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A Tauberian Theorem for (C, α, β)- Convergence of Ces$\dot{a}ro$ Means of Order k of Functions

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Abstract: *The objective of this paper to generalize certain Tauberian results proved by Gehring [3] for summability ($C, k; \alpha$) of sequences to functions. In [1] A. V. Boyd generalized the Tauberian theorem for α convergence of Cesȧro means of sequences. In this paper ,we obtain some Tauberian theorems for (C, α, β) convergence of Cesȧro means of order k of functions and investigate some of its properties .*
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I. Introduction

The notation is similar that are in [3],with the following additional definitions: If $k > -1$ then A_n^k, B_n^k denote the n-th Cesȧro sums of order k for the series $\sum_{n=0}^{\infty} a_n, \sum_{n=0}^{\infty} b_n$ where $b_n = na_n$. A_n^{-1}, B_n^{-1} denote the a_n, b_n. Summability $(C, -1; \alpha)$ of $\sum a_n$ will b$(C, 0; \alpha)$ of $\sum a_n$. Mishra and Srivastava [6] introduced the Summability method (C, α, β) for functions by generalizing (C, α) summability method. In this paper, we discuss some Tauberian theorems for (C, α, β) convergence of Cesȧro means of order k of functions and investigate some of its properties .

II. Definitions and Some Preliminaries

We would like to first introduce Summability method. Summability method is more general than that of ordinary convergence. If we are given a sequence (s_n), we can construct a generalized sequence (σ_n), the arithmetic mean of (s_n) by this sequence (s_n). If (σ_n) is convergent in ordinary sense for all $n > 0$, then we say that (s_n) is summable $(C, 1)$ to the sum s . This $(C, 1)$ is called Cesaro mean of first order.

If $s_n \to s \Rightarrow \sigma_n = \frac{s_0 + s_1 + \cdots \ldots\ldots\ldots + s_n}{n+1} \to s$, ie if a sequence is convergent, it is summable by method of arithmetic mean. Also a series $1 - 1 + 1 + 1 + \cdots \ldots \ldots$ is not convergent , but is summable to the sum $\frac{1}{2}$. The space of summable sequences is larger than space of convergent sequences. If $\sigma_n \to s$ as $n \to \infty$, then we say that sequence (s_n) is summable by method of arithmetic mean.

For example : Consider the series $\sum_{n=0}^{\infty} u_n = u_0 + u_1 + \cdots \ldots \ldots$. (1)

And let $\sigma_n = \frac{s_0 + s_1 + \cdots \ldots\ldots\ldots + s_n}{n+1}$, It may happen that whereas (1) diverges , the quantities (the arithmetic mean of partial sum of series) converges to a definite limit as $n \to \infty$. For example $1 - 1 + 1 + 1 + \cdots \ldots \ldots$ diverges, but in this case $s_0 = 1$, $s_1 = 1 - 1 = 0$, $s_2 = 1 - 1 + 1 = 1$, $s_3 = 0 \ldots \ldots \ldots \ldots \ldots \ldots$ $(s_n) = (1,0,1,0,1 \ldots \ldots..)$. Since $s_n = \frac{1 + (-1)^n}{2}$,

$$\sigma_n = \frac{s_0 + s_1 + \cdots \ldots\ldots\ldots + s_n}{n+1}$$

$$= \frac{1+(-1)^0}{2} + \frac{1+(-1)^1}{2} + \frac{1+(-1)^2}{2} + \cdots \ldots \ldots \ldots \ldots .. + \frac{1+(-1)^n}{2} / (n+1)$$

$$= \frac{(n+1)}{2} + \frac{1}{2}\{1 - 1 + 1 - \cdots \ldots \ldots \ldots (n+1) terms\} / (n+1)$$

$= \frac{1}{2} + \frac{1+(-1)^n}{4(n+1)}$, If n is even then $\sigma_n = \frac{1}{2} + \frac{1}{2(n+1)} \to \frac{1}{2}$ as $n \to \infty$ and if n is odd then $\sigma_n = \frac{1}{2}$. So in either case $\lim_{n\to\infty} \sigma_n = \frac{1}{2}$, $\therefore$ $s_n \notin C$ *but* $s_n \epsilon S$. Therefore space of summable sequences is larger than thar of space of convergent sequences .

Let $f(x)$ be any function which is Lebesgue-measurable, and that $f : [0, +\infty) \to R,$ and integrable in (0, x) for any finite x and which is bounded in some right hand neighbourhood of origin. Integrals of the form $\int_0^\infty$ are throughout to be taken as $\lim_{x\to\infty} \int_0^x$, $\int_0^x$ being a Lebesgue integral.

Let $k > 0.$ If, for $t > 0,$ the integral

$$g(t) = g^{(k)}(t) = kt\int_0^\infty \frac{x^{k-1}}{(x+t)^{k+1}} f(x)dx \quad , \qquad (2.1)$$

exists and if $g(t) \to s$ as $t \to \infty$, we say that function $f(x)$ is summable (D,k) to the sum s and we write $f(x) \to s(D,k)$ as $x \to \infty$.

We note that, for any fixed $t > 0,$ $k > 0,$ it is necessary and sufficient for convergence of (2.1) that

$$\int_1^\infty \frac{f(x)}{x^2} dx \text{, should converge .} \qquad (2.2)$$

The (C,α,β) transform of $f(x)$, which we denote by $\partial_{\alpha,\beta}(x)$ is given by

$$f(x) \quad (\alpha = 0)$$

$$\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)} \frac{1}{x^{\alpha+\beta}} \int_0^x (x-y)^{\alpha-1} y^\beta f(y)dy\,,\ (\alpha > 0, \beta > -1) \qquad (2.3)$$

If this exists for $x > 0$ and $\partial_{\alpha,\beta}(x)$ tends to a limit s as $x \to \infty$, we say that $f(x)$ is summable (C,α,β) to s, and we write $f(x) \to s(C,\alpha,\beta)$. We also write

$$U_{k,\alpha,\beta}(t) = kt\int_0^\infty \frac{x^{k-1}}{(x+t)^{k+1}} \partial_{\alpha,\beta}(x)dx, \qquad (2.4)$$

if this exists, and tends to a limit s as $t \to \infty$, we say that the function $f(x)$ is summable $(D,k)(C,\alpha,\beta)$ to s.

When $\beta = 0$, $(D,k)(C,\alpha,\beta)$ and $(D,k)(C,\alpha)$ denote the same method. Here we give some Gehrings generalized Tauberian theorems.

Theorem 2.1: Suppose that $0 \le \alpha \le 1$ and that $f(x)$ is summable (A,α) to s, then $f(x)$ is (C,α,β) convergent to s if and only if the function $\{f(x), |\partial_{\alpha,\beta}(x)\}$ is (C,α,β) convergent to 0.

Theorem 2.2: Suppose that $0 \le \alpha \le 1$ and that $f(x)$ is (C,α,β) convergent. If the function $x\,\partial_{\alpha,\beta}(x)$ is (C,α,β) convergent to 0,then $f(x)$ is summable (C,k,α) to its sum for every $k > -1$.

III. Now we shall prove the following theorem

Theorem 3.1: Suppose that $0 \le \alpha \le 1$ and that $f(x)$ is summable (A,α) to s. Then for $r \ge -1$, $f(x)$is summable (C,r,α) to s if and only if the function $\frac{f(x)}{\partial_{\alpha,\beta}(x)}$ is (C,α,β) to 0.

Proof : Necessary Condition: If $r = -1$,the theorem immediate follows from the summability of $(C,-1,\alpha)$. If $r > -1$, then by consistency theorem for (C,r,α) summability (Gehring [3,theorem 4.2.1]) it follows that both the functions $f(x)$ and $\partial_{\alpha,\beta}(x)$ are (C,α,β) convergent to s. By Hardy [1, Equation (6.1.6)], $S_r^n = S_{r+1}^n +$

$\frac{1}{r+1}\frac{f(x)}{\partial_{\alpha,\beta}(x)}$, and the result follows since a linear combination of functions summable (C,k,α) to itself. The sufficient conditions to prove the theorem are :
If $r>-1$, it may be shown as in Sz*a*ṡ*z* [4 (1)] , that

$$\frac{1}{y+1}\int_0^\infty \partial_{\alpha,\beta}(y)\ (1-\frac{1}{y+1})^n\ dy=\frac{r+1}{y}\int_0^y\left(1-\frac{u}{y}\right)^r\ \partial_{\alpha,\beta}(u)du \qquad (3.1)$$

Where $\partial_{\alpha,\beta}(u)=f(u)$, $(\alpha=0)$

$$=\frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}\frac{1}{x^{\alpha+\beta}}\int_0^x(x-y)^{\alpha-1}y^\beta f(y)dy\ ,\ (\alpha>0,\beta>-1)\ .$$

Case (a) : $\alpha=0,\ r>-1$ is obvious.
Case (b) : $0\le\alpha\le1, r>-1$, putting

$$g(y)=\frac{1}{y+1}\int_0^\infty \partial_{\alpha,\beta}(y)\,(1-\frac{1}{y+1})^n\ dy\ .$$

We get from (3.1) that $g(y)=(r+1)\int_0^1 \partial_{\alpha,\beta}(vy)\left(1-f(v)\right)^n dv$,
Where $\partial_{\alpha,\beta}(u)$ now has bounded (C,α,β)- variation over $(0,\infty)$. Let

$$V\ =\ \left[\int_1^N\left|\,\partial_{\alpha,\beta}(y_r)-\partial_{\alpha,\beta}(y_{r-1})\right|^{\frac{1}{\alpha}}\right]^\alpha$$

$=(r+1)\left[\int_1^N\left|\int_0^1(1-f(v))\{\partial_{\alpha,\beta}(vy_v)-\}\right|\ \partial_{\alpha,\beta}(vy_{v-1})^{\frac{1}{\alpha}}\right]^\alpha$. Then by theorem 201 of [5], we have $V\le(r+1)M\int_0^1(1-f(v))^r\ dv\ =$ M. .
Where $M=V_\alpha\{\partial_{\alpha,\beta}(x):0\le x\le\infty\}$. Thus $\partial_{\alpha,\beta}(y)$ has bounded (C,α,β)- variation over $(0,\infty)$. It is readily seen from Minkowski's inequality that the sum of two (C,α,β) convergent sequences is also (C,α,β) convergent and we therefore deduce that f(x) is (C,α,β) convergent to s.
Case (c) r=-1,when $\alpha=0$,the result reduces to Tauber's original theorem; when $0\le\alpha\le1$ it follows from above theorem . For $\alpha=1$, the result was proved by Hyslop [2] .

Theorem 3.2 : Let $\alpha>\gamma\ge0,\ \beta>-1$, and suppose that a(x) is summable (C,γ,β) to s and that $\int_1^\infty\frac{\partial_{\gamma,\beta}(x)}{x^2}\ dx$ converges . Then a(x) is summable $(D,k)(C,\alpha,\beta)$ to s . We first prove this theorem under unreasonable definition (2.2). However ,if the result holds with (2.2), then it must also hold under the definition of (2.3). This follows from the following Lemmas.

Lemma 3.1: Let $p\ge1,\gamma>1$. Suppose that $f(x)\in L(0,x)$ for finite $x>0$ .Suppose that $f(x)\in$ $|C,\gamma,\beta|_p$,according to the definition (2.3).

Define $$\bar{f}(x)=\begin{cases}f(x) & for\ x\ge T\\ 0 & for\ x<T\end{cases} \qquad (3.2)$$

Let $\bar{\partial}_{\gamma,\beta}(y)$ denote the expression corresponding to $\partial_{\gamma,\beta}(y)$ but with $f(x)$ replaced by $\bar{f}(x)$.

Then $$\int_0^\infty y^{p-1}\left|\frac{d}{dy}\partial_{\gamma,\beta}(y)\right|^p dy<\infty. \qquad (3.3)$$

Thus $\bar{f}(x)$ is summable $|C,\gamma,\beta|_p$ under the definition (2.3).

Lemma 3.2: Let the hypothesis be as in Lemma 3.1,and define $f(x)$ as above. Let $k>0, \beta>-1$ and $\alpha>0$ .Then $|(D,k)(C,\alpha,\beta)|_p$ summability of $\{f(x)\}$ and $\{\bar{f}(x)\}$ are equivalent.

Proof of Lemma 3.1: We are given that , for some T>0,

$$\int_T^\infty x^{p-1}\left|\frac{d}{dx}\partial_{\alpha,\beta}(x)\right|^p dx<\infty \tag{3.3}$$

But since, if (3.3) holds for given T, it holds for any greater T, it must hold for all sufficiently large T. Now by standard properties of fractional integrals, and since $\gamma>1$,we have

$$\int_0^T (T-u)^{\gamma-2}u^\beta |f(u)|\, du<\infty, \tag{3.4}$$

Since (3.3) holds, this will follow from Minkowski's inequality if we prove that

$$\int_T^\infty x^{p-1}\left|\frac{d}{dx}\left\{\bar{\partial}_{\gamma,\beta}(x)-\partial_{\gamma,\beta}(x)\right\}\right|^p dx<\infty \tag{3.5}$$

Now , it follows at once from the definition that, for $x>T$,

$$\bar{\partial}_{\gamma,\beta}(x)-\partial_{\gamma,\beta}(x)=$$
$$\frac{\Gamma(\gamma+\beta+1)}{\Gamma(\gamma)\Gamma(\beta+1)}\frac{1}{x^{\gamma+\beta}}\int_0^T (x-y)^{\gamma-1}y^\beta\,\bar{f}(y)\,dy-\frac{\Gamma(\gamma+\beta+1)}{\Gamma(\gamma)\Gamma(\beta+1)}\frac{1}{x^{\gamma+\beta}}\int_0^T (x-y)^{\gamma-1}y^\beta\, f(y)\,dy$$

If $\gamma\le 2$, then for $x>T$, we have $(x-y)^{\gamma-2}\le(T-y)^{\gamma-2}$, so that

$$\left|\frac{d}{dx}\left\{\bar{\partial}_{\gamma,\beta}(x)-\partial_{\gamma,\beta}(x)\right\}\right|\le\frac{\Gamma(\gamma+\beta+1)}{\Gamma(\gamma)\Gamma(\beta+1)}\frac{(\beta+\gamma+1)x}{x^{\gamma+\beta}}\int_0^T (x-y)^{\gamma-2}y^\beta|f(y)|\,dy$$

$$=\frac{\text{Const.}}{x^{\beta+\gamma}} \qquad \text{by (3.4).}$$

Proof of Lemma 3.2: We use notations as in Lemma 3.1, and write further $\overline{U}_{k,\alpha,\beta}(y)$ for the expression corresponding to $U_{k,\alpha,\beta}(y)$ but with $f(x)$ replaced by $\bar{f}(x)$.

We know that for any fixed $y>0,k>0,\beta>-1,\alpha>0$ convergence of

$U_{k,\alpha,\beta}(y)=ky\int_0^x \frac{x^{k-1}}{(x+y)^{k+1}}\partial_{\alpha,\beta}(x)\,dx$, is equivalent to the convergence of $\int_1^\infty \frac{\partial_{\alpha,\beta}(x)}{x^2}dx$

.Then the conclusion will follow from Minkowski's inequality, if we show that

$$\int_1^\infty y^{p-1}\left|\frac{d}{dy}\left\{U_{k,\alpha,\beta}(y)-\overline{U}_{k,\alpha,\beta}(y)\right\}\right|^p dy<\infty, \qquad (3.6)$$

where we take (3.6) as including the assertion that the integral defined by $U_{k,\alpha,\beta}(y)-\overline{U}_{k,\alpha,\beta}(y)$ converges for all $y>0$. For large y ,we have

$$\partial_{\alpha,\beta}(y)-\overline{\partial}_{\alpha,\beta}(y)=\frac{\Gamma(\gamma+\beta+1)}{\Gamma(\alpha)\Gamma(\beta+1)}\frac{1}{y^{\alpha+\beta}}\int_0^T (y-x)^{\alpha-1}x^{\beta}f(x)dx \qquad (3.7)$$

Hence the convergence of $ky\int_0^x \frac{x^{k-1}}{(x+y)^{k+1}}\partial_{\alpha,\beta}(x)\left\{\partial_{\alpha,\beta}(x)-\overline{\partial}_{\alpha,\beta}(x)\right\}dx$, follows at once by a result due to ([2]) . Now (3.6) is equivalent to

$$\int_1^\infty y^{p-1}dy\left|c\int_0^\infty \frac{x^{k-1}}{(x+y)^{k+2}}(x-ky)\left\{\partial_{\alpha,\beta}(x)-\overline{\partial}_{\alpha,\beta}(x)\right\}dx\right|^p<\infty. \qquad (3.8)$$

Let T_0 be any sufficiently large constant. Then (3.8) will follow from Minkowski's inequality, if we show that

$$\int_1^\infty y^{p-1}dy\left|c\int_0^{T_0} \frac{x^{k-1}}{(x+y)^{k+2}}(x-ky)\left\{\partial_{\alpha,\beta}(x)-\overline{\partial}_{\alpha,\beta}(x)\right\}dx\right|^p<\infty. \qquad (3.9)$$

$$\int_1^\infty y^{p-1}dy\left|c\int_{T_0}^\infty \frac{x^{k-1}}{(x+y)^{k+2}}(x-ky)\left\{\partial_{\alpha,\beta}(x)-\overline{\partial}_{\alpha,\beta}(x)\right\}dx\right|^p<\infty. \qquad (3.10)$$

By (3.9), we have

$$\int_1^\infty y^{p-1}dy\left|c\int_0^{T_0}\frac{x^{k-1}}{(x+y)^{k+2}}(x-ky)\left\{\partial_{\alpha,\beta}(x)-\overline{\partial}_{\alpha,\beta}(x)\right\}dx\right|^p$$

$$= O(1)\left[y^{-kp-p}\right]_1^\infty = O(1).$$ Hence (3.9) follows .

By (3.7), the expression on the left of (3.10) does not exceed a constant. Thus

$$\int_1^\infty y^{p-1}dy\left|c\int_{T_0}^\infty\frac{x^{k-1}}{(x+y)^{k+2}}(x-ky)\left\{\partial_{\alpha,\beta}(x)-\overline{\partial}_{\alpha,\beta}(x)\right\}dx\right|^p$$

$$=o(1)\int_1^\infty y^{p-1}dy\left|\int_{T_0}^\infty (x+y)^{-2}x^{-\beta-1}\,dx\right|^p \qquad (3.11)$$

By an obvious change of variables the expression (3.11) is equal to

$$o(1)\int_1^\infty y^{p-1}dy\left|\int_y^\infty t^{-2}(t-y)^{-\beta-1}\,dt\right|^p = o(1)\,C = C.$$ The result follows.

Proof of Theorem 3.2 : We divide the proof into the following cases .

Case I . $\alpha>\gamma$, **Case II .** $\alpha=\gamma$, **Case III .** $\alpha<\gamma$

Here we observe that Case I and II follow from case III, with the aid of Theorem 3.1 .

For, if $\alpha\geq\gamma$, Choose any $\gamma^{'}>\alpha$, summability $\left|C,\gamma,\beta\right|_p$ implies summability $\left|C,\gamma^{'},\beta\right|_p$ by Theorem 3.1, and it follows from Case III, that this implies $\left|(D,k)(C,\alpha,\beta)\right|_p$. Hence it is sufficient to consider the case III only.

Proof of Case III : Since $f(x)\to s(C,\alpha,\beta)$ implies that $f(x)\to s(C,\alpha^{'},\beta)$ for $\alpha^{'}>\alpha>o$, there is no loss of generality in considering the Case $\gamma=\alpha+k,\ k$ is a positive integer.

We have , $$\frac{d}{dy}U_{k,\alpha,\beta}(y)=C\int_{T_0}^\infty\frac{x^{k-1}}{(x+y)^{k+2}}(x-ky)\,\partial_{\alpha,\beta}(x)\,dx \qquad (3.12)$$

Now, by definition

$$\partial_{\alpha+p,\beta}(x) = \frac{\Gamma(\alpha+\beta+1)}{\Gamma(\alpha+p+\gamma)(\gamma+\beta+1)} \frac{1}{y^{\alpha+\beta+p}} \int_0^x (x-t)^{\alpha-\gamma+p-1} t^{\gamma+\beta} \partial_{\alpha,\beta}(t)dt.$$

Putting p=1 and $\alpha = \gamma$,we see that $\partial_{\alpha+1,\beta}(x) = \frac{(\alpha+\beta+1)}{x^{\alpha+\beta+1}} \int_0^x t^{\alpha+\beta} \partial_{\alpha,\beta}(t)dt.$ (3.13)

We also write $R_{\alpha,\beta}(x) = \int_x^\infty \frac{\partial_{\alpha,\beta}(t)}{t^2} dt.$

It is clear that, whenever $\int_1^\infty \frac{\partial_{\alpha,\beta}(x)}{x^2} dx$ converges, $R_{\alpha,\beta}(x)$ is defined for $x>0$, and that $R_{\alpha,\beta}(x) \to 0$ as $x \to \infty$. It follows immediately from (3.13) that

$$\partial_{\alpha+1,\beta}(x) = -\frac{(\alpha+\beta+1)}{x^{\alpha+\beta+1}} \int_0^x t^{\alpha+\beta} t^2 dR_{\alpha,\beta}(t)dt$$

$= o(x^1)$ and hence that, for $p \geq 1$, $\partial_{\alpha+1,\beta}(x) = o(x^1)$ (3.14)

Integrating (3.14) by parts k times,we deduce with the help of (3.13) that

$$\frac{d}{dy} U_{k,\alpha,\beta}(y) = (-1)^k C \int_0^\infty x^{\alpha+\beta+k} \partial_{\alpha+k,\beta}(x) \left\{ \frac{d^k}{dx^k} \left[\frac{x^{k-\alpha-\beta-1}}{(x+y)^{k+2}} (x-ky) \right] \right\} dx. \quad (3.15)$$

It is verified that expression in (3.16) is $o\left(\frac{x^{k-\alpha-\beta-1}}{(x+y)^{k+1}} \right)$. (3.16)

Let $R(x,y) = \int_0^x t^{\alpha+\beta+k} \frac{d^k}{dx^k} \left[\frac{t^{k-\alpha-\beta-1}}{(t+y)^{k+2}} (t-ky) \right] dt.$

In fact, for fixed $k > 0$, we have uniformly in $x > 0, y > 0,$

$$R(x,y) = 0\left(\frac{x^k}{(x+y)^{k+1}} \right) . \quad (3.17)$$

This may be proved by induction on k , if $k = 0$,we have

$$R(x,y) = \int_0^x t^{\alpha+\beta}\left[\frac{t^{k-\alpha-\beta-1}}{(t+y)^{k+2}}(t-ky)\right]dt = \frac{x^k}{(x+y)^{k+1}} \quad ,$$

hence the result is evident. Suppose that $k \geq 1,$ and assume the result true for $k-1.$ Integrating by parts ,we have

$$R(x,y) = x^{\alpha+\beta+k}\frac{d^{k-1}}{dx^{k-1}}\left[\frac{x^{k-\alpha-\beta-1}}{(x+y)^{k+2}}(x-ky)\right] - (\alpha+\beta+k)\int_0^x t^{\alpha+\beta+k+1}\frac{\partial^{k-1}}{\partial t^{k-1}}\left\{\frac{t^{k-\alpha-\beta-1}}{(t+y)^{k+2}}(t-ky)\right\}dt.$$

the first term is of required order by (3.17) (with k replaced by k-1), and the second by induction hypothesis.

Now integrating (3.16) by parts, we have

$$\frac{d}{dy}U_{k,\alpha,\beta}(y) = \int_0^\infty R(x,y)\left(\frac{d}{dx}\partial_{\alpha+k,\beta}(x)\right)dx = \int_0^\infty R(x,y)\left(\frac{d}{dx}\partial_{\gamma,\beta}(x)\right)dx \; .$$

Since the integrated term tends to 0 as $\partial_{\gamma,\beta}(x)$ is bounded and $R(x,y) \to 0 \;\; as \;\; x \to \infty$.

Using (3.17) and putting $x = t\,y$, we see that the expression in curly brackets

$$\leq C\int_0^x \frac{x^{k-1}}{(x+y)^{k+1}}dx = \frac{C}{y}\int_0^x \frac{t^{k-1}}{(1+t)^{k+1}}dt = \frac{C}{y} \quad ,$$

Again using (3.18) , the inner integral

$$\leq C\,x^k\int_0^\infty \frac{1}{(x+y)^{k+1}}\,dy \; , \qquad (3.18)$$

on putting $y = xt$, the expression on the right of (3.19) is equal to

$$C\int_0^\infty \frac{1}{(1+t)^{k+1}}\,dt = C$$

(Since the integral converges) . Hence the result follows.

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