The Finite Difference Methods for Fitz Hugh-Nagumo Equation

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Abstract: *we have studied the numerical solutions for FitzHugh-Nagumo equation (FHN) using Finite Difference Methods (FDM) including explicit method, implicit (Crank-Nicholson) method, fully implicit method, Exponential method. A Comparison was made among all the methods by solving two numerical examples with different time steps.*
Keywords: *Fitz Hugh-Nagumo equation, explicit, implicit, fully implicit, Exponential methods.*

I. Introduction

We can classify PDEs in hyperbolic, parabolic and elliptic equations. Hyperbolic PDEs usually describe phenomena in which features propagate in preferred directions, while keeping its strength (like supersonic flow). Elliptic PDEs usually describe phenomena in which features propagate in all directions, while decaying in strength (like subsonic flow). Parabolic PDEs are just a limit case of hyperbolic PDEs.The two types of physical problems (i.e., equilibrium and propagation problems) are discussed [1].

To solve differential equations numerically we can replace the derivatives in the equation with finite difference approximations on a discredited domain. This results in a number of algebraic equations that can be solved one at a time (explicit methods) or simultaneously (implicit methods) to obtain values of the dependent function corresponding to values of the independent function in the discredited domain [2].

II. Indentations and EQUATIONS

2.1 Mathematical model:
The general form of FitzHugh- Nagumo Equation (FHN) is:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(u - a)(1 - u) \tag{1}$$

with the initial condition

$$u(x, 0) = f(x), \qquad 0 \leq x \leq 1$$

and the boundary conditions

$$u(0, t) = g_0, \ u(1, t) = g_1, \qquad 0 < t \leq T$$

Where $0 < a < 1$ is an arbitrary constant. It is a nonlinear equation proposed by Hodgkin and Huxley [3], it is the most widely accepted mathematical description of the excitation and propagation of nerve impulses [4].

The FitzHugh- Nagumo system of equations has been derived by both Fitzhugh and Nagumo [5, 6]. It is an important nonlinear reaction-diffusion equation used in physics circuit, biology and the area of population genetics as mathematical models [4].

Neuronal dynamics and stability of the differential equations described have been solved in wide aspects, beginning with Hodgkin and Huxley (1952), FitzHugh (1961) and Nagumo *et al* . (1962). Mackey and Nechaeva (1995). Zhang *et al* . (2010), finiahing with Tanabe and Pakdaman (2001) and Hasegawa (2003, 2004). Hasegawa solves the dynamics of the Fitzhugh-Nagumo model of neuron ensembles with time-delayed couplings among neurons, noises and stochastic. Tanabe considers the solutions by numerical calculations for single Hodgkin-Huxley neurons. Zhang studies the traveling wave fronts in synaptic coupled neuronal networks more from the mathematical point of view [7]. In 1950 Hodgkin and Huxley developed a system of non-linear partial differential equations while studying a giant axon of a squid to show the action potential of the nerve axon. The Hodgkin-Huxley model is too difficult to solve analytically, so in 1961 FitzHugh and Nagumo created a simplified version. This simplified equation contains two variables opposed to the four variables of the Hodgkin-Huxley model. The FitzHugh-Nagumo equations show the qualitative solution to the nerve action impulse model [8].

2.2 The derivation of the explicit method for solving Fitz Hugh-Nagumo equation

In this method we use forward difference at time t_j and a second-order central difference for the space derivative at position x_i which was devoted by the unknown function $u_{i,j+1}$ at t_{j+1} depending on the known

function $u_{i+1,j}$, $u_{i,j}$, $u_{i-1,j}$ at t_j and $u_{i,j-1}$ at t_{j-1}. Assuming the rectangle $R = \{(x,t) : 0 \le x \le a,\ 0 \le t \le b\}$ is subdivided into $(n-1)$ by $(m-1)$ rectangle with sides $\Delta x = h$ and $\Delta t = k$

Start at the bottom row, where $t = t_1 = 0$ and the solution is $u(x_i t_j) = f(x_i)$. A method for computing the approximations to $u(x,t)$ at grid points in successive rows $\{u(x_i t_j) : i = 1,2,\dots,n\}$, for $j = 2,3,\dots,m$.

we get:

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + u_{i,j}(u_{i,j} - a)(1 - u_{i,j}).$$

Let $r = \frac{k}{h^2}$. Hence

$$u_{i,j+1} = r(u_{i+1,j} + u_{i-1,j}) + (1 - 2r)u_{i,j} + k(u_{i,j}(u_{i,j} - a)(1 - u_{i,j})). \qquad (2)$$

Equation (2) is the explicit difference equation to the FitzHugh-Nagumo equation.

2.3 The derivation of the Semi Implicit (Crank-Nicholson) Method for solving Fitz Hugh-Nagumo equation

This method was developed by John Crank and Phyllis Nicolson in 1947, and is based on numerical approximation for solution. They replaced u_{xx} by the mean of its finite difference presentation on the jth and $(j+1)th$ time rows

$$\frac{u_{i,j+1} - u_{i,j}}{k} = \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1} + u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{2h^2} + u_{i,j}(u_{i,j} - a)(1 - u_{i,j})$$

After rearrangement of the above equation, we get:

$$(1 + r)u_{i,j+1} - \frac{r}{2}(u_{i+1,j+1} + u_{i-1,j+1}) = \frac{r}{2}(u_{i+1,j} + u_{i-1,j}) + (1 - r)u_{i,j} + ku_{i,j}(u_{i,j} - a)(1 - u_{i,j}). \qquad (3)$$

Equation (3) represents the semi implicit difference approximation for FitzHugh-Nagumo equation, where the left side of equation (3) contains three unknown values $\{u_{i-1,j+1}, u_{i,j+1}, u_{i+1,j+1}\}$, while the right side contains three known values $\{u_{i-1,j}, u_{i,j}, u_{i+1,j}\}$ for $i = 2,3,\dots,n-1$.

Hence, equation (3) forms a tridaigonal linear system: $AX = B$ Which can be solved by either direct methods or by iteration methods.

2.4 The derivation of the Fully Implicit Method for solving FitzHugh-Nagumo equation

In this method, we compute the approximations to $u_{i,j}$ at grid points in successive rows $\{u(x_i t_j) : i = 1,2,\dots,n\}$, for $j = 2,3,\dots,m$. which gives:

$$\frac{u_{i,j+1} - u_{ij}}{k} = \frac{u_{i+1,j+1} - 2u_{i,j+1} + u_{i-1,j+1}}{h^2} + u_{i,j+1}(u_{i,j+1} - a)(1 - u_{i,j+1}) \qquad (4)$$

$$-ru_{i-1,j+1} + \left(1 + 2r - k(u_{i,j+1} - a)(1 - u_{i,j+1})\right)u_{i,j+1} - ru_{i+1,j+1} = u_{i,j}. \qquad (5)$$

Equation (5) represents the fully implicit difference approximation for FitzHugh-Nagumo equation, where the left side of equation (5) contains three unknown values $\{u_{i-1,j+1}, u_{i,j+1}, u_{i+1,j+1}\}$ and the right side known value is $u_{i,j}$.

2.5 The derivation of the Exponential Method for solving FitzHugh-Nagumo equation

The exponential finite-difference method that we applied to solve FitzHugh Nagumo equation (1) was originally developed by Bhattachary [9] and used to solve one dimensional heat conduction in a solid slab [10]. It is also used to solve the Korteweg-de Vriesequation [11, 12].

We assume that $F(u)$ denotes any continuous differential function. Multiplying eq. (1) by the derivation of F leads to the following equation

$$\frac{\partial F}{\partial u}\frac{\partial u}{\partial t} = \frac{\partial F}{\partial u}\left(\frac{\partial^2 u}{\partial x^2} + u(u - a)(1 - u)\right). \text{ Thus}$$

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial u}\left(\frac{\partial^2 u}{\partial x^2} + u(u - a)(1 - u)\right). \qquad (6)$$

Using the usual forward difference replacement to $\frac{\partial F}{\partial t}$ obtains the finite difference representation of equation (6) as:

$$\frac{\partial F}{\partial t} = \frac{F(u_{i,j+1}) - F(u_{i,j})}{k}. \text{ This implies that } \frac{F(u_{i,j+1}) - F(u_{i,j})}{k} = \left(\frac{\partial F}{\partial u}\right)_{(i,j)}\left(\left(\frac{\partial^2 u}{\partial x^2}\right)_{(i,j)} + u_{i,j}(u_{i,j} - a)(1 - u_{i,j})\right).$$

Hence

$$F(u_{i,j+1}) = F(u_{i,j}) + k\left(\frac{\partial F}{\partial u}\right)_{(i,j)}\left(\left(\frac{\partial^2 u}{\partial x^2}\right)_{(i,j)} + u_{i,j}(u_{i,j} - a)(1 - u_{i,j})\right) \qquad (7)$$

Assume let $F(u) = \ln(u)$ and $\frac{\partial F}{\partial u} = \frac{1}{u}$ in (7), the exponential finite difference scheme becomes

$$\ln(u_{i,j+1}) = \ln(u_{i,j}) + k(\frac{1}{u_{i,j}})((\frac{\partial^2 u}{\partial x^2})_{(i,j)} + u_{i,j}(u_{i,j} - a)(1 - u_{i,j})).$$

taking the exponential to both sides of the equation

$$\exp(\ln(u_{i,j+1})) = \exp(\ln(u_{i,j}) + (\frac{k}{u_{i,j}})((\frac{\partial^2 u}{\partial x^2})_{(i,j)} + u_{i,j}(u_{i,j} - a)(1 - u_{i,j}))).$$

Hence:

$$u_{i,j+1} = u_{i,j} \exp((\frac{k}{u_{i,j}})((\frac{\partial^2 u}{\partial x^2})_{(i,j)} + u_{i,j}(u_{i,j} - a)(1 - u_{i,j}))). \qquad (8)$$

we get

$$u_{i,j+1} = u_{i,j} \exp((\frac{r}{u_{i,j}})(u_{i+1,j} - 2u_{i,j} + u_{i-1,j} + ku_{i,j}(u_{i,j} - a)(1 - u_{i,j}))) \qquad (9)$$

III.   Figures and Tables

3.1 Numerical Examples

We solved the following examples numerically to illustrate the efficiency of the presented methods

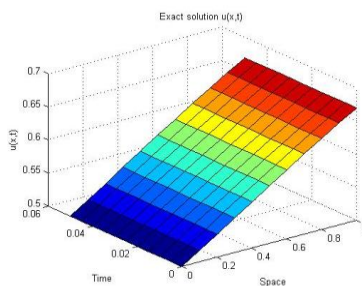**Example 1 [4]:** We take the FitzHugh-Nagumo equation (1):

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(u - a)(1 - u), t \geq 0$$

with the initial condition
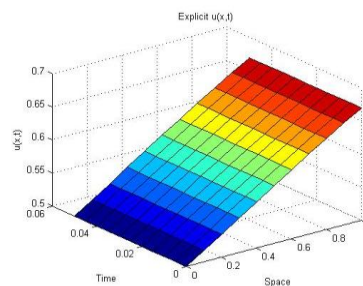$$u(x, 0) = \frac{1}{1 + e^{(-x/\sqrt{2})}}, 0 \leq x \leq 1$$

We take $a = 0.5$, $h = 0.1$, $k = 0.005$, and the exact solution is $u(x, t) = \frac{1}{1 + e^{(-s/\sqrt{2})}}$

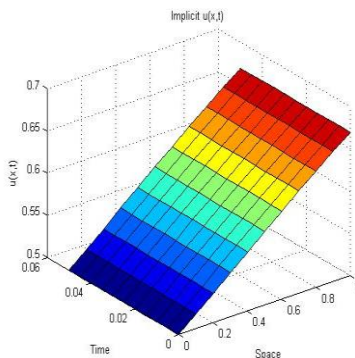where the wave speed $s = x + ct$, $c = \sqrt{2}(\frac{1}{2} - a)$.

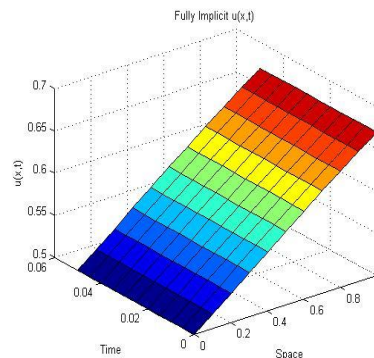The boundary conditions are given by $u(0, t) = \frac{1}{1 + e^{\frac{-ct}{\sqrt{2}}}}$, $u(1, t) = \frac{1}{1 + e^{\frac{-1-ct}{\sqrt{2}}}}$.

Figure 1: Exact Solution (0<*t*<0.05 and 0<*x*<1)   **Figure 2:** EFDM Solution (0<*t*<0.05 and 0<*x*<1)

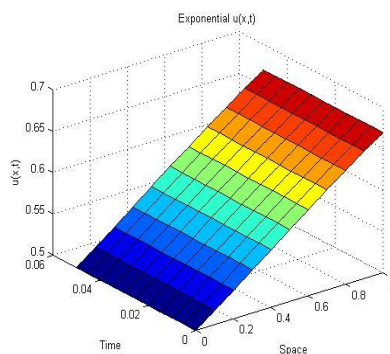**Figure 3:** Implicit Solution (0<*t*<0.05 and 0<*x*<1)   **Figure 4:** Fully implicit Solution (0<*t*<0.05 and 0<*x*<1)

Figure 5: Exponential Solution (0<*t*<0.05 and 0<*x*<1)

Example 2 [12]:
We take the FitzHugh-Nagumo equation (1):
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(u - a)(1 - u), \ t \geq 0$$ With the initial condition $u(x, 0) = \sin(\pi x), \ 0 \leq x \leq 1$
We take $a = 0.5, \quad h = 0.1, \quad k = 0.005,$
The boundary conditions is given by $\quad u(0, t) = 0, \quad\quad u(1, t) = 0.$
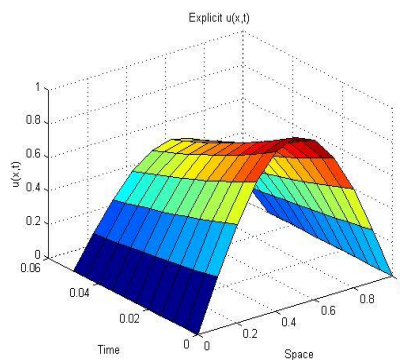
Figure 6: Space-time graph of Explicit solution to
0<*t*<0.05 and 0<*x*<1
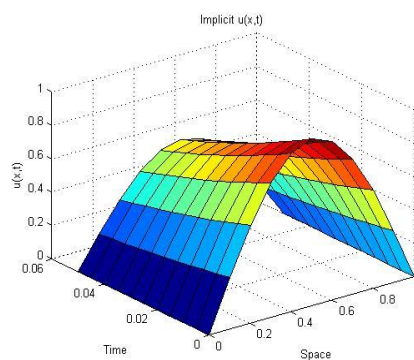
Figure 7: Space-time graph of Implicit solution to
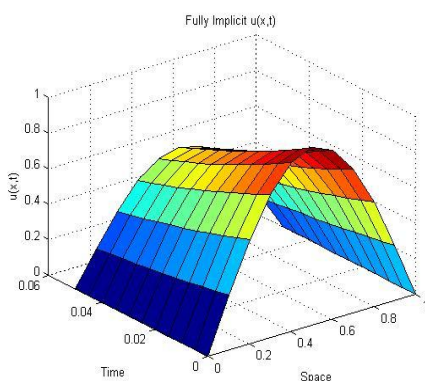0<*t*<0.05 and 0<*x*<1

Figure 8: Space-time graph of Fully implicit solution to **Figure 9:** Space-time graph of Exponential solution to
0<*t*<0.05 and 0<*x*<1   0<*t*<0.05 and 0<*x*<1
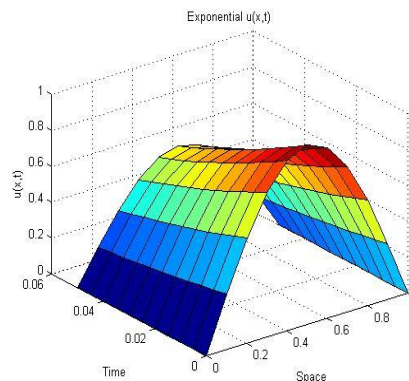
Table 1: Comparison between Exact, EFDM, IFDM, FIFDM, ExpFDM and HPM solution at $t = 0.04$ with $h = 0.1$

x	Exact	EFDM	IFDM	FIFDM	ExpFDM
0	0.500000000000000	0.500000000000000	0.500000000000000	0.500000000000000	0.5000000000000
0.2	0.535296531107333	0.535296812285343	0.535296809648766	0.535296806800963	0.535293428119228
0.4	0.570243014550411	0.570243549975995	0.570243541326771	0.570243533429389	0.570231138974464
0.6	0.604503152468914	0.604503859341221	0.604503841969949	0.604503826644968	0.604478178693358
0.8	0.637767014612595	0.637767653424034	0.637767636334362	0.637767620189916	0.637726627036934
1	0.669761549326657	0.669761549326657	0.669761549326657	0.669761549326657	0.669761549326657
L.S.E	2.5581×10^{-11}	2.4504×10^{-11}	2.3489×10^{-11}	5.8877×10^{-9}	

Table 2: Comparison between EFDM, IFDM, FIFDM, ExpFDM and HPM solution at $t = 0.04$ with $h = 0.1$

x	EFDM	IFDM	FIFDM	ExpFDM
0	0.000000000000000	0.000000000000000	0.000000000000000	0.000000000000000
0.1	0.206683256306262	0.208734652210643	0.210709491135952	0.208969251602732
0.2	0.393590714655683	0.397500259239577	0.401256584364793	0.397617845709237
0.3	0.542226995901918	0.547579653401799	0.552728694567887	0.547497952055164
0.4	0.637725414955059	0.643996145116134	0.650018860773807	0.643804796088075
0.5	0.670648111000106	0.677218857584887	0.683537561660151	0.676999376470964
0.6	0.637725414955059	0.643996145116134	0.650018860773806	0.643804796088075
0.7	0.542226995901918	0.547579653401799	0.552728694567887	0.547497952055164
0.8	0.393590714655683	0.397500259239577	0.401256584364793	0.397617845709237
0.9	0.206683256306262	0.208734652210643	0.210709491135952	0.208969251602732
1	0.000000000000000	0.000000000000000	0.000000000000000	0.000000000000000

IV.   Conclusion

It has shown that from example 1and 2, fully implicit finite differences is more accurate than explicit, implicit and exponential methods as shown in Table (1-2).

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