

## Accuracy Analysis of Numerical solutions of initial value problems (IVP) for ordinary differential equations (ODE)

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**Abstract:** In this paper, we solve numerically initial value problems for ordinary differential equations by Euler method. The proposed method is quite efficient and is practically well suited for solving these problems. Several examples are presented to demonstrate the accuracy and easy implementation of the proposed method. Our approximate solutions are compared with the exact solutions. The approximate solutions converge to the exact solutions monotonically. Finally we investigate and compute the error of proposed method for different step size.

**Key words:** Initial value problem, Euler's method, Error analysis.

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### I. Introduction

Many problems of mathematical physics can be started in the form of differential equations. These equations also occur as reformulations of other mathematical problems such as ordinary differential equations and partial differential equations. Numerical methods are commonly used for solving mathematical problems that are formulated in science and engineering where it is difficult or even impossible to obtain exact solutions. Only a limited number of differential equations can be solved analytically. There are many analytical methods for finding the solution of ordinary differential equations. Even then there exist a large number of ordinary differential equations whose solutions cannot be obtained in closed form by using well known analytical methods, where we have to use the numerical methods to get the approximate solution of a differential equation under the prescribed initial condition or conditions. There are many types of practical numerical methods for solving initial value problems for ordinary differential equations.

From the literature review we may realize that several works in numerical solutions using Euler method and carried out [1-10], many authors have attempted to solve initial value problems (IVP) to obtain high accuracy rapidly by using a numerous methods, such as Euler method's, and also some other methods. In this paper Euler method is applied without any discretization, transformation or restrictive assumptions for solving ordinary differential equation in initial value problems. The Euler Method is traditionally the first numerical technique. It is very simple to understand and geometrically easy. The numerical results are very encouraging. Finally two examples of different kinds of ordinary differential equations are given to verify the proposed formulation. The results of each numerical example indicate the convergence and error analysis are discussed illustrate the efficiency of the method. It is one of the oldest numerical methods used for solving an ordinary initial value differential equation, where the solution will be obtained as a set of tabulated values of variables  $x$  and  $y$ . It is a simple and single step but a crude numerical method of solving first-order ODE, particularly suitable for quick programming because of their great simplicity, although their accuracy is not high.

### II. Problem formulation

In this paper we introduce a first-order differential equation is an Initial value problems (IVP). Euler's method is the most elementary approximation method for solving initial value problems. The object of Euler's method is to obtain approximation to the well-posed initial -value problem

$$y' = f(x, y), x_0 \leq x \leq x_N, y(x_0) = y_0 \quad (2.1)$$

We assume that  $\partial f / \partial y$  is continuous on the strip  $x_0 \leq x \leq x_N$ , thus (2.1) possesses a unique solution. If  $y(x)$  is the exact solution to (2.1), its graph is a curve in the  $xy$  -plane passing through the point  $(x_0, y_0)$ . A discrete numerical solution of (2.1) is defined to be a set of points  $\{(x_n, y_n) : n = 0, 1, 2, \dots, N\}$  and each point  $(x_n, y_n)$  is an approximation to the corresponding point  $(x_n, y(x_n))$  on the solution curve. We again consider (2.1) as the model differential equation and begin by dividing the interval  $[x_0, x_N]$  into  $N$  equally spaced subintervals such that  $x_n = x_0 + nh$  for each  $n = 0, 1, 2, \dots, N$ . The parameter  $h$  is called the step size. If

$y(x)$  is the exact solution of (2.1), then using by Taylor's theorem to derive Euler's method.

$$y(x_{n+1}) = y(x_n) + (x_{n+1} - x_n)y'(x_n) + \frac{(x_{n+1} - x_n)^2}{2!} y''(\xi_n), x_n \leq \xi_n \leq x_{n+1} \quad (2.2)$$

From (2.2) we have

$$y(x_{n+1}) = y(x_n) + hf(x_n, y(x_n)) + \frac{h^2}{2!} f''(\xi_n, y(\xi_n))$$

Consider  $y_n \approx y(x_n)$  and truncating second and higher order term. We have

$$y_{n+1} = y_n + hf(x_n, y_n)$$

This formula is called the Euler method.

### III. Error Analysis

The purpose of examining error of Euler's method is to see how the approximated solution works. In Euler's method the slope of function affects the accuracy of methods. In order to minimize the error that occur, the step size should be chosen very small. In other words, number of points need to be taken enormous between the given interval. Thus when step size is chosen excessively small, error is minimized and approximated solution will be better. In solving an initial value problem, there are two types of errors: Local and global truncation errors.

#### 3.1 Local truncation error

The local truncation error of the Euler method is error made in a single step. It is the difference between the numerical solution after one step,  $y_1$  and the exact solution at time  $x_1 = x_0 + h$ . The numerical solution is given by  $y_1 = y_0 + hf(x_0, y_0)$ . For the exact solution, we use the Taylor expansion of the function  $y$  around

$x_0$ :  $y(x_0+h) = y(x_0) + hy'(x_0) + \frac{1}{2}h^2y''(x_0) + O(h^3)$ . The local truncation error (LTE) introduced by the Euler method is given by the difference between these equations:  
 $LTE = y(x_0 + h) - y_1 = \frac{1}{2}h^2y''(x_0) + O(h^3)$ . This result is valid if  $y$  has a bounded third derivative.

This shows that for small  $h$ , the local truncation error is approximately proportional to  $h^2$ . A slightly different formulation for the local truncation error can be obtained by using the Lagrange form for the remainder term in Taylor's theorem. If  $y$  has a continuous second derivative, then there exists a  $x_0 \leq \xi \leq x_0 + h$  such that

$$LTE = y(x_0 + h) - y_1 = \frac{1}{2}h^2y''(\xi).$$

#### 3.2 Global truncation error

The global truncation error is the error at a fixed point  $x$ , after however many steps the methods needs to take to reach that time from the initial value. The global truncation error is the cumulative effect of the local truncation errors committed in each step. The number of steps is easily determined to be  $\frac{x - x_0}{h}$ , which is proportional to

$\frac{1}{h}$ , and the error committed in each step is proportional to  $h^2$ . Thus, it is to be expected that the global truncation error will be proportional to  $h$ . If the solution  $y$  has a bounded second derivative and  $f$  is Lipchitz continuous in its second argument, then the global truncation error (GTE) is bounded by  $|GTE| \leq \frac{hM}{2L}(e^{L(x-x_0)} - 1)$ . where  $M$  is an upper bound on the second derivative of  $y$  on the given interval and  $L$  is the Lipchitz constant of  $f$ . It shows that the global truncation error is (approximately) proportional to  $h$ .

**IV. Numerical Examples:**

In this section we consider two nonlinear IVP problems to verify accuracy of the proposed Method. Then using this method we find numerical approximations for desired IVP. All the computations are performed by Mathematica software. The convergence of IVP is calculated by  $e_n = |y(x_n) - y_n| < \delta$  where  $y(x_n)$  denotes the approximate solution and  $y_n$  denotes the exact solution and  $\delta$  depends on the problem which varies from  $10^{-7}$ . The absolute error and absolute relative error for this formula is defined by absolute Error =  $|y(x_n) - y_n|$  and absolute Relative Error =  $\left| \frac{y(x_n) - y_n}{y_n} \times 100 \right|$ .

**Example 1:** we consider the initial value problem  $y'(x) = 1 + xy, y(0) = 2$  on the interval  $0 \leq x \leq 1$ . The exact solution of the given problem is given by  $y(x) = \frac{1}{2} e^x \left( 4 + \sqrt{2\pi} \operatorname{Erf} \left[ \frac{\pi}{\sqrt{2}} \right] \right)$ . The results obtained shown in Table 1(a), 1(b) and 1(c) and the graphs of the numerical values of  $y(x)$  are displayed in figure 1.1, 1.2 and 1.3.

**Table 1(a):** Numerical Approximations for different step size.

$x_n$	Numerical Approximations				Exact Solution
	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.0125$	
0.1	2.10000	2.10512	2.10773	2.10904	2.11036
0.2	2.22100	2.23189	2.23745	2.24026	2.24309
0.3	2.36542	2.38301	2.39204	2.39660	2.40122
0.4	2.53638	2.56196	2.57514	2.58184	2.58861
0.5	2.73784	2.77312	2.79139	2.80070	2.81012
0.6	2.97473	3.02199	3.04659	3.05915	3.0719
0.7	3.25321	3.31544	3.34800	3.36467	3.38161
0.8	3.58094	3.66203	3.70471	3.72661	3.74892
0.9	3.96741	4.07250	4.12812	4.15675	4.18597
1.0	4.42448	4.56029	4.63259	4.66994	4.70813

**Table 1(b):** Absolute Errors in different step size

$x_n$	Absolute Errors			
	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.0125$
0.1	0.01036	0.00524	0.00263	0.00132
0.2	0.14309	0.0112	0.00564	0.00283
0.3	0.18022	0.01821	0.00918	0.00462
0.4	0.22319	0.02665	0.01347	0.00677
0.5	0.27374	0.037	0.01873	0.00942
0.6	0.33406	0.04991	0.02531	0.01275
0.7	0.40688	0.06617	0.03361	0.01694
0.8	0.49571	0.08689	0.04421	0.02231
0.9	0.60503	0.11347	0.05785	0.02922
1.0	0.74072	0.14784	0.07554	0.03819

**Table 1(c):** Relative Errors in different step size

$x_n$	Relative Errors			
	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.0125$
0.1	0.490912	0.248299	0.124623	0.062549
0.2	0.984802	0.499311	0.251439	0.126165
0.3	1.490909	0.758364	0.382306	0.192402
0.4	2.0176785	1.02951	0.520356	0.26153
0.5	2.562132	1.31667	0.66652	0.335217
0.6	3.163189	1.624727	0.82392	0.415053
0.7	3.797008	1.95676	0.993905	0.500945
0.8	4.480757	2.317734	1.179273	0.595105
0.9	5.221251	2.710722	1.381997	0.698046
1.0	6.024685	3.1401	1.604459	0.81115

Figure:1.1 Numerical approximation in different step size

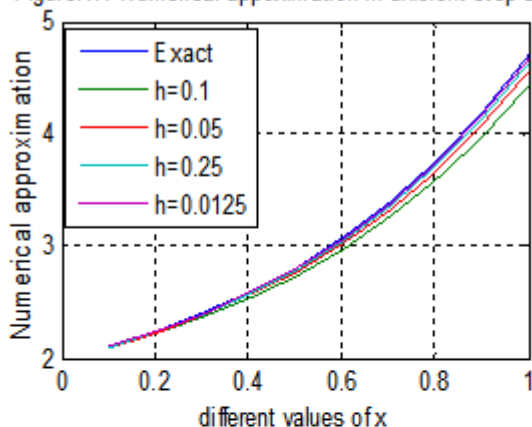


Figure:1.2 Absolute error in different step size

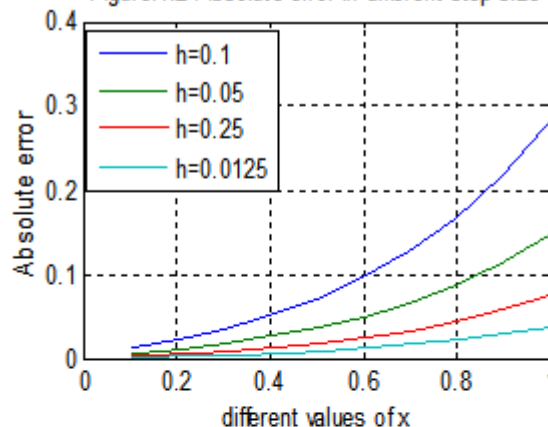
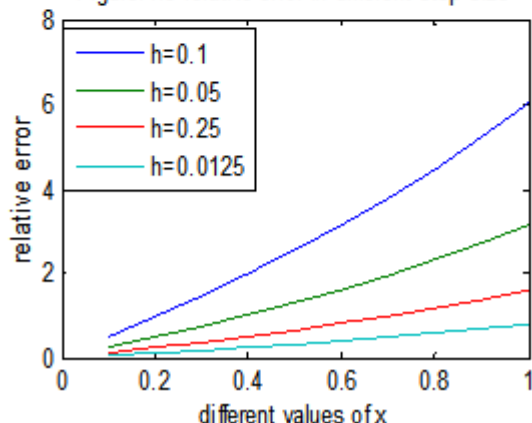


Figure:1.3 relative error in different step size



**Example 2:** we consider the initial value problem  $y'(x) = 1 + \frac{1}{2}y^2$ ,  $y(0) = 1$  on the interval  $0 \leq x \leq 1$ . The exact solution of the given problem is given by  $y(x) = \sqrt{2} \tan\left(\frac{x}{\sqrt{2}} + \tan^{-1}\left(\frac{1}{\sqrt{2}}\right)\right)$ . The results obtained shown in Table 2(a), 2(b) and 2(c) and the graphs of the numerical values of  $y(x)$  are displayed in figure 2.1, 2.2 and 2.3.

Table 2(a): Numerical Approximations for different step size.

$x_n$	Numerical Approximations				Exact solution
	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.0125$	
0.1	1.15000	1.15389	1.15598	1.15706	1.15817
0.2	1.31613	1.32544	1.33049	1.33312	1.33582
0.3	1.50273	1.51972	1.52903	1.53391	1.53895
0.4	1.71560	1.74368	1.75924	1.76746	1.77601
0.5	1.96282	2.0070	2.03206	2.04539	2.05935
0.6	2.25545	2.3242	2.36382	2.38523	2.40786
0.7	2.60980	2.71665	2.77995	2.81473	2.85196
0.8	3.05035	3.21889	3.32243	3.38064	3.44406
0.9	3.61559	3.88908	4.06567	3.16825	4.28301
1.0	4.36921	4.83316	5.15475	5.35106	5.58016

Table 2(b): Absolute Errors in different step size

$x_n$	Absolute Errors			
	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.0125$
0.1	0.00817	0.00428	0.00219	0.00111
0.2	0.019695	0.01038	0.00533	0.0027
0.3	0.03622	0.01923	0.00992	0.00504
0.4	0.06041	0.03233	0.01677	0.00855
0.5	0.09653	0.05235	0.02729	0.01396
0.6	0.15241	0.08366	0.04404	0.02263
0.7	0.24216	0.13531	0.07201	0.03723
0.8	0.39371	0.22517	0.12163	0.06342
0.9	0.66742	0.39393	0.21734	0.11476
1.0	1.21095	0.747	0.42541	0.2291

Table 2(c): Relative Errors in different step size

$x_n$	Relative Errors			
	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = 0.0125$
0.1	0.705423	0.369549	0.189091	0.095841
0.2	1.474375	0.777051	0.399006	0.202123
0.3	2.353553	1.249553	0.644595	0.327496
0.4	3.401445	1.820373	0.944251	0.481416
0.5	4.687401	2.542064	1.325175	0.677884
0.6	6.329687	3.474454	1.82901	0.939839
0.7	8.491003	4.744456	2.52493	1.305418
0.8	11.43157	6.537923	3.531588	1.841431
0.9	15.58297	9.197504	5.074469	2.679424
1.0	21.70099	13.38671	7.623617	4.105617

Figure:2.1 Numerical approximation in different step size

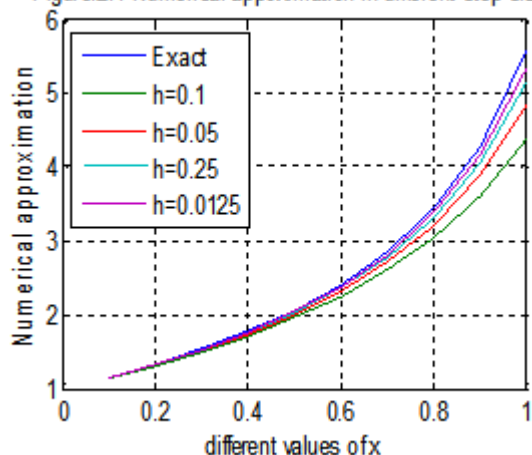


Figure:2.2 Absolute error in different step size

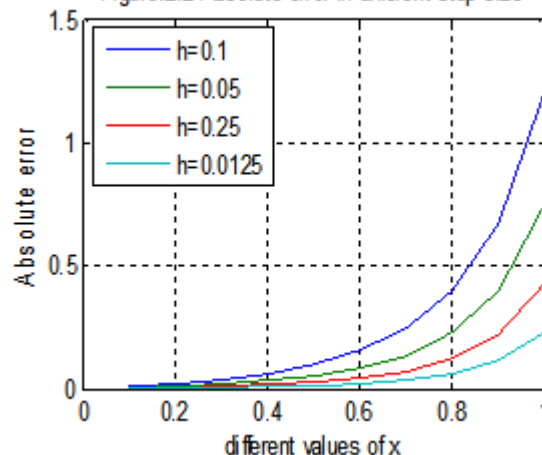
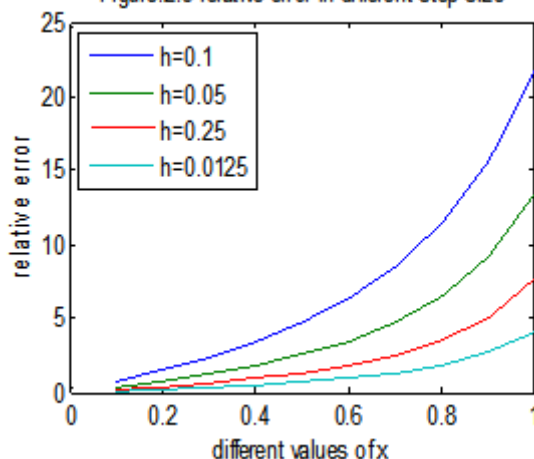


Figure:2.3 relative error in different step size



## V. Discussion of results

The calculated results are displayed in Table 1 and Table 2 and graphically represent in Figure1 and Figure2. Reasonably good results are obtained even for a moderately large step size and the approximation can be improved by decreasing the step size. The proposed methods give very good result when compared with the exact solution. In principle, we say that a numerical solution converges to the exact solution if decreasing the step size leads to decreased errors such that in the limit when the step size to zero the errors go to zero.

## VI. Conclusion

In this paper, we have used Euler method for solving ordinary differential equation in initial value problems (IVP). To achieve the desired accuracy of numerical solution it is necessary to take step size very, very small. Therefore, it is an important task to make a proper balance between accuracy and efficiency of the solutions. The numerical results obtained by the proposed method are in good agreement with exact solutions. The results of each problem guarantee the convergence and stability. We may notice that the accuracy increase with decrease step size, which is shown in Table 1, Table 2. We may realize that this method may be applied to solve IVP to find the desired accuracy.

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