IOSR Journal of Mathematics (IOSR-JM)
e-ISSN: 2278-5728, p-ISSN: 2319-765X. Volume 11, Issue 2 Ver. V (Mar - Apr. 2015), PP 21-28
www.iosrjournals.org

Zeros of Polynomials with Restricted Coefficients

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Abstract: *We extend some existing results on the zeros of polynomials by considering more general coefficient conditions. As special cases the extended results yield much simpler expressions for the upper bounds of zeros than those of the existing results. The zero-free regions of analytic functions subject to similar coefficient conditions are also investigated.*
Mathematics Subject Classification:*30C10, 30C15.*
Keywords: *Zeros of polynomial,Eneström-Kakeya theorem.*

I. Introduction

Many results on the location of zeros of polynomials are available in the literature. A.Joyal, G.Labelle and Q.I.Rahman [3] obtained the generalized results by considering the coefficients to be real, instead of being only positive. In literature [5-11] attempts have been made to extend and generalize the Enestrom-Kakeya theorem. Aziz and Zargar [4] also relaxed the hypothesis of Eneström-Kakeya theorem in a different way. Existing results in the literature also show that there is a need to find bounds for special polynomials, for example, for those having restrictions on the coefficient, there is always need for refinement of results in this subject. Among them the Enestrom-Kakeya theorem [1,2] given below is well known in the theory of zero distribution of polynomials.

Theorem (Enestrom-Kakeyatheorem): Let$P(z)=\sum_{i=0}^{n} a_i z^i$be a polynomial of degree n such that $0 < a_0 \leq a_1 \leq a_2 \leq, \ldots, \leq a_n$ then allthe zeros of P(z) lie in $|z|\leq 1$.

Here we establish more generalized results by using Eneström-Kakeya Theorem.

Theorem 1. Let $P(z) =\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq 2$and $2\leq m \leq n$with real coefficients such that

$$a_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq \cdots \geq a_{n-m+1} \leq a_{n-m} \geq a_{n-m-1} \geq \cdots a_3 \geq a_2 \geq a_1 \geq a_0$$

if both n and (n-m) are even or odd

(OR)

$$a_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq \cdots \leq a_{n-m+1} \geq a_{n-m} \geq a_{n-m-1} \geq \cdots a_3 \geq a_2 \geq a_1 \geq a_0$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|a_n|}\ [\ |a_0| - a_0 + a_n + 2\{(a_{n-2} + a_{n-4} + \cdots + a_{n-m+2} + a_{n-m}) - (a_{n-1} + a_{n-3} + + \cdots + a_{n-m+3} + an-m+1\ ]$$

.if both n and (n-m) are even or odd

(OR)

then all the zeros of P(z) lie in

$$|z| \leq \frac{1}{|a_n|}\ [\ |a_0| - a_0 + a_n + 2\{(a_{n-2} + a_{n-4} + \cdots + a_{n-m+3} + a_{n-m+1}) - (a_{n-1} + a_{n-3} + + \cdots + a_{n-m+4} + an-m+2\ ]$$

.if n is even and (n-m) is odd (or) if n is odd and (n-m) is even.

Corollary 1.. Let $P(z) =\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \geq 2$and $2 \leq m \leq n$with positive real coefficients such that for some

$$a_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq \cdots \geq a_{n-m+1} \leq a_{n-m} \geq a_{n-m-1} \geq \cdots a_3 \geq a_2 \geq a_1 \geq a_0$$

if both n and (n-m) are even or odd

(OR)

$$a_n \geq a_{n-1} \leq a_{n-2} \geq a_{n-3} \leq a_{n-4} \geq \cdots \leq a_{n-m+1} \geq a_{n-m} \geq a_{n-m-1} \geq \cdots a_3 \geq a_2 \geq a_1 \geq a_0$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

then all the zeros of P(z) lie in
$|z| \le \frac{1}{a_n}$ $[a_n + 2\{(a_{n-2} + a_{n-4} + \cdots + a_{n-m+2} + a_{n-m}) - (a_{n-1} + a_{n-3} + + \cdots + a_{n-m+3} + a_{n-m+1})\}]$.
if both n and (n-m) are even or odd

(OR)

then all the zeros of P(z) lie in

$$|z| \le \frac{1}{a_n}[a_n + 2\{(a_{n-2} + a_{n-4} + \cdots + a_{n-m+3} + a_{n-m+1}) - (a_{n-1} + a_{n-3} + + \cdots + a_{n-m+4} + a_{n-m+2})\}].$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

Remark 1. By taking $a_i > 0$ for$i = 0,1,2,\dots,n-1$,in theorem 1,then theorem 1, reduces to Corollary 2.

Theorem 2. Let P(z) =$\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \ge 2$and $2 \le m \le n$with real coefficients such that for some

$$a_n \le a_{n-1} \ge a_{n-2} \le a_{n-3} \ge a_{n-4} \le \cdots \le a_{n-m+1} \ge a_{n-m} \ge a_{n-m-1} \ge \cdots a_3 \ge a_2 \ge a_1 \ge a_0$$

if both n and (n-m) are even or odd

(OR)

$$a_n \le a_{n-1} \ge a_{n-2} \le a_{n-3} \ge a_{n-4} \le \cdots \ge a_{n-m+1} \le a_{n-m} \ge a_{n-m-1} \ge \cdots a_3 \ge a_2 \ge a_1 \ge a_0$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

then all the zeros of P(z) lie in
$|z| \le \frac{1}{|a_n|}$ $[\,|a_0| - a_0 - a_n + 2\{(a_{n-1} + a_{n-3} + + \cdots + a_{n-m+3} + a_{n-m+1}) - (a_{n-2} + a_{n-4} + \cdots + a_{n-m+4} +$ an−m+2].if both n and (n-m) are even or odd

(OR)

then all the zeros of P(z) lie in

$|z| \le \frac{1}{|a_n|}[|a_0| - a_0 - a_n + 2\{(a_{n-1} + a_{n-3} + + \cdots + a_{n-m+2} + a_{n-m}) - (a_{n-2} + a_{n-4} + \cdots + a_{n-m+3} +$ an−m+1 .if n is even and (n-m) is odd (or) if n is odd and (n-m) is even .

Corollary 2. Let P(z) =$\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \ge 2$and $2 \le m \le n$with positive realcoefficients such that for some

$$a_n \le a_{n-1} \ge a_{n-2} \le a_{n-3} \ge a_{n-4} \le \cdots \le a_{n-m+1} \ge a_{n-m} \ge a_{n-m-1} \ge \cdots a_3 \ge a_2 \ge a_1 \ge a_0$$

if both n and (n-m) are even or odd

(OR)

$$a_n \le a_{n-1} \ge a_{n-2} \le a_{n-3} \ge a_{n-4} \le \cdots \ge a_{n-m+1} \le a_{n-m} \ge a_{n-m-1} \ge \cdots a_3 \ge a_2 \ge a_1 \ge a_0$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

then all the zeros of P(z) lie in
$|z| \le \frac{1}{a_n}$ $[\,2\{(a_{n-1} + a_{n-3} + + \cdots + a_{n-m+3} + a_{n-m+1}) - (a_{n-2} + a_{n-4} + \cdots + a_{n-m+4} + a_{n-m+2})\} - a_n]$. if both n and (n-m) are even or odd

(OR)

then all the zeros of P(z) lie in

$|z| \le \frac{1}{a_n}$ $[\,2\{(a_{n-1} + a_{n-3} + + \cdots + a_{n-m+2} + a_{n-m}) - (a_{n-2} + a_{n-4} + \cdots + a_{n-m+3} + a_{n-m+1})\} - a_n]$. if n is even and (n-m) is odd (or) if n is odd and (n-m) is even .

Remark 2. By taking $a_i > 0\, fori = 0,1,2,\dots,n-1$, in theorem 2, then theorem 2, reduces to Corollary 2.

Theorem 3. Let P(z) =$\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \ge 2$and $2 \le m \le n$with real coefficients such that for some

$$a_n \ge a_{n-1} \le a_{n-2} \ge a_{n-3} \le a_{n-4} \ge \cdots \ge a_{n-m+1} \le a_{n-m} \le a_{n-m-1} \le \cdots a_3 \le a_2 \le a_1 \le a_0$$

if both n and (n-m) are even or odd

(OR)

$$a_n \ge a_{n-1} \le a_{n-2} \ge a_{n-3} \le a_{n-4} \ge \cdots \le a_{n-m+1} \ge a_{n-m} \le a_{n-m-1} \le \cdots a_3 \le a_2 \le a_1 \le a_0$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

then all the zeros of P(z) lie in
$|z| \le \frac{1}{|a_n|}\ [\,|a_0| + a_0 + a_n + 2\{(a_{n-2} + a_{n-4} + \cdots + a_{n-m+4} + a_{n-m+2}) - (a_{n-1} + a_{n-3} + + \cdots + a_{n-m+3} + a_{n-m+1}\,]$. if both n and (n-m) are even or odd

(OR)

then all the zeros of P(z) lie in

$|z| \le \frac{1}{|a_n|}\ [\,|a_0| + a_0 + a_n + 2\{(a_{n-2} + a_{n-4} + \cdots + a_{n-m+3} + a_{n-m+1}) - (a_{n-1} + a_{n-3} + + \cdots + a_{n-m+2} + a_{n-m}\,]$. if n is even and (n-m) is odd (or) if n is odd and (n-m) is even.

Corollary 3. Let P(z) $=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \ge 2$and $2 \le m \le n$with positive real coefficients such that for some

$$a_n \ge a_{n-1} \le a_{n-2} \ge a_{n-3} \le a_{n-4} \ge \cdots \ge a_{n-m+1} \le a_{n-m} \le a_{n-m-1} \le \cdots a_3 \le a_2 \le a_1 \le a_0$$

if both n and (n-m) are even or odd

(OR)

$$a_n \ge a_{n-1} \le a_{n-2} \ge a_{n-3} \le a_{n-4} \ge, \ldots, \le a_{n-m+1} \ge a_{n-m} \le a_{n-m-1} \le \cdots a_3 \le a_2 \le a_1 \le a_0$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

then all the zeros of P(z) lie in
$|z| \le \frac{1}{a_n}\ [\, a_n + 2\{(a_0 + a_{n-2} + a_{n-4} + \cdots + a_{n-m+4} + a_{n-m+2}) - (a_{n-1} + a_{n-3} + + \cdots + a_{n-m+3} + a_{n-m+1}\,]$. if both n and (n-m) are even or odd

(OR)

then all the zeros of P(z) lie in

$|z| \le \frac{1}{a_n}\ [\, a_n + 2\{(a_0 + a_{n-2} + a_{n-4} + \cdots + a_{n-m+3} + a_{n-m+1}) - (a_{n-1} + a_{n-3} + + \cdots + a_{n-m+2} + a_{n-m}\,]$. if n is even and (n-m) is odd (or) if n is odd and (n-m) is even.

Remark 3. By taking $a_i > 0\ for i = 0,1,2,\ldots,n-1$, in theorem 3, then theorem 3, reduces to Corollary 3.

Theorem 4. Let P(z) $=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \ge 2$and $2 \le m \le n$with real coefficients such that for some

$$a_n \le a_{n-1} \ge a_{n-2} \le a_{n-3} \ge a_{n-4} \le \cdots \le a_{n-m+1} \ge a_{n-m} \le a_{n-m-1} \le \cdots a_3 \le a_2 \le a_1 \le a_0$$

if both n and (n-m) are even or odd

(OR)

$$a_n \le a_{n-1} \ge a_{n-2} \le a_{n-3} \ge a_{n-4} \le \cdots \ge a_{n-m+1} \le a_{n-m} \le a_{n-m-1} \le \cdots a_3 \le a_2 \le a_1 \le a_0$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

then all the zeros of P(z) lie in
$|z| \le \frac{1}{|a_n|}\ [\,|a_0| + a_0 - a_n + 2\{(a_{n-1} + a_{n-3} + + \cdots + a_{n-m+3} + a_{n-m+1}) - (a_{n-2} + a_{n-4} + \cdots + a_{n-m+2} + a_{n-m}\,]$. if both n and (n-m) are even or odd

(OR)

then all the zeros of P(z) lie in
$|z| \le \frac{1}{|a_n|}[\,|a_0| + a_0 - a_n + 2\{(a_{n-1} + a_{n-3} + + \cdots + a_{n-m+4} + a_{n-m+2}) - (a_{n-2} + a_{n-4} + \cdots + a_{n-m+3} + a_{n-m+1}$ .if n is even and (n-m) is odd (or) if n is odd and (n-m) is even .

Corollary 4. Let P(z) $=\sum_{i=0}^{n} a_i z^i$ be a polynomial of degree $n \ge 2$and $2 \le m \le n$with real coefficients such that for some

$$a_n \le a_{n-1} \ge a_{n-2} \le a_{n-3} \ge a_{n-4} \le \cdots \le a_{n-m+1} \ge a_{n-m} \le a_{n-m-1} \le \cdots a_3 \le a_2 \le a_1 \le a_0$$

if both n and (n-m) are even or odd

(OR)

$$a_n \le a_{n-1} \ge a_{n-2} \le a_{n-3} \ge a_{n-4} \le \cdots \ge a_{n-m+1} \le a_{n-m} \le a_{n-m-1} \le \cdots a_3 \le a_2 \le a_1 \le a_0$$

if n is even and (n-m) is odd (or) if n is odd and (n-m) is even

then all the zeros of P(z) lie in

$|z| \le \frac{1}{|a_n|}$ [$|a_0| + a_0 - a_n + 2\{(a_{n-1} + a_{n-3} + + \cdots + a_{n-m+3} + a_{n-m+1}) - (a_{n-2} + a_{n-4} + \cdots + a_{n-m+2} + an-m$] . if both n and (n-m) are even or odd

(OR)

then all the zeros of P(z) lie in

$|z| \le \frac{1}{|a_n|}[|a_0| + a_0 - a_n + 2\{(a_{n-1} + a_{n-3} + + \cdots + a_{n-m+4} + a_{n-m+2}) - (a_{n-2} + a_{n-4} + \cdots + a_{n-m+3} + an-m+1$ .if n is even and (n-m) is odd (or) if n is odd and (n-m) is even .

Remark 4. By taking $a_i > 0\ for i = 0,1,2,\ldots,n-1$, in theorem 4, then theorem 4 , reduces to Corollary 4.

II. Proofs Of The Theorems

Proof of the Theorem 1.

LetP(z)=$a_n z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + a_{n-3}z^{n-3} + \cdots + a_{n-m}z^{n-m} + \cdots + a_2 z^2 + a_1 z + a_0$.
be a polynomial of degree n≥ 2

Letus consider the polynomial Q(z)=(1-z) P(z) so that

Q(z)=(1-z)($a_n z^n + a_{n-1}z^{n-1} + \cdots + a_{m+1}z^{m+1} + a_m z^m + a_{m-1}z^{m-1} + \cdots + a_1 z + a_0$)

$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_{n-m+1} - a_{n-m})z^{n-m+1} + (a_{n-m} - a_{n-m-1})z^{n-m} + (a_{n-m-1} - a_{n-m-2})z^{n-m-1} + \cdots + (a_3 - a_2)z^3 + (a_2 - a_1)z^2 + (a_1 - a_0)z + a_n$

Also if $|z| > 1$ then$\frac{\mathbf{1}}{|\boldsymbol{z}|^{\boldsymbol{n-i}}} < for i = 0,1,2,\ldots,n-1$.

Now$|Q(z)| \ge |a_n||z|^{n+1} - \{ \ |a_n - a_{n-1}||z|^n + |a_{n-1} - a_{n-2}||z|^{n-1} + \cdots + |a_{n-m+1} - a_{n-m}||z|^{n-m+1} + |a_{n-m} - a_{n-m-1}||z|^{n-m} + |a_{n-m-1} - a_{n-m-2}||z|^{n-m-1} + \cdots + |a_3 - a_2||z|^3 + |a_2 - a_1||z|^2 + |a_1 - a_0||z| + |a_0| \}$

$\ge |a_n||z|^n[|z| - \frac{1}{|a_n|}\{ |a_n - a_{n-1}| + \frac{|a_{n-1}-a_{n-2}|}{|z|} + \frac{|a_{n-2}-a_{n-3}|}{|z|^2} + \frac{|a_{n-3}-a_{n-4}|}{|z|^3} + \cdots + \frac{|a_{n-m+1}-a_{n-m}|}{|z|^{m-1}} + \frac{|a_{n-m}-a_{n-m-1}|}{|z|^m} + \frac{|a_{n-m-1}-a_{n-m-2}|}{|z|^{m+1}} + \cdots + \frac{|a_3-a_2|}{|z|^{n-3}} + \frac{|a_2-a_1|}{|z|^{n-2}} + \frac{|a_1-a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\}]$

$\ge |a_n||z|^n[|z| - \frac{1}{|a_n|}\{ |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}| + |a_{n-3} - a_{n-4}| + \cdots + |a_{n-m+1} - an-m + |an-m-an-m-1 + |an-m-1-an-m-2 + |a3-a2 + |a2-a1 + |a1-a0 + a0]$

$\ge |a_n||z|^n[|z| - \frac{1}{|a_n|}\{ (a_n - a_{n-1}) + (a_{n-2} - a_{n-1}) + (a_{n-2} - a_{n-3}) + + (a_{n-4} - a_{n-3}) + \cdots + (a_{n-m} - a_{n-m+1}) + (a_{n-m} - a_{n-m-1}) + (a_{n-m-1} - a_{n-m-2}) + \cdots + (a_3 - a_2) + (a_2 - a_1) + (a_1 - a_0) + |a_0| \}]$.if both n and (n-m) are even or odd

$= |a_n||z|^n[|z| - \frac{1}{|a_n|}\{|a_0| - a_0 + a_n + 2\{(a_{n-2} + a_{n-4} + \cdots + a_{n-m+2} + a_{n-m}) - (a_{n-1} + a_{n-3} + + \cdots + an-m+3 + an-m+1\}]$

> 0 if

$|z| > \frac{1}{|a_n|}$ [$|a_0| - a_0 + a_n + 2\{(a_{n-2} + a_{n-4} + \cdots + a_{n-m+2} + a_{n-m}) - (a_{n-1} + a_{n-3} + + \cdots + a_{n-m+3} + an-m+1$].

This shows that if Q(z)> 0 provided

$|z| > \frac{1}{|a_n|}$ [$|a_0| - a_0 + a_n + 2\{(a_{n-2} + a_{n-4} + \cdots + a_{n-m+2} + a_{n-m}) - (a_{n-1} + a_{n-3} + + \cdots + a_{n-m+3} + an-m+1$].

Hence all the zeros of Q(z) with |z|> 1 lie in

z| $\le \frac{1}{|a_n|}$ [$|a_0| - a_0 + a_n + 2\{(a_{n-2} + a_{n-4} + \cdots + a_{n-m+2} + a_{n-m}) - (a_{n-1} + a_{n-3} + + \cdots + a_{n-m+3} + an-m+1$].
if both n and (n-m) are even or odd.

But those zeros of Q(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Since all the zeros of P(z) are also the zeros of Q(z) lie in the circle defined by the above inequality and this completes the proof of the Theorem 1, if both n and (n-m) are even or odd.

Similarly we can also prove for n is even and (n-m) is odd (or) if n is odd and (n-m) is even degree polynomials. For this we can rearrange the terms of the given polynomial and compute accordingly. That is if n is even and (n-m) is odd (or) if n is odd and (n-m) is even then all the zeros P(z) lie in

z| $\leq \frac{1}{|a_n|}$ [$[\,|a_0| - a_0 + a_n + 2\{(a_{n-2} + a_{n-4} + \cdots + a_{n-m+3} + a_{n-m+1}) - (a_{n-1} + a_{n-3} + + \cdots + a_{n-m+4} +$ $an - m + 2]$.

This completes the proof of the Theorem 1.

Proof of the Theorem 2.

LetP(z)=$a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + a_{n-3} z^{n-3} + \cdots + a_{n-m} z^{n-m} + \cdots + a_2 z^2 + a_1 z + a_0$.
be a polynomial of degree n$\geq$ 2

Letus consider the polynomial Q(z)= (1-z) P(z) so that

Q(z)=(1-z)($a_n z^n + a_{n-1} z^{n-1} + \cdots + a_{m+1} z^{m+1} + a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$)

= $-a_n z^{n+1} + (a_n - a_{n-1}) z^n + (a_{n-1} - a_{n-2}) z^{n-1} + \cdots + (a_{n-m+1} - a_{n-m}) z^{n-m+1} + (a_{n-m} - a_{n-m-1}) z^{n-m} + (a_{n-m-1} - a_{n-m-2}) z^{n-m-1} + \cdots + (a_3 - a_2) z^3 + (a_2 - a_1) z^2 + (a_1 - a_0) z + a_n$

Also if $|z| > 1$ then$\frac{\mathbf{1}}{|\mathbf{z}|^{\mathbf{n-i}}} < fori = 0,1,2, \ldots, n-1$.

Now $|Q(z)| \geq |a_n||z|^{n+1} - \{\ \ |a_n - a_{n-1}||z|^n + |a_{n-1} - a_{n-2}||z|^{n-1} + \cdots + |a_{n-m+1} - a_{n-m}||z|^{n-m+1} + |a_{n-m} - a_{n-m-1}||z|^{n-m} + |a_{n-m-1} - a_{n-m-2}||z|^{n-m-1} + \cdots + |a_3 - a_2||z|^3 + |a_2 - a_1||z|^2 + |a_1 - a_0||z| + |a_0|\,\}$

$\geq |a_n||z|^n[|z| - \frac{1}{|a_n|}\{\,|a_n - a_{n-1}| + \frac{|a_{n-1} - a_{n-2}|}{|z|} + \frac{|a_{n-2} - a_{n-3}|}{|z|^2} + \frac{|a_{n-3} - a_{n-4}|}{|z|^3} + \cdots + \frac{|a_{n-m+1} - a_{n-m}|}{|z|^{m-1}} + \frac{|a_{n-m} - a_{n-m-1}|}{|z|^m} + \frac{|a_{n-m-1} - a_{n-m-2}|}{|z|^{m+1}} + \cdots + \frac{|a_3 - a_2|}{|z|^{n-3}} + \frac{|a_2 - a_1|}{|z|^{n-2}} + \frac{|a_1 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\}]$

$\geq |a_n||z|^n[|z| - \frac{1}{|a_n|}\{\,|a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}| + |a_{n-3} - a_{n-4}| + \cdots + \ |a_{n-m+1} -$ $an - m + |an - m - an - m - 1 + |an - m - 1 - an - m - 2 + |a3 - a2 + |a2 - a1 + |a1 - a0 + a0]$

$\geq |a_n||z|^n[|z| - \frac{1}{|a_n|}\{\ (a_{n-1} - a_n) + (a_{n-1} - a_{n-2}) + \ (a_{n-3} - a_{n-2}) + + (a_{n-3} - a_{n-4}) + \cdots + \ (a_{n-m+1} - a_{n-m}) + \ (a_{n-m} - a_{n-m-1}) + \ (a_{n-m-1} - a_{n-m-2}) + \cdots + (a_3 - a_2) + \ (a_2 - a_1) + \ (a_1 - a_0) + |a_0|\ \}]$.if both n and (n-m) are even or odd

$= |a_n||z|^n[|z| - \frac{1}{|a_n|}\ \{[\,|a_0| - a_0 - a_n + 2\{(a_{n-1} + a_{n-3} + + \cdots + a_{n-m+3} + a_{n-m+1}) - (a_{n-2} + a_{n-4} + \cdots +$ $an - m + 4 + an - m + 2\}\ ]$

> 0 if

|z| $> \frac{1}{|a_n|}$ [$|a_0| - a_0 - a_n + 2\{(a_{n-1} + a_{n-3} + + \cdots + a_{n-m+3} + a_{n-m+1}) - (a_{n-2} + a_{n-4} + \cdots + a_{n-m+4} +$ $an - m + 2]$.

This shows that if Q(z)$>$ 0 provided

|z| $> \frac{1}{|a_n|}$ [$|a_0| - a_0 - a_n + 2\{(a_{n-1} + a_{n-3} + + \cdots + a_{n-m+3} + a_{n-m+1}) - (a_{n-2} + a_{n-4} + \cdots + a_{n-m+4} +$ $an - m + 2]$.

Hence all the zeros of Q(z) with |z|$>$ 1 lie in

z| $\leq \frac{1}{|a_n|}$ [$[\,|a_0| - a_0 - a_n + 2\{(a_{n-1} + a_{n-3} + + \cdots + a_{n-m+3} + a_{n-m+1}) - (a_{n-2} + a_{n-4} + \cdots + a_{n-m+4} +$ $an - m + 2]$.

if both n and (n-m) are even or odd.

But those zeros of Q(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Since all the zeros of P(z) are also the zeros of Q(z) lie in the circle defined by the above inequality and this completes the proof of the Theorem 2, if both n and (n-m) are even or odd.

Similarly we can also prove for n is even and (n-m) is odd (or) if n is odd and (n-m) is even degree polynomials. For this we can rearrange the terms of the given polynomial and compute accordingly. That is if n is even and (n-m) is odd (or) if n is odd and (n-m) is even then all the zeros P(z) lie in

$$|z| \le \frac{1}{|a_n|}[|a_0| - a_0 - a_n + 2\{(a_{n-1} + a_{n-3} + + \cdots + a_{n-m+2} + a_{n-m}) - (a_{n-2} + a_{n-4} + \cdots + a_{n-m+3} + a_{n-m+1})\}]$$

This completes the proof of the Theorem 2.

Proof of the Theorem 3.

LetP(z)=$a_n z^n + a_{n-1} z^{n-1} + a_{n-2} z^{n-2} + a_{n-3} z^{n-3} + \cdots + a_{n-m} z^{n-m} + \cdots + a_2 z^2 + a_1 z + a_0$.
be a polynomial of degree n≥ 2

Letus consider the polynomial Q(z)= (1-z) P(z) so that

Q(z)=(1-z)$(a_n z^n + a_{n-1} z^{n-1} + \cdots + a_{m+1} z^{m+1} + a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0)$

$= -a_n z^{n+1} + (a_n - a_{n-1}) z^n + (a_{n-1} - a_{n-2}) z^{n-1} + \cdots + (a_{n-m+1} - a_{n-m}) z^{n-m+1} + (a_{n-m} - a_{n-m-1}) z^{n-m} + (a_{n-m-1} - a_{n-m-2}) z^{n-m-1} + \cdots + (a_3 - a_2) z^3 + (a_2 - a_1) z^2 + (a_1 - a_0) z + a_n$

Also if $|z| > 1$ then$\frac{1}{|z|^{n-i}} < fori = 0,1,2,\ldots,n-1$.

Now $|Q(z)| \ge |a_n||z|^{n+1} - \{ |a_n - a_{n-1}||z|^n + |a_{n-1} - a_{n-2}||z|^{n-1} + \cdots + |a_{n-m+1} - a_{n-m}||z|^{n-m+1} + |a_{n-m} - a_{n-m-1}||z|^{n-m} + |a_{n-m-1} - a_{n-m-2}||z|^{n-m-1} + \cdots + |a_3 - a_2||z|^3 + |a_2 - a_1||z|^2 + |a_1 - a_0||z| + |a_0| \}$

$\ge |a_n||z|^n[|z| - \frac{1}{|a_n|}\{ |a_n - a_{n-1}| + \frac{|a_{n-1}-a_{n-2}|}{|z|} + \frac{|a_{n-2}-a_{n-3}|}{|z|^2} + \frac{|a_{n-3}-a_{n-4}|}{|z|^3} + \cdots + \frac{|a_{n-m+1}-a_{n-m}|}{|z|^{m-1}} + \frac{|a_{n-m}-a_{n-m-1}|}{|z|^m} + \frac{|a_{n-m-1}-a_{n-m-2}|}{|z|^{m+1}} + \cdots + \frac{|a_3-a_2|}{|z|^{n-3}} + \frac{|a_2-a_1|}{|z|^{n-2}} + \frac{|a_1-a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n}\}]$

$\ge |a_n||z|^n[|z| - \frac{1}{|a_n|}\{ |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}| + |a_{n-3} - a_{n-4}| + \cdots + |a_{n-m+1} -$
an−m+|an−m−an−m−1+|an−m−1−an−m−2+|a3−a2+|a2−a1+|a1−a0+a0]

$\ge |a_n||z|^n[|z| - \frac{1}{|a_n|}\{ (a_n - a_{n-1}) + (a_{n-2} - a_{n-1}) + (a_{n-2} - a_{n-3}) + + (a_{n-4} - a_{n-3}) + \cdots + (a_{n-m} - a_{n-m+1}) + (a_{n-m-1} - a_{n-m}) + (a_{n-m-2} - a_{n-m-1}) + \cdots + (a_2 - a_3) + (a_1 - a_2) + (a_0 - a_1) + |a_0| \}]$.if both n and (n-m) are even or odd

$= |a_n||z|^n[|z| - \frac{1}{|a_n|} \{|a_0| + a_0 + a_n + 2\{(a_{n-2} + a_{n-4} + \cdots + a_{n-m+4} + a_{n-m+2}) - (a_{n-1} + a_{n-3} + + \cdots +$
an−m+3+an−m+1}]

> 0 if
$|z| > \frac{1}{|a_n|}$ [$|a_0| + a_0 + a_n + 2\{(a_{n-2} + a_{n-4} + \cdots + a_{n-m+4} + a_{n-m+2}) - (a_{n-1} + a_{n-3} + + \cdots + a_{n-m+3} +$
an−m+1].

This shows that if Q(z)$>$ 0 provided
$|z| > \frac{1}{|a_n|}$ [$|a_0| + a_0 + a_n + 2\{(a_{n-2} + a_{n-4} + \cdots + a_{n-m+4} + a_{n-m+2}) - (a_{n-1} + a_{n-3} + + \cdots + a_{n-m+3} +$
an−m+1].

Hence all the zeros of Q(z) with |z|$>$ 1 lie in

z| $\leq \frac{1}{|a_n|}$ [$|a_0| + a_0 + a_n + 2\{(a_{n-2} + a_{n-4} + \cdots + a_{n-m+4} + a_{n-m+2}) - (a_{n-1} + a_{n-3} + + \cdots + a_{n-m+3} + a_{n-m+1}]$.
if both n and (n-m) are even or odd.

But those zeros of Q(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Since all the zeros of P(z) are also the zeros of Q(z) lie in the circle defined by the above inequality and this completes the proof of the Theorem 3, if both n and (n-m) are even or odd.
Similarly we can also prove for n is even and (n-m) is odd (or) if n is odd and (n-m) is even degree polynomials. For this we can rearrange the terms of the given polynomial and compute accordingly. That is if n is even and (n-m) is odd (or) if n is odd and (n-m) is even then all the zeros P(z) lie in
z| $\leq \frac{1}{|a_n|}|a_0| + a_0 + a_n + 2\{(a_{n-2} + a_{n-4} + \cdots + a_{n-m+3} + a_{n-m+1}) - (a_{n-1} + a_{n-3} + + \cdots + a_{n-m+2} + a_{n-m}]$.

This completes the proof of the Theorem 3.

Proof of the Theorem 4.
LetP(z)=$a_n z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + a_{n-3}z^{n-3} + \cdots + a_{n-m}z^{n-m} + \cdots + a_2 z^2 + a_1 z + a_0$.
be a polynomial of degree n≥ 2

Letus consider the polynomial Q(z)= (1-z) P(z) so that

Q(z)=(1-z)($a_n z^n + a_{n-1}z^{n-1} + \cdots + a_{m+1}z^{m+1} + a_m z^m + a_{m-1}z^{m-1} + \cdots + a_1 z + a_0$)

$= -a_n z^{n+1} + (a_n - a_{n-1})z^n + (a_{n-1} - a_{n-2})z^{n-1} + \cdots + (a_{n-m+1} - a_{n-m})z^{n-m+1} + (a_{n-m} - a_{n-m-1})z^{n-m} + (a_{n-m-1} - a_{n-m-2})z^{n-m-1} + \cdots + (a_3 - a_2)z^3 + (a_2 - a_1)z^2 + (a_1 - a_0)z + a_n$
Also if $|z| > 1$ then$\frac{\mathbf{1}}{|\mathbf{z}|^{\mathbf{n-i}}} < for i = 0,1,2, \dots, n-1$.

Now $|Q(z)| \geq |a_n||z|^{n+1} - \{ |a_n - a_{n-1}||z|^n + |a_{n-1} - a_{n-2}||z|^{n-1} + \cdots + |a_{n-m+1} - a_{n-m}||z|^{n-m+1} + |a_{n-m} - a_{n-m-1}||z|^{n-m} + |a_{n-m-1} - a_{n-m-2}||z|^{n-m-1} + \cdots + |a_3 - a_2||z|^3 + |a_2 - a_1||z|^2 + |a_1 - a_0||z| + |a_0| \}$

$\geq |a_n||z|^n[|z| - \frac{1}{|a_n|}\{ |a_n - a_{n-1}| + \frac{|a_{n-1}-a_{n-2}|}{|z|} + \frac{|a_{n-2}-a_{n-3}|}{|z|^2} + \frac{|a_{n-3}-a_{n-4}|}{|z|^3} + \cdots + \frac{|a_{n-m+1}-a_{n-m}|}{|z|^{m-1}} + \frac{|a_{n-m}-a_{n-m-1}|}{|z|^m} + \frac{|a_{n-m-1}-a_{n-m-2}|}{|z|^{m+1}} + \cdots + \frac{|a_3-a_2|}{|z|^{n-3}} + \frac{|a_2-a_1|}{|z|^{n-2}} + \frac{|a_1-a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n} \}]$

$\geq |a_n||z|^n[|z| - \frac{1}{|a_n|}\{ |a_n - a_{n-1}| + |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}| + |a_{n-3} - a_{n-4}| + \cdots + |a_{n-m+1} - a_{n-m}| + |a_{n-m} - a_{n-m-1}| + |a_{n-m-1} - a_{n-m-2}| + |a_3 - a_2| + |a_2 - a_1| + |a_1 - a_0| + a_0]$

$\geq |a_n||z|^n[|z| - \frac{1}{|a_n|}\{ (a_{n-1} - a_n) + (a_{n-1} - a_{n-2}) + (a_{n-3} - a_{n-2}) + + (a_{n-3} - a_{n-4}) + \cdots + (a_{n-m+1} - a_{n-m}) + (a_{n-m-1} - a_{n-m}) + (a_{n-m-2} - a_{n-m-1}) + \cdots + (a_2 - a_3) + (a_1 - a_2) + (a_0 - a_1) + |a_0| \}]$.if both n and (n-m) are even or odd

$= |a_n||z|^n[|z| - \frac{1}{|a_n|} \{|a_0| + a_0 - a_n + 2\{(a_{n-1} + a_{n-3} + + \cdots + a_{n-m+3} + a_{n-m+1}) - (a_{n-2} + a_{n-4} + \cdots + a_{n-m+2} + a_{n-m}\}]$

> 0 if
|z| $> \frac{1}{|a_n|}$ [$|a_0| + a_0 - a_n + 2\{(a_{n-1} + a_{n-3} + + \cdots + a_{n-m+3} + a_{n-m+1}) - (a_{n-2} + a_{n-4} + \cdots + a_{n-m+2} + a_{n-m}]$.

This shows that if Q(z)> 0 provided
|z| $> \frac{1}{|a_n|}|a_0| + a_0 - a_n + 2\{(a_{n-1} + a_{n-3} + + \cdots + a_{n-m+3} + a_{n-m+1}) - (a_{n-2} + a_{n-4} + \cdots + a_{n-m+2} + a_{n-m}]$.

Hence all the zeros of Q(z) with |z|> 1 lie in

$z| \ \leq \frac{1}{|a_n|} \ [\ \ |a_0| + \ a_0 - a_n + 2\{(a_{n-1} + a_{n-3} + + \cdots + a_{n-m+3} + a_{n-m+1}) - (a_{n-2} + a_{n-4} + \cdots + a_{n-m+2} +$ an−m].
if both n and (n-m) are even or odd.

But those zeros of Q(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Since all the zeros of P(z) are also the zeros of Q(z) lie in the circle defined by the above inequality and this completes the proof of the Theorem 4, if both n and (n-m) are even or odd.
Similarly we can also prove for n is even and (n-m) is odd (or) if n is odd and (n-m) is even degree polynomials. For this we can rearrange the terms of the given polynomial and compute accordingly. That is if n is even and (n-m) is odd (or) if n is odd and (n-m) is even then all the zeros P(z) lie in

$$|z| \leq \frac{1}{|a_n|}[|a_0| + \ a_0 - a_n + 2\{(a_{n-1} + a_{n-3} + + \cdots + a_{n-m+4} + a_{n-m+2}) - (a_{n-2} + a_{n-4} + \cdots + a_{n-m+3} + a_{n-m+1})\}]$$

This completes the proof of the Theorem 4.

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