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Pseudospherical 3- Planes In $\mathbb{R}^5$ and Evolution Equations Of Type $u_{tt} = \psi\left(u, u_x, \ldots\ldots, \frac{\partial^k u}{\partial x^k}, u_y, \ldots\ldots, \frac{\partial^{k'} u}{\partial y^{k'}}, u_t\right)$

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Abstract:*In this paper, evolution equations with two or more spatial variables, which may describe pseudospherical planes in higher dimensions, are considered. Necessary and sufficient conditions for equations of type*$u_{tt} = \psi\left(u, u_x, \ldots\ldots, \frac{\partial^k u}{\partial x^k}, u_y, \ldots\ldots, \frac{\partial^{k'} u}{\partial y^{k'}}, u_t\right)$
to describe a 3-dimensionalpseudospherical plane of R^5*are given . Such equations are characterized.*
Keywords:*Evolution equations, Pseudospherical surfaces, Riemannian manifold, Solitons and* differential equations.

I. Introduction

It has been observed that exactly solvable nonlinear differential equations with two independent variables are obtained as compatibility conditions for linear systems. Moreover, obtaining a spectral linear problem associated with a nonlinear equation[14-16] has been useful in order to solve the initial value problem by the inverse scattering method, [1]. In [2,13] the notion of a differential equation which describe pseudospherical surfaces (surfaces with constant negative Gaussian curvature inR^5) was given . Studies are made in this direction concerning non linear evolution equations of type

$$u_{xt} = \psi\left(u, u_x, u_{xx}, \ldots\ldots, \frac{\partial^k u}{\partial x^k}\right) \quad \text{in [2,11]}$$

$$u_t = \psi\left(u, u_x, u_{xx}, \ldots\ldots, \frac{\partial^k u}{\partial x^k}\right) \quad \text{in [1,10]}$$

And $\quad u_{tt} = \psi(u, u_x,, u_{xx}, u_t) \quad$ in [8].

Then, we [6,7] generalized these studies to evolution equations with three independent variables which are related to pseudospherical planes (P.S.P) in R^5 (3-dim planes of R^5 with constant negative sectional curvature) where equations of types

$$u_{xt} = \psi\left(u, u_x, u_{xx}, \ldots\ldots, \frac{\partial^k u}{\partial x^k}, u_y, u_{yy}, \ldots\ldots, \frac{\partial^{k'} u}{\partial y^{k'}}\right) \tag{1}$$

and

$$u_t = \psi\left(u, u_x, u_{xx}, \ldots\ldots, \frac{\partial^k u}{\partial x^k}, u_y, u_{yy}, \ldots\ldots, \frac{\partial^{k'} u}{\partial y^{k'}}\right) \tag{2}$$

Are studied . Here we provide a similar study for another class of evolution equations with two spatial variables plus the time variable which have the from

$$u_{tt} = \psi\left(u, u_x, \ldots\ldots, \frac{\partial^k u}{\partial x^k}, u_y, \ldots\ldots, \frac{\partial^{k'} u}{\partial y^{k'}}, u_t\right) \tag{3}$$

II. Basic notations and Preliminaries

A differential equation E-for a real function $u(x, y, t)$ describes a 3-dimensional pseudospherical plane in R^5(simply P.S.P.) if it is the necessary and sufficient condition for the existence of differentiable functions $f_{\alpha i}$, $1 \le \alpha \le 6$ and$1 \le i \le 3$, depending on u and its derivatives, such that the 1-forms

$$\omega_\alpha = f_{\alpha 1} dx + f_{\alpha 2} dy + f_{\alpha 3} dt$$

satisfy the structure equations of a 3-plane of constant sectional curvature -1 in R^5 i.e. equations (4).

$$\left.\begin{aligned} d\omega_1 &= \omega_4 \wedge \omega_2 + \omega_5 \wedge \omega_3 \\ d\omega_2 &= -\omega_4 \wedge \omega_1 + \omega_6 \wedge \omega_3 \\ d\omega_3 &= -\omega_5 \wedge \omega_1 - \omega_6 \wedge \omega_2 \\ d\omega_4 &= \omega_1 \wedge \omega_2 \\ d\omega_5 &= \omega_1 \wedge \omega_3 \\ d\omega_6 &= \omega_2 \wedge \omega_3 \end{aligned}\right\} \tag{4}$$

where we have written

$\omega_4 = \omega_{12}$ $\omega_5 = \omega_{13,}$ and

$\omega_6 = \omega_{23}$ with $\omega_{ij} = -\omega_{ji}$, $i,j = 1,2,3$, $\omega_{ii} = 0$

We shall define such 3-dimensional P.S.P to be a two-parameters 3-dimensional P.S.P f_{31}=f_{41}= ζ and $f_{22} = f_{42}$= ξ, with ζ and ξ constant parameters.

To study equation (3).we first write

$$z_0 = u, \quad z_1 = u_x, z_2 = u_{xx}, \dots\dots, z_k = \frac{\partial^k u}{\partial x^k},$$

$$z_{1'} = u_y, \quad z_{2'} = u_{yy}, \dots\dots\dots\dots\dots.. z_{k'} = \frac{\partial^{k'} u}{\partial y^{k'}} \text{and} \quad \mathrm{p} = u_t$$

Thus equation (3) becomes

$$p_t = \psi(z_0, z_1, \dots., z_k, z_{1'}, \dots\dots, z_{k'}, p) \tag{5}$$

In Particular, we shall consider equation (5) with the following assumptions

$$\left.\begin{array}{l} z_{i,t} = z_{i',t} = 0 \\ z_{i',x} = z_{i,y} = 0 \\ z_{0,t} = p \text{for} \quad 1 \le i \le k \ , 1 \le i' \le k' \end{array}\right\} \tag{6}$$

where the comma denotes partial differentiation with respect to the shown variable. Now consider the following ideal I of forms on the space of variables $x, y, t, z_0, z_1, \dots\dots., z_k, z_{1'}, z_{2'}, \dots\dots, z_{k'}, p$:

$$\left.\begin{array}{ll} \Omega_i = dz_i \wedge dt - z_{i+1} dx \wedge dt, & 0 \le i \le k-1 \\ \Omega_{i'} = dz_{i'} \wedge dt - z_{i'+1} dy \wedge dt, & 0 \le i' \le k'-1 \\ \Omega_p = dp \wedge dt + dz_1 \wedge dx & \\ \Omega_{p'} = dp \wedge dt + dz_{1'} \wedge dy & \\ \Omega_k = dp \wedge dx \wedge dy - \psi dx \wedge dy \wedge dt & \\ \Omega = dz_0 \wedge dx \wedge dy - p dx \wedge dy \wedge dt & \end{array}\right\} \tag{7}$$

Note that assumptions (6) mean that u has no (xy) terms. Now, if we apply Cartan-Kahler theory, for equation (5) and using the notation above we can obtain the following result which relates solutions of the differential equation (3) with integral manifolds of the ideal I formed by the forms in (7).

Lemma 2.1

Let $p_t = \psi(z_0, z_1, \dots., z_k, z_{1'}, \dots\dots, z_{k'}, p)$,be a differential equation which describe an (ζ, ξ) 3-dirnensional P.S.P with the associated 1-forms $\omega_\alpha = f_{\alpha 1} dx + f_{\alpha 2} dy + f_{\alpha 3} dt, \alpha = 1,2,\dots,6$ where $f_{\alpha i}$ and ψ are real differentiable (C$^\infty$) functions defined on an open connected subset $U \subset R^{k+k'+1}$ with no explicit dependence on x, y and t. Then

$$\left.\begin{array}{c} f_{11,z_i} = f_{12,z_i} = f_{21,z_i} = f_{22,z_i} = f_{51,z_i} = f_{52,z_i} = f_{61,z_i} = f_{62,z_i} = 0 \ \forall i \neq 1 \\ f_{11,z_{k'}} = f_{13,z_{k'}} = f_{21,z_{k'}} = f_{23,z_{k'}} = f_{33,z_{k'}} = f_{43,z_{k'}} = \\ f_{51,z_{k'}} = f_{53,z_{k'}} = f_{61,z_{k'}} = f_{63,z_{k'}} = 0 \\ f_{33,z_{k'-1}} = f_{43,z_{k'-1}} = 0 \\ f_{12,z_k} = f_{13,z_k} = f_{22,z_k} = f_{23,z_k} = f_{33,z_k} = f_{43,z_k} = \\ f_{52,z_k} = f_{53,z_k} = f_{62,z_k} = f_{63,z_k} = 0 \\ f_{33,z_{k-1}} = f_{43,z_{k-1}} = 0 \\ f_{13,p} = f_{23,p} = f_{33,p} = f_{43,p} = f_{53,p} = f_{63,p} = 0 \\ f_{33,p} = f_{43,p} = 0, \quad f_{33,z_1} = f_{43,z_1} = 0 \\ f_{33,z_{1'}} = f_{43,z_{1'}} = 0, \quad f_{13,p} = f_{11,z_1} = f_{12,z_{1'}} \\ f_{23,p} = f_{21,z_1} = f_{22,z_{1''}}, \quad f_{53,p} = f_{51,z_1} = f_{52,z_{1'}} \\ f_{63,p} = f_{61,z_1} = f_{62,z_{1'}} \\ f_{11,p}^2 + f_{12,p}^2 + f_{21,p}^2 + f_{22,p}^2 + f_{51,p}^2 + f_{52,p}^2 + f_{61,p}^2 + f_{62,p}^2 \neq 0 \end{array}\right\} \tag{8}$$

In U, and

$$-\sum_{i'=1}^{k'-1} z_{i'+1} f_{11,z_{i'}} + \sum_{i=0}^{k-1} z_{i+1} f_{12,z_i} = \zeta f_{22} - \xi f_{21} + \xi f_{51} + \zeta f_{52} \tag{9}$$

$$-\psi f_{11,p} - p f_{11,z_0} + \sum_{i=0}^{k-1} z_{i+1} f_{13,z_i} = \zeta f_{23} - f_{43} f_{21} + f_{51} f_{33} - \zeta f_{53} \quad (10)$$

$$-\psi f_{12,p} - p f_{12,z_0} + \sum_{i=0}^{k'-1} z_{i'+1} f_{13,z_{i'}} = \xi f_{23} - f_{43} f_{22} + f_{52} f_{33} - \xi f_{53} \quad (11)$$

$$-\sum_{i'=1}^{k'-1} z_{i'+1} f_{21,z_{i'}} + \sum_{i=0}^{k-1} z_{i+1} f_{22,z_i} = \xi f_{11} - \zeta f_{12} + \xi f_{61} - \zeta f_{62} \quad (12)$$

$$-\psi f_{21,p} - p f_{21,z_0} + \sum_{i=0}^{k-1} z_{i+1} f_{23,z_i} = f_{11} f_{43} - \zeta f_{13} + f_{61} f_{33} - \zeta f_{63} \quad (13)$$

$$-\psi f_{22,p} - p f_{22,z_0} + \sum_{i'=1}^{k'-1} z_{i'+1} f_{23,z_{i'}} = f_{12} f_{43} - \zeta f_{13} + f_{62} f_{33} - \zeta f_{63} \quad (14)$$

$$f_{11} f_{52} - f_{12} f_{51} = f_{22} f_{61} - f_{21} f_{62} \quad (15)$$

$$\sum_{i=0}^{k-2} z_{i+1} f_{33,z_i} = f_{11} f_{53} - f_{13} f_{51} + f_{21} f_{63} - f_{23} f_{61} \quad (16)$$

$$\sum_{i'=1}^{k'-2} z_{i'+1} f_{33,z_{i'}} = f_{12} f_{53} - f_{13} f_{52} + f_{22} f_{63} - f_{23} f_{62} \quad (17)$$

$$f_{11} f_{22} - f_{12} f_{21} = 0 \quad (18)$$

$$\sum_{i=0}^{k-2} z_{i+1} f_{43,z_i} = f_{11} f_{23} - f_{13} f_{21} \quad (19)$$

$$\sum_{i'=1}^{k'-2} z_{i'+1} f_{43,z_{i'}} = f_{12} f_{23} - f_{13} f_{22} \quad (20)$$

$$-\sum_{i'=1}^{k'-1} z_{i'+1} f_{51,z_{i'}} + \sum_{i=0}^{k-1} z_{i+1} f_{52,z_i} = \xi f_{11} - \zeta f_{12} \quad (21)$$

$$-\psi f_{51,p} - p f_{51,z_0} + \sum_{i=0}^{k-1} z_{i+1} f_{53,z_i} = f_{11} f_{33} - \zeta f_{13} \quad (22)$$

$$-\psi f_{52,p} - p f_{52,z_0} + \sum_{i'=1}^{k'-1} z_{i'+1} f_{53,z_{i'}} = f_{12} f_{33} - \xi f_{13} \quad (23)$$

$$-\sum_{i'=1}^{k'-1} z_{i'+1} f_{61,z_{i'}} + \sum_{i=0}^{k-1} z_{i+1} f_{62,z_i} = f_{21} f_{32} - f_{22} f_{31} \quad (24)$$

$$-\psi f_{61,p} - p f_{61,z_0} + \sum_{i=0}^{k-1} z_{i+1} f_{63,z_i} = f_{21} f_{33} - \zeta f_{23} \quad (25)$$

$$-\psi f_{62,p} - p f_{62,z_0} + \sum_{i'=1}^{k'-1} z_{i'+1} f_{63,z_{i'}} = f_{22} f_{33} - \xi f_{23} \quad (26)$$

with the assumptions (6).

Proof:

In the space of variables $(x, y, t, z_0, z_1, \cdots\cdots, z_k, z_{1'}, \cdots\cdots, z_{k'}, p)$ we consider the ideal I generated by $\Omega_i, \Omega_{i'}, \Omega_k, \Omega_p, \Omega_{p'} and \Omega$ defined by equations (7) with ψ given by equation(53) then
$\Omega_i = \Omega_{i'} = \Omega_k = \Omega_p = \Omega_{p'} = \Omega = 0$, when restricted to each integral manifold of E.

Hence, for $z_0, z_1, \ldots., z_k, z_{1'}, \ldots\ldots, z_{k'}, p$ satisfying (5), we have

$$dz_i \wedge dt = z_{i+1} dx \wedge dt, \qquad i = 0,1,\ldots..,k-1$$
$$dz_{i'} \wedge dt = z_{i'+1} dy \wedge dt, \qquad i' = 0,1,\ldots..,k'-1$$
$$dp \wedge dt = -dz_1 \wedge dx$$
$$dp \wedge dt = -dz_{1'} \wedge dy$$

$$dp \wedge dx \wedge dy = \psi dx \wedge dy \wedge dt$$
$$dz_0 \wedge dx \wedge dy = p dx \wedge dy \wedge dt$$

Where from assumptions (6) we have

$$\left.\begin{array}{c} dz_i \wedge dx \wedge dy = z_{i,t} dx \wedge dy \wedge dt \\ dz_i \wedge dx \wedge dt = 0 \\ dz_i \wedge dy \wedge dt = z_{i+1}\, dx \wedge dy \wedge dt \\ dz_{i'} \wedge dx \wedge dy = 0 \\ dz_{i'} \wedge dx \wedge dt = -z_{i'+1}\, dx \wedge dy \wedge dt \\ dz_{i'} \wedge dy \wedge dt = 0 \\ dp \wedge dx \wedge dy = p_t dx \wedge dy \wedge dt \\ dp \wedge dy \wedge dt = 0 \\ dp \wedge dx \wedge dt = 0 \end{array}\right\} \quad (27)$$

And also we have

$$\left.\begin{array}{c} z_{i',t} = z_{i',t} = 0 \\ z_{i',x} = z_{i,y} = 0 \\ z_{0,t} = p for \quad 1 \le i \le k \ , 1 \le i' \le k' \\ f_{11,z_i} \equiv f_{12,z_i} \equiv f_{21,z_i} \equiv f_{22,z_i} \equiv f_{51,z_i} \equiv f_{52,z_i} \equiv f_{61,z_i} \equiv f_{62,z_i} \equiv 0 \quad \forall i \neq 1 \\ f_{11,z_{k'}} \equiv f_{13,z_{k'}} \equiv f_{21,z_{k'}} \equiv f_{23,z_{k'}} \equiv f_{33,z_{k'}} \equiv f_{43,z_{k'}} \equiv f_{51,z_{k'}} \equiv f_{53,z_{k'}} \equiv f_{61,z_{k'}} \equiv f_{63,z_{k'}} \equiv 0 \\ f_{12,z_k} \equiv f_{13,z_k} \equiv f_{22,z_k} \equiv f_{23,z_k} \equiv f_{33,z_k} \equiv f_{43,z_k} \equiv f_{52,z_k} \equiv f_{53,z_k} \equiv f_{62,z_k} \equiv f_{63,z_k} \equiv 0 \end{array}\right\} (28)$$

At the beginning by using assumptions (6) and (8) we have
The 1-forms ω_α satisfy the structure equations (4) therefore,

$$\sum_{i=0}^{k} f_{11,z_i} dz_i \wedge dx + \sum_{i'=1}^{k'} f_{11,z_{i'}} dz_{i'} \wedge dx + f_{11,p} dp \wedge dx + \sum_{i=0}^{k} f_{12,z_i} dz_i \wedge dy + \sum_{i'=1}^{k'} f_{12,z_{i'}} dz_{i'} \wedge dy + f_{12,p} dp \wedge dy$$
$$+ \sum_{i=0}^{k} f_{13,z_i} dz_i \wedge dt + \sum_{i'=1}^{k'} f_{13,z_{i'}} dz_{i'} \wedge dt + f_{13,p} dp \wedge dt$$
$$= (\zeta f_{22} - \xi f_{21} + \xi f_{51} - \zeta f_{52}) dx \wedge dy + (\zeta f_{23} - f_{43} f_{21} + f_{51} f_{33} - \zeta f_{53}) dx \wedge dt$$
$$+ (\xi f_{23} - f_{43} f_{22} + f_{52} f_{33} - \xi f_{53}) dy \wedge dt \qquad (*)$$

From the above equation$(*)$we can obtain the following equations by simple calculations and by using equations (27)

$$-\psi f_{12,p} - p f_{12,z_0} + \sum_{i=0}^{k'-1} z_{i'+1} f_{13,z_{i'}} = \xi f_{23} - f_{43} f_{22} + f_{52} f_{33} - \xi f_{53}$$
$$-\psi f_{11,p} - p f_{11,z_0} + \sum_{i=0}^{k-1} z_{i+1} f_{13,z_i} = \zeta f_{23} - f_{43} f_{21} + f_{51} f_{33} - \zeta f_{53}$$
$$-\sum_{i'=1}^{k'-1} z_{i'+1} f_{11,z_{i'}} + \sum_{i=0}^{k-1} z_{i+1} f_{12,z_i} = \zeta f_{22} - \xi f_{21} + \xi f_{51} + \zeta f_{52}$$

In similar way by using assumptions (6) and (8) we have the 1-forms ω_α satisfy the structure equations (4) then

$$\sum_{i=0}^{k} f_{21,z_i} dz_i \wedge dx + \sum_{i'=1}^{k'} f_{21,z_{i'}} dz_{i'} \wedge dx + f_{21,p} dp \wedge dx + \sum_{i=0}^{k} f_{22,z_i} dz_i \wedge dy$$
$$+ \sum_{i'=1}^{k'} f_{22,z_{i'}} dz_{i'} \wedge dy + f_{22,p} dp \wedge dy + \sum_{i=0}^{k} f_{23,z_i} dz_i \wedge dt + \sum_{i'=1}^{k'} f_{23,z_{i'}} dz_{i'} \wedge dt + f_{23,p} dp \wedge dt$$
$$= (-\zeta f_{12} + \xi f_{11} + \xi f_{61} - \zeta f_{62}) dx \wedge dy + (-\zeta f_{13} + f_{43} f_{11} + f_{61} f_{33} - \zeta f_{63}) dx \wedge dt$$
$$+ (-\xi f_{13} + f_{43} f_{12} + f_{62} f_{33} - \xi f_{63}) dy \wedge dt \qquad (**)$$

From the above equation$(**)$we can obtain the following equations by simple calculations and by using equations (27)

$$-\psi f_{22,p} - p f_{22,z_0} + \sum_{i'=1}^{k'-1} z_{i'+1} f_{23,z_{i'}} = f_{12} f_{43} - \zeta f_{13} + f_{62} f_{33} - \zeta f_{63}$$

$$-\psi f_{21,p} - p f_{21,z_0} + \sum_{i=0}^{k-1} z_{i+1} f_{23,z_i} = f_{11} f_{43} - \zeta f_{13} + f_{61} f_{33} - \zeta f_{63}$$
$$-\sum_{i'=1}^{k'-1} z_{i'+1} f_{21,z_{i'}} + \sum_{i=0}^{k-1} z_{i+1} f_{22,z_i} = \xi f_{11} - \zeta f_{12} + \xi f_{61} - \zeta f_{62}$$

In similar way by using assumptions(6) and(8) we have the 1-forms $\omega_\alpha$satisfy the structure equations (4) then

$$\sum_{i=0}^{k} f_{31,z_i} dz_i \wedge dx + \sum_{i'=1}^{k'} f_{31,z_{i'}} dz_{i'} \wedge dx + f_{31,p} dp \wedge dx + \sum_{i=0}^{k} f_{32,z_i} dz_i \wedge dy + \sum_{i'=1}^{k'} f_{32,z_{i'}} dz_{i'} \wedge dy + f_{32,p} dp \wedge dy + \sum_{i=0}^{k} f_{33,z_i} dz_i \wedge dt + \sum_{i'=1}^{k'} f_{33,z_{i'}} dz_{i'} \wedge dt + f_{33,p} dp \wedge dt$$
$$= (-f_{51} f_{12} + f_{52} f_{11} - f_{61} f_{22} + f_{62} f_{21}) dx \wedge dy + (-f_{52} f_{13} + f_{53} f_{12} - f_{62} f_{23} + f_{22} f_{63}) dy \wedge dt + (-f_{51} f_{13} + f_{53} f_{11} - f_{61} f_{23} - f_{63} f_{21}) dx \wedge dt \qquad (***)$$

From the above equation$(***)$we can obtain the following equations by simple calculations and by using equations (27)

$$\sum_{i'=1}^{k'-2} z_{i'+1} f_{33,z_{i'}} = f_{12} f_{53} - f_{13} f_{52} + f_{22} f_{63} - f_{23} f_{62}$$
$$f_{31,z_i} = f_{31,z_{i'}} = f_{32,z_i} = f_{32,z_{i'}} = 0 \text{where} f_{31} = \zeta, f_{32} = \xi$$
$$\sum_{i=0}^{k-2} z_{i+1} f_{33,z_i} = f_{11} f_{53} - f_{13} f_{51} + f_{21} f_{63} - f_{23} f_{61}$$
$$f_{31,z_i} = f_{31,z_{i'}} = f_{32,z_i} = f_{32,z_{i'}} = 0 \text{where} f_{31} = \zeta, f_{32} = \xi$$
$$f_{11} f_{52} - f_{12} f_{51} = f_{22} f_{61} - f_{21} f_{62}$$
$$f_{31,z_i} = f_{31,z_{i'}} = f_{32,z_i} = f_{32,z_{i'}} = 0 \text{where} f_{31} = \zeta, f_{32} = \xi$$

Similarly by using assumptions(6) and(8) we have the 1-forms ω_α satisfy the structure equations (4) then

$$\sum_{i=0}^{k} f_{41,z_i} dz_i \wedge dx + \sum_{i'=1}^{k'} f_{41,z_{i'}} dz_{i'} \wedge dx + f_{41,p} dp \wedge dx + \sum_{i=0}^{k} f_{42,z_i} dz_i \wedge dy + \sum_{i'=1}^{k'} f_{42,z_{i'}} dz_{i'} \wedge dy + f_{42,p} dp \wedge dy + \sum_{i=0}^{k} f_{43,z_i} dz_i \wedge dt + \sum_{i'=1}^{k'} f_{43,z_{i'}} dz_{i'} \wedge dt + f_{43,p} dp \wedge dt$$
$$= (f_{11} f_{22} - f_{12} f_{21}) dx \wedge dy + (f_{12} f_{23} - f_{13} f_{22}) dy \wedge dt + (f_{11} f_{23} - f_{13} f_{21}) dx \wedge dt \qquad (\cdot)$$

From the above equation$(\cdot)$we can obtain the following equations by simple calculations and by using equations (27)

$f_{41,z_i} = f_{41,z_{i'}} = f_{42,z_i} = f_{42,z_{i'}} = 0 \text{where} f_{41} = \zeta, f_{42} = \xi$ we have

$$\sum_{i'=1}^{k'-2} z_{i'+1} f_{43,z_{i'}} = f_{12} f_{23} - f_{13} f_{22}$$

$f_{41,z_i} = f_{41,z_{i'}} = f_{42,z_i} = f_{42,z_{i'}} = 0 \text{where} f_{41} = \zeta, f_{42} = \xi$ then we have

$$\sum_{i=0}^{k-2} z_{i+1} f_{43,z_i} = f_{11} f_{23} - f_{13} f_{21}$$

$f_{41,z_i} = f_{41,z_{i'}} = f_{42,z_i} = f_{42,z_{i'}} = 0 \text{where} f_{41} = \zeta, f_{42} = \xi$ then we have

$$f_{11} f_{22} - f_{12} f_{21} = 0$$

Similarly by using assumptions (6) and (8) we have the 1-forms ω_α satisfy the structure equations (4) then

$$\sum_{i=0}^{k} f_{51,z_i} dz_i \wedge dx + \sum_{i'=1}^{k'} f_{51,z_{i'}} dz_{i'} \wedge dx + f_{51,p} dp \wedge dx + \sum_{i=0}^{k} f_{52,z_i} dz_i \wedge dy + \sum_{i'=1}^{k'} f_{52,z_{i'}} dz_{i'} \wedge dy + f_{52,p} dp \wedge dy + \sum_{i=0}^{k} f_{53,z_i} dz_i \wedge dt + \sum_{i'=1}^{k'} f_{53,z_{i'}} dz_{i'} \wedge dt + f_{53,p} dp \wedge dt$$
$$= (f_{11} f_{32} - f_{12} f_{31}) dx \wedge dy + (f_{12} f_{33} - f_{13} f_{32}) dy \wedge dt + (f_{11} f_{33} - f_{13} f_{31}) dx \wedge dt \qquad (\cdot\cdot)$$

From the above equation(··)we can obtain the following equations by simple calculations and by using equations (27)

$$-\psi f_{52,p} - pf_{52,z_0} + \sum_{i'=1}^{k'-1} z_{i'+1} f_{53,z_{i'}} = f_{12}f_{33} - \xi f_{13}$$

$$-\psi f_{51,p} - pf_{51,z_0} + \sum_{i=0}^{k-1} z_{i+1} f_{53,z_i} = f_{11}f_{33} - \zeta f_{13}$$

$$-\sum_{i'=1}^{k'-1} z_{i'+1} f_{51,z_{i'}} + \sum_{i=0}^{k-1} z_{i+1} f_{52,z_i} = \xi f_{11} - \zeta f_{12}$$

Finally by using assumptions (6) and (8) we have the 1-forms ω_α satisfy the structure equations (4) then

$$\sum_{i=0}^{k} f_{61,z_i} dz_i \wedge dx + \sum_{i'=1}^{k'} f_{61,z_{i'}} dz_{i'} \wedge dx + f_{61,p} dp \wedge dx + \sum_{i=0}^{k} f_{62,z_i} dz_i \wedge dy$$
$$+ \sum_{i'=1}^{k'} f_{62,z_{i'}} dz_{i'} \wedge dy + f_{62,p} dp \wedge dy + \sum_{i=0}^{k} f_{63,z_i} dz_i \wedge dt + \sum_{i'=1}^{k'} f_{63,z_{i'}} dz_{i'} \wedge dt + f_{63,p} dp \wedge dt$$
$$= (f_{21}f_{32} - f_{22}f_{31}) dx \wedge dy + (f_{22}f_{33} - f_{23}f_{32}) dy \wedge dt + (f_{21}f_{33} - f_{23}f_{31}) dx \wedge dt \qquad (\cdots)$$

From the above equation(···)we can obtain the following equations by simple calculations and by using equations (27)

$$-\psi f_{62,p} - pf_{62,z_0} + \sum_{i'=1}^{k'-1} z_{i'+1} f_{63,z_{i'}} = f_{22}f_{33} - \xi f_{23}$$

$$-\psi f_{61,p} - pf_{61,z_0} + \sum_{i=0}^{k-1} z_{i+1} f_{63,z_i} = f_{21}f_{33} - \zeta f_{23}$$

$$-\sum_{i'=1}^{k'-1} z_{i'+1} f_{61,z_{i'}} + \sum_{i=0}^{k-1} z_{i+1} f_{62,z_i} = f_{21}f_{32} - f_{22}f_{31} \blacksquare$$

To justify the last equation of (8) we observe that if $f_{11,p}, f_{12,p}, f_{21,p}, f_{22,p}, f_{51,p}, f_{52,p}, f_{61,p}$, and $f_{62,p}$ vanish simulataneously, it follows that equation (5) can not be the necessary and sufficient condition for the forms ω_α to satisfy the structure equations of a 3-dim. p.s.p. Therefore, this condition is added, and that completes the proof of the lemma.

III. Characterization of this type of equations:

Now, by the lemma (2.1), necessary conditions for an equation of the type

$$p_t = \psi(z_0, z_1, \dots, z_k, z_{1'}, \dots\dots, z_{k'}, p)$$

To describe a 3-dim.P.S.P, is that the functions $f_{\alpha i}$ satisfy (8) to (26). Therefore we shall assume these conditions in order to characterize all such equations. We consider quantities $L_1 \neq 0,\ L_2 \neq 0,\ L_1' \neq 0$ and $L_2' \neq 0$ where

$$\left.\begin{array}{ll} L_{1,p} = f_{11}f_{51,p} - f_{51}f_{11,p}, & L_{2,p} = f_{21}f_{61,p} - f_{61}f_{21,p} \\ L_{1,z_i} = f_{51,z_i}f_{11} - f_{11,z_i}f_{51}, & L_{1,pz_i} = f_{51,p}f_{11,z_i} - f_{11,p}f_{51,z_i} \\ L_{2,z_i} = f_{61,z_i}f_{21} - f_{21,z_i}f_{61}, & L_{2,pz_i} = f_{61,p}f_{21,z_i} - f_{21,p}f_{61,z_i} \\ S_1 = f_{51}^2 - f_{11}^2, & S_2 = f_{61}^2 - f_{21}^2 \end{array}\right\} \qquad (29)$$

Also, we consider the following

$$\left.\begin{array}{ll} L'_{1,p} = f_{12}f_{52,p} - f_{52}f_{12,p}, & L'_{2,p} = f_{22}f_{62,p} - f_{62}f_{22,p} \\ L'_{1,z_i} = f_{52,z_i}f_{12} - f_{12,z_i}f_{52}, & L'_{1,pz_i} = f_{52,p}f_{12,z_i} - f_{12,p}f_{52,z_i} \\ L'_{2,z_i} = f_{62,z_i}f_{22} - f_{22,z_i}f_{62}, & L'_{2,pz_i} = f_{62,p}f_{22,z_i} - f_{22,p}f_{62,z_i} \\ S'_1 = f_{52}^2 - f_{12}^2, & S'_2 = f_{62}^2 - f_{22}^2 \end{array}\right\} \qquad (30)$$

Now we state the following theorem

Theorem 3.1

Let $f_{\alpha i}$, $1 \leq \alpha \leq 6$, $1 \leq i \leq 3$,be differentiable functions of $p, z_0, z_1, \dots\dots z_k, z_{1'}, \dots\dots z_{k'}$ such that quations (8) hold and $f_{31} = f_{41} = \zeta$, $f_{32} = f_{42} = \xi$are parameters. Suppose $L_{1,P}, L'_{1,p}, L_{2,P}$ and $L'_{2,p}$ as given before are non zero.Then theequation $p_t = \psi(z_0, z_1, \dots, z_k, z_{1'}, \dots\dots, z_{k'}, p)$describes a two-parameters 3-dimensional P.S.P with associated 1-forms$\omega_\alpha = f_{\alpha1}dx + f_{\alpha2}dy + f_{\alpha3}dt$If and only if the function ψ is given by

$$\psi = \frac{1}{(L_{1,P})^2}\Bigg[z_{j+1}\left(L_{1,P}\sum_{i=0}^{k-2} f_{33,z_i} - \sum_{i=0}^{k-2} f_{33,z_i} z_{i+1}L_{1,Pz_{i+1}} + L_{1,z_i}L_{1,z_1}\right) + PL_{1,z_0}L_{1,P} - L_{1,P}\sum_{i=0}^{k-2} f_{33,z_iz_i}z_{i+1} - L_{1,z_i}L_{1,z_1} + \sum_{i=0}^{k-2} f_{33,z_i}z_{i+1}L_{1,Pz_i} + \frac{1}{2}\zeta S_{1,P}\sum_{i=0}^{k-2} f_{33,z_i}z_{i+1} + S_1(\zeta L_{1,z_1} - f_{33}L_{1,P})\Bigg]$$
$$+ \frac{1}{(L_{2,P})^2}\Bigg[z_{j+1}\left(L_{2,P}\sum_{i=0}^{k-2} f_{33,z_i} - \sum_{i=0}^{k-2} f_{33,z_i} z_{i+1}L_{2,Pz_{i+1}} + L_{2,z_i}L_{2,z_1}\right) + PL_{2,z_0}L_{2,P} - L_{2,P}\sum_{i=0}^{k-2} f_{33,z_iz_i}z_{i+1} - L_{2,z_i}L_{2,z_1} + \sum_{i=0}^{k-2} f_{33,z_i}z_{i+1}L_{2,Pz_i} + \frac{1}{2}\zeta S_{2,P}\sum_{i=0}^{k-2} f_{33,z_i}z_{i+1} + S_2(\zeta L_{2,z_1} - f_{33}L_{2,P})\Bigg]$$
$$+ \frac{1}{(L'_{1,P})^2}\Bigg[z_{j'+1}\left(L'_{1,P}\sum_{i'=1}^{k'-2} f_{33,z_{i'}} - \sum_{i'=1}^{k'-2} f_{33,z_{i'}} z_{i'+1}L'_{1,Pz_{i'+1}} + L'_{1,z_{i'}}L'_{1,z_{1'}}\right) + PL'_{1,z_0}L'_{1,P} - L'_{1,P}\sum_{i'=1}^{k'-2} f_{33,z_{i'}z_{i'}}z_{i'+1} - L'_{1,z_{i'}}L'_{1,z_1} + \sum_{i'=1}^{k'-2} f_{33,z_{i'}}z_{i'+1}L'_{1,Pz_{i'}} + \frac{1}{2}\xi S'_{1,P}\sum_{i'=1}^{k'-2} f_{33,z_{i'}}z_{i'+1} + S'_1(\xi L'_{1,z_1} - f_{33}L'_{1,P})\Bigg]$$
$$+ \frac{1}{(L'_{2,P})^2}\Bigg[z_{j'+1}\left(L'_{2,P}\sum_{i'=1}^{k'-2} f_{33,z_{i'}} - \sum_{i'=1}^{k'-2} f_{33,z_{i'}} z_{i'+1}L'_{2,Pz_{i'+1}} + L'_{2,z_{i'}}L'_{2,z_{1'}}\right) + PL'_{2,z_0}L'_{2,P} - L'_{2,P}\sum_{i'=1}^{k'-2} f_{33,z_{i'}z_{i'}}z_{i'+1} - L'_{2,z_{i'}}L'_{2z_1} + \sum_{i'=1}^{k'-2} f_{33,z_{i'}}z_{i'+1}L'_{2,Pz_{i'}} + \frac{1}{2}\xi S'_{2,P}\sum_{i'=1}^{k'-2} f_{33,z_{i'}}z_{i'+1} + S'_2(\xi L'_{2,z_1} - f_{33}L'_{2,P})\Bigg] + f_{51,P}(f_{43}f_{21} - \zeta f_{23}) + f_{61,P}(\zeta f_{13} - f_{11}f_{43}) + f_{52,P}(f_{43}f_{22} - \xi f_{23}) + f_{62,P}(\xi f_{13} - f_{12}f_{43}) \quad (31)$$

Where $1 \le j \le k-1 \quad , \quad 1 \le j' \le k' - 1$

Moreover

$$f_{13} = \frac{1}{L_{1,p}}\left[f_{11,p}\sum_{i=0}^{k-2} f_{33,z_i} z_{i+1} + f_{11}\left(L_{1,z_1} + L_{2,z_1}\right) - f_{23}\left(f_{11}, f_{61,p} - f_{61}f_{11,p}\right) - f_{63}\left(f_{21}, f_{11,p} - f_{11}f_{21,p}\right)\right] (32)$$

$$f_{53} = \frac{1}{L_{1,p}}\left[f_{51,p}\sum_{i=0}^{k-2} f_{33,z_i} z_{i+1} + f_{51}\left(L_{1,z_1} + L_{2,z_1}\right) - f_{23}\left(f_{51}, f_{61,p} - f_{61}f_{51,p}\right) - f_{63}\left(f_{21}, f_{51,p} - f_{51}f_{21,p}\right)\right] (33)$$

$$f_{23} = \frac{1}{L_{2,p}}\left[f_{21,p}\sum_{i=0}^{k-2} f_{33,z_i} z_{i+1} + f_{21}\left(L_{1,z_1} + L_{2,z_1}\right) - f_{13}\left(f_{21}, f_{51,p} - f_{51}f_{21,p}\right) - f_{53}\left(f_{11}, f_{21,p} - f_{21}f_{11,p}\right)\right] (34)$$

$$f_{63} = \frac{1}{L_{2,p}}\left[f_{61,p}\sum_{i=0}^{k-2} f_{33,z_i} z_{i+1} + f_{61}\left(L_{1,z_1} + L_{2,z_1}\right) - f_{13}\left(f_{61}, f_{51,p} - f_{51}f_{61,p}\right) - f_{53}\left(f_{11}, f_{61,p} - f_{61}f_{11,p}\right)\right] (35)$$

It is noted that by similar construction, one may obtain

$$f_{13} = \frac{1}{L'_{1,p}}\left[f_{12,p}\sum_{i'=1}^{k'-2} f_{33,z_{i'}} z_{i'+1} + f_{12}\left(L'_{1,z_1} + L'_{2,z_1}\right) - f_{23}\left(f_{12}, f_{62,p} - f_{62}f_{12,p}\right) - f_{63}\left(f_{22}, f_{12,p} - f_{12}f_{22,p}\right)\right] (36)$$

$$f_{53} = \frac{1}{L'_{1,p}}\left[f_{52,p}\sum_{i'=1}^{k'-2} f_{33,z_{i'}} z_{i'+1} + f_{52}\left(L'_{1,z_1} + L'_{2,z_1}\right) - f_{23}\left(f_{52}, f_{62,p} - f_{62}f_{52,p}\right) - f_{63}\left(f_{22}, f_{52,p} - f_{52}f_{22,p}\right)\right] \quad (37)$$

$$f_{23} = \frac{1}{L'_{2,p}}\left[f_{22,p}\sum_{i'=1}^{k'-2} f_{33,z_{i'}} z_{i'+1} + f_{22}\left(L'_{1,z_1} + L'_{2,z_1}\right) - f_{13}\left(f_{22}, f_{52,p} - f_{52}f_{22,p}\right) - f_{53}\left(f_{12}f_{22,p} - f_{22}, f_{12,p}\right)\right] \quad (38)$$

$$f_{63} = \frac{1}{L'_{2,p}}\left[f_{62,p}\sum_{i'=1}^{k'-2} f_{33,z_{i'}} z_{i'+1} + f_{62}\left(L'_{1,z_1} + L'_{2,z_1}\right) - f_{13}\left(f_{62}, f_{52,p} - f_{52}f_{62,p}\right) - f_{53}\left(f_{12}, f_{62,p} - f_{62}f_{12,p}\right)\right] \quad (39)$$

Proof

Suppose the equation $P_t = \psi$ describes a 3-dim. P.S.P. Then it follows from the lemma that equations (9) $\rightarrow$ (26) are satisfied. Now, consider eqns. (16), (17) and their derivatives with respect to p, it follows from (8) that

$$\left.\begin{aligned} f_{11}f_{53} - f_{13}f_{51} + f_{21}f_{63} - f_{23}f_{61} &= \sum_{i=0}^{k-2} f_{33,z_i} z_{i+1} \\ -f_{11,p}f_{53} + f_{51,p}f_{13} - f_{21,p}f_{63} + f_{61,p}f_{23} &= L_{1,z_1} + L_{2,z_1} \end{aligned}\right\} \quad (40)$$

$$\left.\begin{aligned} f_{12}f_{23} - f_{13}f_{52} + f_{22}f_{63} - f_{23}f_{62} &= \sum_{i'=1}^{k'-2} f_{33,z_{i'}} z_{i'+1} \\ -f_{12,p}f_{53} + f_{52,p}f_{13} - f_{22,p}f_{63} + f_{62,p}f_{23} &= L'_{1,z_1} + L'_{2,z_1} \end{aligned}\right\} \quad (41)$$

Using notations (29), (30) and from (40) one gets f_{53}, f_{53}, f_{53} and f_{53} as given by(32) $\rightarrow$ (35).
Also, by the same way, (41) gives, the formulas (36) $\rightarrow$ (39) for these functions.
Thus we have the following result:
Now, consider the derivative with respect to z_{j+1} of eqns (10), (13), (22), and (25) as well as the derivative with respect to $z_{j'+1}$ of eqns (11), (14), (23), and (26) . Then it follows from (8) that

$$\left.\begin{aligned} -\psi_{z_{j+1}} f_{11,P} + f_{13,z_j} &= 0 \\ -\psi_{z_{j+1}} f_{12,P} + f_{23,z_j} &= 0 \\ -\psi_{z_{j+1}} f_{51,P} + f_{53,z_j} &= 0 \\ -\psi_{z_{j+1}} f_{61,P} + f_{63,z_j} &= 0 \end{aligned}\right\} \quad (42)$$

And

$$\left.\begin{aligned} -\psi_{z_{j'+1}} f_{12,P} + f_{13,z_{j'}} &= 0 \\ -\psi_{z_{j'+1}} f_{22,P} + f_{23,z_{j'}} &= 0 \\ -\psi_{z_{j'+1}} f_{52,P} + f_{53,z_{j'}} &= 0 \\ -\psi_{z_{j'+1}} f_{62,P} + f_{63,z_{j'}} &= 0 \end{aligned}\right\} \quad (43)$$

Therefore (42), and (43) give $\psi_{z_{j+1}z_{j+1}} = 0$and $\psi_{z_{j'+1}z_{j'+1}} = 0$. Thus$\psi$is of the form :

$$\psi = Az_{j+1} + Bz_{j'+1} + C \quad (44)$$

Where A is independent of z_{j+1} , B is independent of $z_{j'+1}$, while C is independent of both z_{j+1} , and$z_{j'+1}$. From eqns (42), (43), and (44) we get

$$f_{52,z_i}f_{11,P} - f_{13,z_i}f_{51,P} = 0 \quad , \quad f_{63,z_i}f_{21,P} - f_{23,z_i}f_{61,P} = 0 \quad (45)$$

$$f_{53,z_{i'}}f_{12,P} - f_{13,z_{i'}}f_{52,P} = 0 \quad , \quad f_{63,z_{i'}}f_{22,P} - f_{23,z_{i'}}f_{62,P} = 0 \quad (46)$$

$$\left.\begin{aligned} \left(f_{53,z_i}f_{11} - f_{13,z_i}f_{51}\right) - \psi_{z_{i+1}} L_{1,P} &= 0 \\ \left(f_{63,z_i}f_{21} - f_{23,z_i}f_{61}\right) - \psi_{z_{i+1}} L_{2,P} &= 0 \end{aligned}\right\} \quad (47)$$

$$\left.\begin{aligned} \left(f_{53,z_{i'}}f_{12} - f_{13,z_{i'}}f_{52}\right) - \psi_{z_{i'+1}} L'_{1,P} &= 0 \\ \left(f_{63,z_{i'}}f_{22} - f_{23,z_{i'}}f_{62}\right) - \psi_{z_{i'+1}} L'_{2,P} &= 0 \end{aligned}\right\} \quad (48)$$

Where $1 \le j \le k-1 \quad , \quad 1 \le j' \le k' - 1$

Now from equations (44), (47) and (48) one gets:

$$A = \frac{1}{L_{1,P}}\left(f_{53,z_j}f_{11} - f_{13,z_j}f_{51}\right) + \frac{1}{L_{2,P}}\left(f_{63,z_j}f_{21} - f_{23,z_j}f_{61}\right) \quad (49)$$

$$B = \frac{1}{L'_{1,P}}\left(f_{53,z_{j'}}f_{12} - f_{13,z_{j'}}f_{52}\right) + \frac{1}{L'_{2,P}}\left(f_{63,z_{j'}}f_{22} - f_{23,z_{j'}}f_{62}\right) \quad (50)$$

Hence, by (32) → (35) and (36) → (39) we get

$$A=\frac{1}{(L_{1,P})^2}\left[L_{1,P}\sum_{i=0}^{k-2}f_{33,z_i}-\sum_{i=0}^{k-2}f_{33,z_i}\ z_{i+1}\,L_{1,Pz_{i+1}}+L_{1,z_i}\,L_{1,z_1}\right]$$

$$+\frac{1}{(L_{2,P})^2}\left[L_{2,P}\sum_{i=0}^{k-2}f_{33,z_i}-\sum_{i=0}^{k-2}f_{33,z_i}\ z_{i+1}\,L_{2,Pz_{i+1}}+L_{2,z_i}L_{2,z_1}\right] \quad (51)$$

$$B=\frac{1}{(L'_{1,P})^2}\left[L'_{1,P}\sum_{i'=1}^{k'-2}f_{33,z_{i'}}-\sum_{i'=1}^{k'-2}f_{33,z_{i'}}\ z_{i'+1}\,L'_{1,Pz_{i'+1}}+L'_{1,z_{i'}}L'_{1,z_{1'}}\right]$$

$$+\frac{1}{(L'_{2,P})^2}\left[L'_{2,P}\sum_{i'=1}^{k'-2}f_{33,z_{i'}}-\sum_{i'=1}^{k'-2}f_{33,z_{i'}}\ z_{i'+1}\,L'_{2,Pz_{i'+1}}+L'_{2,z_{i'}}\ L'_{2,z_{1'}}\right](52)$$

Now, it follows from (*44*) → (*50*) that equations (10) and (22) are equivalent to thefollowing

$$CL_{1,P}+PL_{1,z_0}-\sum_{i=0}^{k-1}z_{i+1}\left(f_{53,z_i}f_{11}-f_{13,z_i}f_{51}\right)-f_{33}S_1+\zeta(f_{53}f_{51}-f_{13}f_{11})f_{33}S_1+f_{51}(f_{43}f_{21}-\zeta f_{23})=0(53)$$

$$-PL_{1,Pz_0}-\sum_{i=0}^{k-1}z_{i+1}\left(f_{53,z_i}f_{11,P}-f_{13,z_i}f_{51,P}\right)+\zeta\left(f_{53}f_{51,P}-f_{13}f_{11,P}\right)-\frac{1}{2}f_{33}S_{1,P}+f_{51,P}(f_{43}f_{21}-\zeta f_{23})=0(54)$$

Also equations (13) and (25) are equivalent to:

$$CL_{2,P}+PL_{2,z_0}-\sum_{i=0}^{k-1}z_{i+1}\left(f_{63,z_i}f_{21}-f_{23,z_i}f_{61}\right)-f_{33}S_2+\zeta(f_{63}f_{61}-f_{23}f_{21})f_{33}S_2+f_{61}(\zeta f_{13}-f_{11}f_{43})=0(55)$$

$$-PL_{2,Pz_0}-\sum_{i=0}^{k-1}z_{i+1}\left(f_{63,z_i}f_{21,P}-f_{23,z_i}f_{61,P}\right)+\zeta\left(f_{63}f_{61,P}-f_{23}f_{21,P}\right)-\frac{1}{2}f_{33}S_{2,P}+f_{61,P}(\zeta f_{13}-f_{11}f_{43})=0(56)$$

Similarly, eqns (11) and (23) are equivalent to the following :

$$CL'_{1,P}+PL'_{1,z_0}-\sum_{i'=1}^{k'-1}z_{i'+1}\left(f_{53,z_{i'}}f_{12}-f_{13,z_{i'}}f_{52}\right)-f_{33}S'_1+\xi(f_{52}f_{53}-f_{13}f_{12})f_{33}S'_1+f_{52}(f_{43}f_{22}-\xi f_{23})=0\ (57)$$

$$-PL'_{1,Pz_0}-\sum_{i'=1}^{k'-1}z_{i'+1}\left(f_{53,z_{i'}}f_{12,P}-f_{13,z_{i'}}f_{52,P}\right)+\xi\left(f_{52,P}f_{53}-f_{13}f_{12,P}\right)-\frac{1}{2}f_{33}S'_{1,P}+f_{52,P}(f_{43}f_{22}-\xi f_{23})=0(58)$$

Also, equations (14) and (26) are equivalent to:

$$CL'_{2,P}+PL'_{2,z_0}-\sum_{i'=1}^{k'-1}z_{i'+1}\left(f_{63,z_{i'}}f_{22}-f_{23,z_{i'}}f_{62}\right)-f_{33}S'_2+\xi(f_{63}f_{62}-f_{22}f_{23})f_{33}S'_2+f_{62}(\xi f_{13}-f_{12}f_{43})=0(59)$$

$$-PL'_{2,Pz_0}-\sum_{i'=1}^{k'-1}z_{i'+1}\left(f_{63,z_{i'}}f_{22,P}-f_{23,z_{i'}}f_{62,P}\right)+\xi\left(f_{63}f_{62,P}-f_{23}f_{22,P}\right)-\frac{1}{2}f_{33}S'_{2,P}+f_{62,P}(\xi f_{13}-f_{12}f_{43})=0(60)$$

Therefore, using (32) → (35) and (36) → (39) in ((55) → (60), we get :

$$C = \frac{1}{(L_{1,P})^2}\left[PL_{1,z_0}L_{1,P} - L_{1,P}\sum_{i=0}^{k-2} f_{33,z_iz_i}z_{i+1} - L_{1,z_i}\,L_{1,z_1} + \sum_{i=0}^{k-2} f_{33,z_i}z_{i+1}L_{1,Pz_i} + \frac{1}{2}\zeta S_{1,P}\sum_{i=0}^{k-2} f_{33,z_i}z_{i+1} + S_1(\zeta L_{1,z_1} - f_{33}L_{1,P})\right]$$
$$+ \frac{1}{(L_{2,P})^2}\left[PL_{2,z_0}L_{2,P} - L_{2,P}\sum_{i=0}^{k-2} f_{33,z_iz_i}z_{i+1} - L_{2,z_i}\,L_{2,z_1} + \sum_{i=0}^{k-2} f_{33,z_i}z_{i+1}L_{2,Pz_i} + \frac{1}{2}\zeta S_{2,P}\sum_{i=0}^{k-2} f_{33,z_i}z_{i+1} + S_2(\zeta L_{2,z_1} - f_{33}L_{2,P})\right]$$
$$+ \frac{1}{(L'_{1,P})^2}\left[PL'_{1,z_0}L'_{1,P} - L'_{1,P}\sum_{i'=1}^{k'-2} f_{33,z_{i'}z_{i'}}z_{i'+1} - L'_{1,z_{i'}}L'_{1,z_1} + \sum_{i'=1}^{k'-2} f_{33,z_{i'}}z_{i'+1}L'_{1,Pz_{i'}} + \frac{1}{2}\xi S'_{1,P}\sum_{i'=1}^{k'-2} f_{33,z_{i'}}z_{i'+1} + S'_1(\xi L'_{1,z_1} - f_{33}L'_{1,P})\right]$$
$$+ \frac{1}{(L'_{2,P})^2}\left[PL'_{2,z_0}L'_{2,P} - L'_{2,P}\sum_{i'=1}^{k'-2} f_{33,z_{i'}z_{i'}}z_{i'+1} - L'_{2,z_{i'}}L'_{2z_1} + \sum_{i'=1}^{k'-2} f_{33,z_{i'}}z_{i'+1}L'_{2,Pz_{i'}} + \frac{1}{2}\xi S'_{2,P}\sum_{i'=1}^{k'-2} f_{33,z_{i'}}z_{i'+1} + S'_2(\xi L'_{2,z_1} - f_{33}L'_{2,P})\right] + f_{51,P}(f_{43}f_{21} - \zeta f_{23})$$
$$+ f_{61,P}(\zeta f_{13} - f_{11}f_{43}) + f_{52,P}(f_{43}f_{22} - \xi f_{23}) + f_{62,P}(\xi f_{13} - f_{12}f_{43}) \qquad (61)$$

Thus, form (44),(51),(52), and (61) we obtain ψ as given by (31) in the theorem. Conversely, given functions $f_{11}, f_{51}, f_{21}, f_{61}, f_{12}, f_{52}, f_{22}$ and f_{62} of $p, z_0, z_1, \ldots\ldots z_k, z_{1'}, \ldots\ldots z_{k'}$ and the functions $f_{13}, f_{53}, f_{23}, f_{63}$ and ψ as given by (32) → (35), or(36) → (39) and (31), then straightforward computations show that the equation $P_t = \psi$ describes two-parameters 3-dimensional p.s.p with associated 1-forms $\omega_\alpha = f_{\alpha 1}dx + f_{\alpha 2}dy + f_{\alpha 3}dt$, $1 \le \alpha \le 6$ which satisfy equations (4). This completes the proof of the theorem.

As a matter of fact, this model of evolution equations with more than two spatial variables, which we are considering here, fits many equations of physical interest namely the higher dimension sine-Gordon equations, [*3,12*]

IV. Conclusion

In this paper, we extended the notion of P.S.P to higher dimensions i.e. 3-dim plane of constant sectional curvature-1 imbedded in R^5 and we studied the change in the results and properties.

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