Expected waiting time of the customer in the system W_S, when the system is in faster (slower) rate of arrivals either with feedback or without feedback.

The probability that the system is in faster rate of arrivals either with feedback or without feedback is,

$$P(0)=\sum_{n=0}^{r}P_n(0)+\sum_{n=r+1}^{R-1}P_n(0) \qquad \text{... (27)}$$

From the equations (15), (18), (19) and (27) we get

$$P(0)=\sum_{n=0}^{r}(R+S)^{n}P_0(0)+\sum_{n=r+1}^{R-1}\left[(R+S)^{n}P_0(0)-\frac{\left[1-(R+S)^{R-r}\right]}{1-(R+S)}\frac{\bar{\rho}_1}{\bar{\rho}_0}P_{r+1}(1)\right]$$

$$=\frac{1-(R+S)^{r+1}}{1-(R+S)}P_0(0)+\frac{(R+S)^{r+1}\left[1-(R+S)^{R-r-1}\right]}{1-(R+S)}P_0(0)$$

$$-\frac{1}{1-(R+S)}\left[(R-r-1)-(R+S)\frac{\left[1-(R+S)^{R-r-1}\right]}{1-(R+S)}\right]\frac{\bar{\rho}_1}{\bar{\rho}_0}P_{r+1}(1)$$

$$=\frac{\left[1-(R+S)^{r+1}+(R+S)^{r+1}-(R+S)^{R}\right]}{1-(R+S)}P_0(0)$$

$$-\left[\frac{(R-r-1)}{1-(R+S)}-\frac{\left[(R+S)\left(1-(R+S)^{R-r-1}\right)\right]}{\left[1-(R+S)\right]^2}\right]\frac{\bar{\rho}_0}{\bar{\rho}_0}P_{r+1}(1)$$

$$=\frac{1-(R+S)^R}{1-(R+S)}P_0(0)-\left[\frac{(R-r-1)\left[1-(R+S)\right]-\left[(R+S)\left(1-(R+S)\right)^{R-r-1}\right]}{\left[1-(R+S)\right]^2}\right]\frac{\bar{\rho}_1}{\bar{\rho}_0}P_{r+1}(1)$$

$$P(0)=\left[\frac{1}{1-(R+S)}-\frac{(R-r)(R+S)^{R+r}}{(R+S)^r-(R+S)^R}\right]P_0(0) \qquad \text{... (28)}$$

The probability that the system is in slower rate of arrivals either with feedback or without feedback is

$$P(1)=\sum_{n=r+1}^{R}P_n(1)+\sum_{n=R+1}^{k}P_n(1) \qquad \text{... (29)}$$

Using the results (19), (22), (26) in (28) we get

$$P(1)=\sum_{n=r+1}^{R}\frac{\left[1-(T+U)^{n-r}\right]}{1-(T+U)}P_{r+1}(1)+\sum_{n=R+1}^{k}\left[\frac{1-(T+U)^{n-r}}{1-(T+U)}-\left(\frac{\alpha'+\beta'}{R+S}\right)\frac{\left[1-(T+U)^{n-R}\right]}{1-(T+U)}\frac{\bar{\rho}_1}{\bar{\rho}_0}\right]P_{r+1}(1)$$

$$=\left\{\sum_{n=r+1}^{R}\frac{\left[1-(T+U)^{n-r}\right]}{1-(T+U)}+\sum_{n=R+1}^{k}\left[\frac{1-(T+U)^{n-r}}{1-(T+U)}\right.\right.$$

$$\left.\left.-\left(\frac{\alpha'+\beta'}{R+S}\right)\frac{\left[1-(T+U)^{n-R}\right]}{1-(T+U)}\frac{\bar{\rho}_1}{\bar{\rho}_0}\right]\right\}P_{r+1}(1)$$

$$=\left[\frac{1}{\left[1-(T+U)\right]}\left\{(R-r)-\frac{(T+U)\left[1-(T+U)^{R-r}\right]}{1-(T+U)}\right\}\right.$$

$$+\frac{1}{1-(T+U)}\left\{(k-R)-\frac{(T+U)^{R-r+1}\left[1-(T+U)^{k-R}\right]}{1-(T+U)}\right\}$$

$$\left.-\frac{\bar{\rho}_0}{\bar{\rho}_0}\left(\frac{\alpha'+\beta'}{R+S}\right)\frac{1}{1-(T+U)}\left\{(k-R)-\frac{(T+U)\left[1-(T+U)^{k-R}\right]}{\left[1-(T+U)\right]^2}\right\}\right]P_{r+1}(1)$$

$$P(1)=\left[\frac{(k-r)-(k-R)\dfrac{\alpha'+\beta'}{R+S}\dfrac{\bar{\rho}_1}{\bar{\rho}_0}}{1-(T+U)}\right.$$

$$+\frac{(T+U)\left[\frac{\alpha'+\beta'}{R+S}\left(1-(T+U)^{k-R}\right)\frac{\bar{\rho}_1}{\bar{\rho}_0}-\left(1-(T+U)^{k-r}\right)\right]}{\left[1-(T+U)\right]^2}\Bigg]$$

$$\times\frac{(R+S)^{R+r}\left(1-(R+S)\right)}{(R+S)^r-(R+S)^R}\frac{\bar{\rho}_0}{\bar{\rho}_1}P_0(0)$$

and hence

$$P(1)=\left\{\frac{A}{1-(T+U)}+\frac{(T+U)B}{\left[1-(T+U)\right]^2}\right\}C\,P_0(0) \qquad \text{... (30)}$$

where $A=(k-r)-(k-R)\frac{\alpha'+\beta'}{R+S}\frac{\bar{\rho}_1}{\bar{\rho}_0}$

$$B=\left(\frac{\alpha'+\beta'}{R+S}\right)\left(1-(T+U)^{k-R}\right)\frac{\bar{\rho}_1}{\bar{\rho}_0}-\left(1-(T+U)^{k-r}\right)$$

$$C=\frac{(R+S)^{R+r}\left(1-(R+S)\right)\frac{\bar{\rho}_0}{\bar{\rho}_1}}{(R+S)^r-(R+S)^R}$$

The probability $[P_0(0)]$ that the system is empty can be calculated from the normalizing condition

P(0) + P(1) = 1 ... (31)

From equations (28), (30) and (31) we get

$$P_0(0)=\left[\frac{1}{1-(R+S)}-\frac{(R-r)(R+S)^{R+r}}{(R+S)^r-(R+S)^R}+\left\{\frac{A}{1-(T+U)}+\frac{(T+U)B}{\left[1-(T+U)\right]^2}\right\}C\right]^{-1} \qquad \text{... (32)}$$

Expected number of customers in the system L_{S_0}, when the system is in faster rate of arrivals either with feedback or without feedback, is

$$L_{S_0}=\sum_{n=0}^{r}n\,P_n(0)+\sum_{n=r+1}^{R-1}n\,P_n(0) \qquad \text{... (33)}$$

Using the results (15), (18), (19) in equation (33) we get

$$L_{S_0}=\sum_{n=0}^{r}n(R+S)^n P_0(0)+\sum_{n=r+1}^{R-1}n\left[(R+S)^n P_0(0)-\frac{\left[1-(R+S)^{n-r}\right]}{1-(R+S)}\frac{\bar{\rho}_1}{\bar{\rho}_0}P_{r+1}(1)\right]$$

$$=\frac{(R+S)\left[1-(r+1)(R+S)^r+r(R+S)^{r+1}\right]}{\left[1-(R+S)\right]^2}P_0(0)+\Bigg\{\frac{r(R+S)^{r+1}\left[1-(R+S)^{R-r-1}\right]}{1-(R+S)}$$

$$+\frac{(R+S)^{r+1}\left[1-(R-r)(R+S)^{R-r-1}+(R-r-1)(R+S)^{R-r}\right]}{\left[1-(R+S)\right]^2}\Bigg\}P_0(0)$$

$$-\left[\frac{(R-r)(R+r-1)}{2(1-(R-S))}-\left\{\frac{r(R+S)\left(1-(R+S)^{R-r-1}\right)}{\left[1-(R+S)\right]^2}\right.\right.$$

$$\left.\left.+\frac{(R+S)\left[1-(R-r)(R+S)^{R-r-1}+(R-r-1)(R+S)^{R-S}\right]}{\left[1-(R+S)\right]^3}\right\}\right]\frac{\bar{\rho}_1}{\bar{\rho}_1}P_{r+1}(1)$$

$$=\left[\frac{\left[(R+S)-R(R+S)^R-(R-1)(R+S)^{R+1}\right]}{\left[1-(R+S)\right]^2}\right.$$

$$+\left\{\frac{r[R+S]-R[R+S]^{R-r}}{1-(R+S)}+\frac{(R+S)\left[1-(R+S)^{R-r}\right]}{\left[1-(R+S)\right]^2}\right.$$

$$\left.\left.-\frac{(R+r)(R-r-1)}{2}\right\}\frac{(R+S)^{R+r}}{(R+S)^r-(R+S)^R}\right]P_0(0)$$

$$L_{S_0}=\left[\frac{D}{\left[1-(R+S)\right]^2}+\frac{E(R+S)^{R+r}}{(R+S)^r-(R+S)^R}\right]P_0(0) \qquad \text{... (34)}$$

Where $D=(R+S)-R(R+S)^R-(R-1)(R+S)^{R+1}$

$$E=\frac{r[R+S]-R[R+S]^{R-r}}{1-(R+S)}+\frac{(R+S)\left(1-(R+S)^{R-r}\right)}{\left[1-(R+S)\right]^2}-\frac{(R+r)(R-r-1)}{2}$$

Expected number of customers in the system L_{S_1}, when the system is in slower rate of arrivals either with feedback or without feedback is

$$L_{S_1}=\sum_{n=r+1}^{R-1}nP_n(1)+\sum_{n=R}^{k}n\,P_n(1) \qquad \text{... (35)}$$

From the equations (19), (22), (26) and (34) we get

$$L_{S_1}=\sum_{n=r+1}^{R-1}n\frac{\left[1-(T+U)^{n-r}\right]}{1-(T+U)}P_{r+1}(1)+\sum_{n=R}^{k}n\left[\frac{1-(T+U)^{n-r}}{1-(T+U)}\right.$$

$$\left.-\left(\frac{\alpha'+\beta'}{R+S}\right)\frac{\left[1-(T+U)^{n-R}\right]}{1-(T+U)}\frac{\bar{\rho}_1}{\bar{\rho}_0}\right]P_{r+1}(1)$$

$$=\left[\sum_{n=r+1}^{R-1}n\frac{\left[1-(T+U)^{n-r}\right]}{1-(T+U)}+\sum_{n=R}^{k}n\frac{\left[1-(T+U)^{n-r}\right]}{1-(T+U)}\right.$$

$$-\sum_{n=R}^{k} n\frac{\alpha'+\beta'}{R+S}\frac{\left[1-(T+U)^{n-R}\right]}{1-(T+U)}\frac{\bar{\rho}_1}{\bar{\bar{\rho}}_0}\Bigg] P_{r+1}(1)$$

$$=\left[\left\{\frac{(R-r-1)(R+r)+\left[1+\left(\frac{\alpha'+\beta'}{R+S}\right)\frac{\bar{\rho}_1}{\bar{\bar{\rho}}_0}\right](k-R)(k-R+1)}{2\left(1-(T+U)\right)}\right\}\right.$$

$$+\left\{\frac{k\left[(T+U)^{k-r+1}+\left(\frac{\alpha'+\beta'}{R+S}\right)(T+U)^{k-R-1}\frac{\bar{\rho}_1}{\bar{\bar{\rho}}_0}\right]-\left[\left\{r+R\left(\frac{\alpha'+\beta'}{R+S}\right)\frac{\bar{\rho}_1}{\bar{\bar{\rho}}_0}\right\}(T+U)\right]}{\left[1-(T+U)\right]^2}\right\}$$

$$\left.+\left\{\frac{\left[(T+U)^{k-r+1}+\left(\frac{\alpha'+\beta'}{R+S}\right)(T+U)^{k-R+1}\frac{\bar{\rho}_1}{\bar{\bar{\rho}}_0}\right]-\left[1+\left(\frac{\alpha'+\beta'}{R+S}\right)\frac{\bar{\rho}_1}{\bar{\bar{\rho}}_0}\right](T+U)}{\left[1-(T+U)\right]^3}\right\}\right] P_{r+1}(1)$$

$$=\left[\frac{F}{2\left(1-(T+U)\right)}+\frac{G}{\left[1-(T+U)\right]^2}+\frac{H}{\left[1-(T+U)\right]^3}\right] P_{r+1}(1)$$

$$L_{S_1}=\left\{\frac{F}{2\left(1-(T+U)\right)}+\frac{G}{\left[1-(T+U)\right]^2}+\frac{H}{\left[1-(T+U)\right]^3}\right\} P_{r+1}(1) \qquad \text{... (36)}$$

where $P_{r+1}(1)$ is given by (19)

$$F=(R-r-1)(R+r)+\left[1+\left(\frac{\alpha'+\beta'}{R+S}\right)\frac{\bar{\rho}_1}{\bar{\bar{\rho}}_0}\right](k-R)(k-R+1)$$

$$G=k\left[(T+U)^{k-r+1}+\left(\frac{\alpha'+\beta'}{R+S}\right)(T+U)^{k-R+1}\frac{\bar{\rho}_1}{\bar{\bar{\rho}}_0}\right]-\left[\left\{r+R\left(\frac{\alpha'+\beta'}{R+S}\right)\frac{\bar{\rho}_1}{\bar{\bar{\rho}}_0}\right\}(T+U)\right]$$

$$H=\left[(T+U)^{k-r+1}+\left(\frac{\alpha'+\beta'}{R+S}\right)(T+U)^{k-R-1}\frac{\bar{\rho}_1}{\bar{\bar{\rho}}_0}\right]-\left[1+\left(\frac{\alpha'+\beta'}{R+S}\right)\frac{\bar{\rho}_1}{\bar{\bar{\rho}}_0}\right](T+U)$$

Expected number of customers in the system L_S, when the system is in faster (slower) rate of arrivals either with feedback or without feedback, is

$$L_S=L_{S_0}+L_{S_1} \qquad \text{... (37)}$$

Using the results (34) and (36) in (37) we get

$$L_S=\left[\frac{D}{\left[1-(R+S)\right]^2}+\frac{E(R+S)^{R+r}}{(R+S)^r-(R+S)^R}+\left\{\frac{F}{2\left(1-(T+U)\right)}\right.\right.$$

$$+\frac{G}{\left[1-(T+U)\right]^2}+\frac{H}{\left[1-(T+U)\right]^3}\Bigg\}C\Bigg]P_0(0) \qquad \text{... (38)}$$

where $P_0(0)$ is given by (32)

Using Little's formula, the expected waiting time of the customers in the system W_S when the system is in faster and slower rate of arrivals either with feedback or without feedback is given by

$$W_S=\frac{L_S}{\bar{\lambda}},\ \text{where } \bar{\lambda}=(\lambda_0-\varepsilon)P(0)+(\lambda_1-\varepsilon)P(1) \qquad \text{... (39)}$$

Note

This model includes the certain models as particular cases For example, when $\rho_0=p_0$, $\rho_1=p_1$, $p=q=1$, $1=$ c and $k\to\infty$, this model reduces to Geo/Geo/c/∞ interdependent queueing model with controllable arrival rates which was discussed by Thiagarajan and Srinivasan, when $\lambda_0=\lambda_1=\lambda$, $k\to\infty$, $p=q=1$ and $\varepsilon=0$, this model reduces to the conventional Geo/Geo/1/∞ model discussed by Hunter (1983)

V. Numerical Illustrations

For various values of λ_0, λ_1, μ_1, ε, k while r, R are fixed values, computed and tabulated the values of $P_0(0)$, P(0) and P(1) by taking $p=q=\frac{1}{2}$.

Table 1

R	R	k	λ_0	λ_1	μ	ε	$P_0(0)$	P(0)	P(1)
4	8	15	6	5	5	1.0	0.497529660	0.986767159	0.013232573
4	8	20	8	6	5	0.5	0.616116034	0.999974808	0.000017692
4	8	15	6	5	5	0.5	0.419303234	0.939226648	0.060773350
4	8	15	7	5	5	0.5	0.554479100	0.999999927	0.000342617
4	8	15	5	4	4	0.5	0.570596976	0.995856393	0.000999999
4	8	20	5	4	5	0.5	0.305198362	0.964795991	0.035204006
4	8	15	8	6	6	0.5	0.455121423	0.972050156	0.027949841

For various values of λ_0, λ_1, μ, ε, k while r, R are fixed values, computed and tabulated the values of L_{S_0}, L_{S_1}, L_S and W_S by taking $p=q=\frac{1}{2}$.

Table 2

r	R	K	λ_0	λ_1	μ	ε	L_{S_0}	L_{S_1}	L_S	W_S
4	8	15	6	5	5	1.0	0.883309494	0.558968576	1.44227807	0.289221119
4	8	20	8	6	5	0.5	0.612664316	0.074227168	0.686891484	0.09158665
4	8	15	6	5	5	0.5	0.832481928	1.101296424	1.933778352	0.355524503
4	8	15	5	4	4	0.5	0.717485471	0.185390031	0.902875502	0.201316597
4	8	15	8	6	5	0.5	0.616257980	0.010429804	0.627055602	0.083623012
4	8	15	8	6	6	0.5	0.928934438	0.313281134	1.242215572	0.166872992

VI. Conclusion

It is observed from the Table 1 and Table 2 that

i. When the mean dependence rate increases and the other parameters are kept fixed, L_S and W_S decrease (either with feedback or without feedback).

ii. When the arrival rates and service rates increases and the other parameters are kept fixed, L_S and W_S increases (either with feedback or wither feedback).

iii. When the system size and arrival rates increases and the other parameters are kept fixed, L_S and W_S decrease (either with feedback or without feedback).

iv. When the service rate increases and the other parameters are kept fixed, L_S and W_S increase.

References

[1]. Goswami, V. and Gupta, U.C. (2002) Distribution of the Number of Customers Served During a Busy Period in a Discrete Time Geom/Geom/1 Queue, Indian J. Pure Appl. Maths., 33(9), 1405-1408.

[2]. Kalyanaraman, R. and Renganathan, N. (1996) A Single Server Instantaneous Bernoulli Feedback Queue with Multiple Vacation, Optimization, 37(1), 87-96.

[3]. Kalyanaraman, R. and Sumathy, S. (2009) A Feedback Queue with Multiple Servers and Batch Service, International Review of Pure and Applied Mathematics, 5(1), 23-27.

[4]. Rani, G. and Srinivasan, A. (2013) Geo/Geo/c/k Interdependent Queueing Model with Controllable Arrival Rates and Feedback, International Journal of Computational Science and Mathematics, 5(2), 33-42, ISSN: 0974-3189.

[5]. Rani, G. and Srinivasan, A. (2013) M/M/c/k Loss and Delay Interdependent Queueing Model with Controllable Arrival Rates, No Passing and Feedback, International Journal of Mathematical Sciences and Engineering Applications, 7(IV), 403-419, ISSN 0973-9424.

[6]. Takacs, L. (1963) A Single Server Queue with Feedback, Bell Systems Technical Journal, 42, 509-519.

[7]. Thiagarajan, M. and Srinivasan, A. (2007) Geo/Geo/c/∞ Interdependent Queueing Model with Controllable Arrival Rates, The Proceedings of the National Conference on Mathematical and Computational Models, Narosa Publishing House, 113-122.

[8]. Hunter, J.J. (1983) Mathematical Techniques of Applied Probability, Vol. 2, Discrete Time Models, Techniques and Applications, Academic Press, New York.