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Adomian decomposition method for solving a class of hyperbolic equations with nonlocal conditions

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Abstract: *In this paper, we show that with a few modifications and techniques the Adomian Decomposition Method (ADM) for solving a class of hyperbolic equations with nonlocal conditions can be applied to obtain the exact solutions to this type of problems. The clue of this one consists in transforming the boundary value nonlocal problem into a classical problem whose solution is obtained.*
Keywords: *Hyperbolic equation, Nonlocal conditions, Adomian decomposition method*

I. Introduction

Boundary value problems for parabolic and hyperbolic equations with nonlocal conditions have been investigated in [3, 4, 5, 6, 7, 9, 10, 11, 12, 14, 15, 16, 18, 19]. The consideration of these nonlocal conditions is motivated by a number of physical problems in various fields: plasma physics [15], heat conduction [10], dynamics of ground water [14], thermo-elasticity [14]. Such problems constitute a very interesting and important class of problems. In [4, 5, 6] a simple boundary value problem for hyperbolic wave equation with an integral condition $\int_0^1 u(t,x)dx = 0$ has been investigated and several results concerning existence and uniqueness of solutions by using Fourier's method and Adomian decomposition method [1, 2] have been established.

In [8, 17], boundary value problems with an integral conditions $\int_0^1 u\,(t,x)dx = 0$ and $\int_0^1 x\,u\,(t,x)dx = 0$ for a second-order hyperbolic equation with the Bessel operator

$$-\frac{1}{x}\frac{\partial}{\partial x}\left(x\frac{\partial u}{\partial x}\right)$$

have received considerable attention.

Motivated by this, we study a class of hyperbolic equations with nonlocal conditions

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{x^r}\frac{\partial}{\partial x}\left(x^{r+2}\frac{\partial u}{\partial x}\right) + k\,u = f\,(t,x), \quad k \text{ and } 0 \neq r \in \Re\,, \tag{1.1}$$

in the rectangular domain $\Omega = (0,\,T)\times(0,\,1)$.

To equation (1.1) we attach the initial conditions

$$u(0,x) = \phi(x)\,, \tag{1.2}$$

$$\frac{\partial u}{\partial t}(0,x) = \psi(x)\,, \tag{1.3}$$

and the nonlocal boundary conditions

$$\int_0^1 u\,(t,x)dx = 0\,, \tag{1.4}$$

$$\int_0^1 x^r\,u\,(t,x)dx = 0\,, \tag{1.5}$$

where $\phi(x), \psi(x) \in L_2(0,1)$ are known functions and satisfy compatibility conditions

$$\int_0^1 \phi(x)dx = \int_0^1 x^r \phi(x)dx = 0$$

and

$$\int_0^1 \psi(x)dx = \int_0^1 x^r \psi(x)dx = 0.$$

The general difficult which arises to us is the presence of two integral conditions which complicates the application of standard functional or numerical methods, owing to the fact that the elliptic differential operator with integral condition is no longer positive definite in the usual function spaces, which poses the main source of difficulty.

As stated in [4], the clue of this one consists in transforming (1.1)-(1.5) to the following problem.

Lemma 1:

Problem (1.1)-(1.5) is equivalent to the following problem

$$(\mathrm{Pr})_1 \quad \begin{cases} \dfrac{\partial u^2}{\partial t^2} - \dfrac{1}{x^r}\dfrac{\partial}{\partial x}\left(x^{r+2}\dfrac{\partial u}{\partial x}\right) + k\,u = f(t,x), \\ u(0,x) = \phi(x), \\ \dfrac{\partial u}{\partial t}(0,x) = \psi(x), \\ u(t,1) = -\dfrac{1}{r}\displaystyle\int_0^1 f(t,x)dx + \dfrac{1}{r}\int_0^1 x^r f(t,x)dx, \\ \dfrac{\partial u}{\partial x}(t,1) = -\displaystyle\int_0^1 x^r f(t,x)dx. \end{cases}$$

Proof:

Let $u(t,x)$ be a regular solution of (1.1) and satisfy the conditions (1.2)-(1.5).

Integrating Eq. (1.1) with respect to x over $(0,1)$ and taking into account of (1.4), we obtain

$$\left[x^2 \frac{\partial u}{\partial x}\right]_0^1 + r\int_0^1 x\frac{\partial u}{\partial x}dx = -\int_0^1 f(t,x)dx$$

and so

$$\frac{\partial u}{\partial x}(t,1) + r\,u(t,1) = -\int_0^1 f(t,x)dx \tag{1.6}$$

To eliminate the second nonlocal condition $\int_0^1 x^r u(t,x)dx = 0$, multiplying both sides of (1.1) by x^r and integrating the resulting over $(0,1)$, and taking into account of (1.5), we obtain

$$\frac{\partial u}{\partial x}(t,1) = -\int_0^1 x^r f(t,x)dx. \tag{1.7}$$

Substituting (1.7) into (1.6)

$$u(t,1) = -\frac{1}{r}\int_0^1 f(t,x)dx + \frac{1}{r}\int_0^1 x^r f(t,x)dx$$

and

$$\frac{\partial u}{\partial x}(t,1) = -\int_0^1 x^r f(t,x)dx.$$

Let now $u(t,x)$ be a regular solution of $(pr)_1$, we are required to prove that

$$\int_0^1 u(t,x)dx = 0$$

and

$$\int_0^1 x^r u(t,x)dx = 0 \ .$$

We integrate Eq. (1.1) with respect to x over $(0, 1)$, and taking into account $u(t, 1) = -\frac{1}{r}\int_0^1 f(t,x)dx + \frac{1}{r}\int_0^1 x^r f(t,x)dx$ and $\frac{\partial u}{\partial x}(t,1) = -\int_0^1 x^r f(t,x)dx$, we obtain

$$\frac{d^2}{dt^2}\int_0^1 u(t,x)dx + k\int_0^1 u(t,x)dx = 0, \quad t \in (0,T).$$

By multiplying by x^r and integrating over $(0, 1)$, we get

$$\frac{d^2}{dt^2}\int_0^1 x^r u(t,x)dx + k\int_0^1 x^r u(t,x)dx = 0, \quad t \in (0,T).$$

By virtue of the above compatibility conditions, we get

$$\int_0^1 u(t,x)dx = 0, \text{ and } \int_0^1 x^r u(t,x)dx = 0.$$

Introduce now the new unknown function $v = u - \frac{x^{-r}}{r}\int_0^1 x^r f(t,x)dx + \frac{1}{r}\int_0^1 f(t,x)dx$.

Then we have the following lemma.
Lemma 2:
$(pr)_1$ can be transformed into the following problem

$$(pr)_2 \quad \begin{cases} \frac{\partial^2 v}{\partial t^2} - \frac{1}{x^r}\frac{\partial}{\partial x}\left(x^{r+2}\frac{\partial v}{\partial x}\right) + k\,v = F(t,x), \\ v(0,x) = \Phi(x), \\ \frac{\partial v}{\partial t}(0,x) = \Psi(x), \\ v(t,1) = 0, \\ \frac{\partial v}{\partial x}(t,1) = 0 \cdot \end{cases}$$

where

$$F(t,x) = f(t,x) - \frac{1}{x^r}\int_0^1 x^r f(t,x)dx - k\left(\frac{1}{r\,x^r}\int_0^1 x^r f(t,x)dx - \frac{1}{r}\int_0^1 f(t,x)dx\right)$$
$$-\frac{1}{r\,x^r}\int_0^1 x^r f_{tt}(t,x)dx + \frac{1}{r}\int_0^1 f_{tt}(t,x)dx,$$

$$\Phi(x) = \phi(x) - \frac{1}{r\,x^r}\int_0^1 x^r f(0,x)dx + \frac{1}{r}\int_0^1 f(0,x)dx$$

and

$$\Psi(x) = \psi(x) - \frac{1}{r\,x^r}\int_0^1 x^r f_t(0,x)dx + \frac{1}{r}\int_0^1 f_t(0,x)dx.$$

II. Adomian Decomposition Method

Now we shall use the Adomian's decomposition method [1, 2] for solving (1.1)-(1.5).
The solution proposed by Adomian [1, 2] is to take L as the highest-order derivative of the linear part. For example, for the boundary value problem $\frac{\partial^2 v}{\partial t^2}-\frac{\partial^2 v}{\partial x^2}=F(t,x)$ with and $v(t,0)=\alpha$ and $v_x(t,0)=\beta.$

Adomian defines L_{xx}^{-1} for $L_{xx}=\frac{\partial^2}{\partial x^2}$ as the two-fold integration operator from 0 to x. We have

$$\left(L_{xx}^{-1}L_{xx}\right)v(t,x)=v(t,x)-v(t,0)-xv_x(t,0)$$

and therefore

$$v(t,x)=v(t,0)+xv_x(t,0)+L_{xx}^{-1}F+L_{xx}^{-1}\frac{\partial^2 v}{\partial t^2}.$$

Adomian's point of view was to take L_{xx} as the simplest easily invertible operator. With this choice the method has given many beautiful results. Hear in order to find the classical solutions in x direction for $(pr)_2$ this approach is not sufficient. Therefore, an efficient modification of this approach is introduced by choosing a general differential operator.
Consider the given hyperbolic equation in an operator form as

$$L_{xx}v=\frac{\partial^2 v}{\partial t^2}+k\,v-F(t,x)\,, \tag{2.1}$$

where

$$L_{xx}v=\frac{1}{x^r}\frac{\partial}{\partial x}\left(x^{r+2}\frac{\partial v}{\partial x}\right).$$

A formal inverse of (2.1) can be easily found. We choose it as

$$L_{xx}^{-1}\,v(t,x)=\int_x^1\frac{d\xi}{p(\xi)}\int_\xi^1\frac{v(t,s)ds}{h(s)}\,,$$

where $p(x)=x^{r+2}$ and $h(x)=\frac{1}{x^r}$. Notice that

$$L_{xx}^{-1}\,L_{xx}\neq L_{xx}\,L_{xx}^{-1}.$$

Appling L_{xx}^{-1} to Eq. (2.1), we see that

$$\left(L_{xx}^{-1}\,L_{xx}\right)v(t,x)=\int_x^1\frac{d\xi}{p(\xi)}\int_\xi^1\frac{1}{h(s)}\left(h(s)\frac{\partial}{\partial s}\left(p(s)\frac{\partial v}{\partial s}\right)\right)ds,$$

$$\left(L_{xx}^{-1}\,L_{xx}\right)v(t,x)=\int_x^1\frac{d\xi}{p(\xi)}\left[p(s)\frac{\partial v}{\partial s}\right]_\xi^1,$$

so that

$$\left(L_{xx}^{-1}\,L_{xx}\right)v(t,x)=v(t,x)-v(t,1)+p(1)\frac{\partial v}{\partial x}(t,1)\int_x^1\frac{d\xi}{p(\xi)}.$$

Therefore, we obtain

$$v(t,x)=v(t,1)-p(1)v_x(t,1)\int_x^1\frac{d\xi}{p(\xi)}+L_{xx}^{-1}\left(\frac{\partial^2 v}{\partial t^2}+k\,v\right)-L_{xx}^{-1}\left(F(t,x)\right). \tag{2.2}$$

Following the Adomian decomposition method [1, 2] the unknown solution $v(t,x)$ is assumed to be given by a series of the form

(2.3) $$v(t,x)=\sum_{n=0}^{\infty} v_n(t,x) \ ,$$

where the components $v_n(t,x)$ are going to be determined recurrently.

We recall the following theorem, which guarantees the convergence of Adomian decomposition method for the general operator equation given by $Lu+Ru+Nu=g$, where L is the highest order differential operator, R the linear term, N represent the nonlinear part and g is a given function in a Hilbert space H .

Theorem:

Let $Tu=-Ru-Nu$ be a hemi-continuous operator in H and satisfy the following hypothesis:

1. $(Tu-Tv,u-v)_H \geq K\|u-v\|_H^2, K>0, \forall u,v \in H.$
2. $\forall M>0, \exists C(M)>0$ such that $\|u\|_H \leq M$ and $\|v\|_H \leq M$, we have $(Tu-Tv,w)_H \leq C(M)\|u-v\|_H \|w\|_H, \forall w \in H.$

Then for every $g \in H^*$ the nonlinear function equation $Lu+Ru+Nu=g$ admits a unique solution $u \in H$. Furthermore, if the solution u can be expressed in a series form given by $u=\sum_{n=0}^{\infty} u_n$, then the Adomian decomposition method scheme corresponding to the functional equation under consideration converges strongly to $u \in H$, which is the unique solution to the functional equation.

Proof:

See [20, 21].

Therefore, we can write the solutions of Eq. (2.2) in x directions as

$$\begin{cases} v_0(t,x) = v(t,1)-p(1)v_x(t,1)\int_x^1 \dfrac{d\xi}{p(\xi)} - L_{xx}^{-1}(F(t,x)) = -L_{xx}^{-1}(F(t,x)), \\ v_{n+1}(t,x) = L_{xx}^{-1}\dfrac{\partial^2 v_n}{\partial t^2} + L_{xx}^{-1} k\, v_n, \quad n \geq 0, \end{cases}$$

Using the scheme described above, we obtained series solutions for $(pr)_2$.

Notice that the given boundary conditions $v(t,1)=0$ and $v_x(t,1)=0$ in $(pr)_2$ are sufficient to carry out the solution and the other initial conditions $v(0,x)=\Phi(x)$ and $v_t(0,x)=\psi(x)$ can be used to show that the obtained solution satisfies these given conditions.

In order to demonstrate the feasibility and efficiency of the ADM, two simple examples with closed form solution are studied carefully.

Example 1:

Consider problem (1.1)-(1.5) with

$$k=-1, \quad f(t,x)=2\pi\left[3x\sin(2\pi x)+2\pi x^2\cos(2\pi x)\right]e^t, \quad r=1$$

and

$$\phi(x)=\psi(x)=\cos(2\pi x).$$

By lemma 2 this nonlocal boundary problem can be transformed into the form of $(pr)_2$, and straightforward computation yields

$$F(t,x)= 2\pi\left[3x\sin(2\pi x)+2\pi x^2\cos(2\pi x)\right] e^t.$$

To obtain the solution in x direction, we express recurrent scheme of ADM (2-5) as

$$\begin{cases} v_0(t,x) = v(t,1)-p(1)v_x(t,1)\int_x^1 \dfrac{d\zeta}{p(\zeta)} - L_{xx}^{-1}(F(t,x)), \\ v_{n+1}(t,x) = L_{xx}^{-1}\dfrac{\partial^2 v_n}{\partial t^2} + L_{xx}^{-1} k\, v_n, \; n \geq 0, \end{cases}$$

taking into account $v(t,1)=0$ and $v_x(t,1)=0$, and direct calculation produces

$$\begin{cases} v_0(t,x) &= e^t\left(\cos(2\pi x)-1\right), \\ v_{n+1}(t,x) &= 0,\ n\geq 0. \end{cases}$$

Then the terms of $v(t,x)$ can be written as

(2.4) $$v(t,x)=\sum_{n=0}^{\infty} v_n(t,x)=e^t\left(\cos(2\pi x)-1\right)$$

and from

(2.5) $$v=u-\frac{x^{-r}}{r}\int_0^1 x^r f(t,x)dx+\frac{1}{r}\int_0^1 f(t,x)dx,$$

we get

(2.6) $$u(t,x)=e^t\cos(2\pi x),$$

which is exact solution of (1.1)-(1.5).

Example 2:

Consider problem (1.1)-(1.5) with

$$k=-1,\quad f(t,x)=\left[6\pi x^2\sin(2\pi x)+4\pi^2x^3\cos(2\pi x)-x\cos(2\pi x)\right]e^t,\quad r=-1$$

and

$$\phi(x)=\psi(x)=x\cos(2\pi x).$$

By lemma 2 this nonlocal boundary problem can be transformed into the form of $(pr)_2$, and straightforward computation yields

$$F(t,x)=\left[6\pi x^2\sin(2\pi x)+4\pi^2x^3\cos(2\pi x)-x\cos(2\pi x)+x\right]e^t\cdot$$

To obtain the solution in x direction, we express the recurrent scheme of ADM as

$$\begin{cases} v_0(t,x) &= xe^t\left(\cos(2\pi x)-1\right), \\ v_{n+1}(t,x) &= 0,\ n\geq 0. \end{cases}$$

Then the terms of $v(t,x)$ can be written as

(2.7) $$v(t,x)=\sum_{n=0}^{\infty} v_n(t,x)=xe^t\left(\cos(2\pi x)-1\right).$$

So that

(2.8) $$u(t,x)=xe^t\cos(2\pi x),$$

which is the exact solution of (1.1)-(1.5).

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