IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 11, Issue 2 Ver. VI (Mar - Apr. 2015), PP 39-42
www.iosrjournals.org

On Tensor product of R-Algebra and R-Homomorphism with Projectivity

Dr. Sumit Kumar Dekate
Associate Professor Sagar Institute of Science and Technology (SISTec) Bhopal M.P. India

Abstract: *In this paper we take a natural homomorphism*

$f_M : U \otimes Hom_R(M, N) \rightarrow Hom_U(U \otimes M, U \otimes N)$

where U is an R-Algebra , R=CenU and M , N are any two R-modules.Here we have shown that

1. *If P is any projective module then f_P is a monomorphism.*
2. *If G be any generator then $f_{G^{(n)}}$ is an epimorphism.*
3. *If M be a generator and N is any free module then $f_{Tr_N(M)}$ is always a monomorphism.*
4. *If M is a generator and N is any finitely generated free R-module then $f_{Tr_N(M)}$ is an isomorphism.*
5. *If M be any artinian module such that it is subdirect product of simple module then f_M is monomorphism iff for each submodule K , f_K is monomorphism.*
6. *Let M be a generator and N is any free module if U is finitely presented then $f_{Tr_N(M)}$ is an isomorphism.*
7. *Let M be any left R-module such that $f_{M/\mathrm{Re}\,j_M(N)}$ is an isomorphism then f_M is an isomorphism.*

Keywords: *Projective module, Generator, Finitely Presented Module, Monomorphism, Epimorphism, Isomorphism.*

I. Introduction

Our purpose to study the relation between the groups

$U \underset{R}{\otimes} Hom_R(M, N)$ and $Hom_U(U \otimes M, U \otimes N)$ via the natural homomorphism

$$f_M : U \underset{R}{\otimes} Hom_R(M, N) \rightarrow Hom_U(U \otimes M, U \otimes N)$$

defined by

$$f_M :(a \otimes f)\{(b \otimes x)\} = ab \otimes f(x) \quad , \; a,b \in U , x \in M$$

$f \in Hom_R(M, N)$ where R is any commutative ring with identity and U is an R- algebra . For our said purpose we will find the condition under which f_M is isomorphism, epimorphism or monomorphism.

Throughout this paper we will assume all rings have unity and all modules are unitary.

Theorem1.1: If P is any projective module then f_P is a monomorphism.

Proof: Given that P be any projective modules then

$$R^{(A)} = P \oplus P_1 \quad \text{for any } A \neq \emptyset$$

then

$f_1: U \otimes Hom_R(R^{(A)}, N) \rightarrow U \otimes Hom_R(P \oplus P_1, N)$

$f_{R^{(A)}} \downarrow \qquad \downarrow f_{P \otimes P_1}$

$f_2: Hom_U(U \otimes R^{(A)}, U \otimes N) \rightarrow Hom_U(U \otimes (P \oplus P_1), N)$

here f_1 and f_2 are obvious isomorphism and via commutative diagram

$g: U \otimes Hom_R(R^{(A)}, N) \rightarrow Hom_U(U \otimes R^{(A)}, U \otimes N)$

$h_1 \downarrow \qquad \downarrow h_2$

$g_1: U \otimes \prod_A N \quad \rightarrow \quad \prod_A U \otimes N$

$f_{R^{(A)}}$ is monomorphism and h_1, h_2 are isomorphism then $f_{P \otimes P_1}$ is monomorphism

and as a result f_P is a monomorphism.

Corollary1.2: If ${}_R G$ be any generator then $f_{G^{(n)}}$ is an epimorphism.

Proof: Let ${}_R G$ be the generator then there is a split epimorphism

$G^{(n)} \rightarrow R \rightarrow 0$

then from Proposition 17.6[1] P 195

$Hom(G^{(n)}, N) \rightarrow Hom(R, N) \rightarrow 0$ is also be split epimorphism.

Now from Proposition 19.13 [1] P 226 we have

$U \otimes Hom(G^{(n)}, N) \rightarrow U \otimes Hom(R, N) \rightarrow 0$

and

$U \otimes G^{(n)} \rightarrow U \otimes R \rightarrow 0.$

Finally we can construct a commutative diagram

$\varnothing_1: U \otimes Hom_R(G^{(n)}, N) \rightarrow U \otimes Hom_R(R, N) \rightarrow 0$

$f_{G^{(n)}} \downarrow \qquad \downarrow f_R$

$\varnothing_2: Hom_U(U \otimes G^{(n)}, U \otimes N) \rightarrow Hom_U(U \otimes R, U \otimes N) \rightarrow 0$

here in above diagram f_R is an isomorphism and from the commutativity of diagram we have

$\varnothing_2 \circ f_{G^{(n)}} = f_R \circ \varnothing_1$

since here f_R is an isomorphism then it is an epimorphism and $\varnothing_1$ is an epimorphism then composition $f_R \circ Ø_1$ is an epimorphism and from above

equation $Ø_2 \circ f_{G^{(n)}}$ is also be an epimorphism and $Ø_2$ is given an epimorphism

therefore $f_{G^{(n)}}$ is an epimorphism and consequently from the diagram

$$f_{G^{(n)}} : U \otimes \mathrm{Hom}_R(G^{(n)}, N) \rightarrow \mathrm{Hom}_U(U \otimes G^{(n)}, U \otimes N)$$

$$\psi_1 \downarrow \qquad\qquad \psi_2 \downarrow$$

$$\prod_n f_G : \prod_n U \otimes \mathrm{Hom}_R(G, N) \rightarrow \prod_n \mathrm{Hom}_U(U \otimes G, U \otimes N)$$

both ψ_1 and ψ_2 are isomorphism then $f_{G^{(n)}}$ is an isomorphism.

Corollary1.3: Let M be a generator and N is any free module then $f_{Tr_N(M)}$ is always a monomorphism.

Proof : Given that M be any generator and N is any free module then

$$N \cong R^{(A)} \quad \text{for any } A \neq Ø$$

now

$$Tr_N(M) \cong Tr_{R^{(A)}}(M)$$

$$= Tr_R(M)^{(A)}$$

$$= R^{(A)}.$$

Then from Corollary 1.3 [5], $f_{Tr_N(M)}$ is a monomorphism.

Corollary1.4: If M is a generator and N is any finitely generated free R-module then $f_{Tr_N(M)}$ is an isomorphism.

Proof: It is an easy consequence of above corollary and Lemma1.1 [5].

Corollary1.5: If $_R M$ be any artinian module such that it is subdirect product of simple module then f_M is monomorphism iff for each submodule K≤M, f_K is a monomorphism.

Proof: Let $_R M$ be any artinian module such that $_R M$ is subdirect product of simples then from Exercise 9.11[1],

$$\mathrm{Rad} M = 0$$

and from Proposition10.15[1] P 129, $_R$M is semisimple so from Proposition1.2[5]

P 67, f $_M$ is monomorphism iff for each submodule K≤ M , f $_K$ is monomorphism.

Corollary1.6: Let M be a generator and N is any free module if U is finitely

presented then f $_{Tr_N(M)}$ is an isomorphism.

Proof: It is an easy consequence of Corollary1.3 and Corollary to Proposition2 [4]

P 161.

Example1.7: Let M be any left R-module such that f $_{M/\mathrm{Re}j_M(N)}$ is an isomorphism

then f $_M$ is an isomorphism.

Proof: If M is any left R-module and we are given that

$$f_M : U\otimes \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_U(U\otimes M, U\otimes N)$$

then

$$f_{M/\mathrm{Re}j_M(N)} : U\otimes \mathrm{Hom}_R(M/\mathrm{Rej}_M(N), N) \to \mathrm{Hom}_U(U\otimes M/\mathrm{Rej}_M(N), U\otimes N).$$

Now we consider a commutative diagram

$$h : U\otimes \mathrm{Hom}_R(M/\mathrm{Rej}_M(N), N) \to U\otimes \mathrm{Hom}_R(M, N)$$

$$f_{M/\mathrm{Re}j_M(N)}\downarrow \qquad\qquad\qquad \downarrow f_M$$

$$h_1 : \mathrm{Hom}_U(U\otimes M/\mathrm{Rej}_M(N), U\otimes N) \to \mathrm{Hom}_U(U\otimes M, U\otimes N).$$

Now from Exercise 8.7[1] P 112 and from definition of Tensor product of homomorphism h and h $_1$ are isomorphism since f $_{M/\mathrm{Re}j_M(N)}$ is isomorphism by assumption then by commutativity of diagram f $_M$ is isomorphism.

References:

Books:

[1]. F. W. ANDERSON AND K. R. FULLER, Rings and Catagories of modules, New York Springer – Verlag Inc. 1973.
[2]. LOUIS HALLE ROWEN, Ring Theory.
[3]. T. Y. LAM, A first course in non commutative rings Springer – Verlag.

Proceedings Papers:

[4]. R. S. SINGH, Finitely presented modules, Proc. Math. Soc. , B. H. U. Vol 3 (1987)
[5]. R. S. SINGH AND BHUPENDRA, Tensor product of R-algebra and R- homomorphisms, Prog. of Math. Vol 37(nos 1 and 2) 2003.