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Intersections and Pullbacks

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Abstract: *Ever since fuzzy sets were introduced by Lotfi Zadeh in the year 1965 [1], many algebraic structures were introduced by many authors. One such structure is fuzzy groups introduced in [2]and [3]. In [4] the authors introduced a novel definition of fuzzy group homomorphism between any two fuzzy groups and gave element wise characterization of some special morphisms in the category of fuzzy groups.*
In this article we prove the existence of the intersection of a family of fuzzy subgroups, and also study some properties of pullbacks.
Keywords: *Injective, Intersection, Equalizer, Fuzzy morphism, Fuzzy group homomorphism, Morphism, Monomorphism, Strong monomorphism,*

I. Introduction

Definition 1.1: Let (X, μ) be a fuzzy group and $\{ (u_i, \alpha_i): (X_i , \mu_i) \to (X, \mu)\}_{i\in I}$ be a family of fuzzy subgroups (X, μ). Then a strong fuzzy group homomorphism $(u,\alpha):(X',\mu')\to(X,\mu)$ is called an ***intersection*** of the family if

(i). for each $i \in I$, there exists morphisms $(\vartheta_i , \beta_i): (X',\mu')\to(X_i ,\mu_i)$ such that

$(u,\alpha) = (u_i ,\alpha_i)(\vartheta_i ,\beta_i)$

(ii). $(h,\gamma): (Y,\eta)\to(X,\mu)$ is any fuzzy group homomorphism such that $(h ,\gamma) = (u_i , \alpha_i) (h_i , \gamma_i)$ for morphisms $(h_i ,\gamma_i): (Y,\eta)\to(X_i ,\mu_i)$ for each $i \in I$, then there exists a unique morphism (k, δ): $(Y, \eta) \to (X', \mu')$ such that $(h,\gamma)= (u,\alpha)(k,\delta)$.

Theorem 1.3: The category of fuzzy groups $\mathcal{F}$ has intersections.

Proof. We have to prove that the intersection of every set of sub objects of (X, μ) in $\mathcal{F}$ exists. Let $\{(u_i ,\alpha_i): (X_i ,\mu_i) \to(X,\mu)\}_{i\in I}$ be any set of sub objects of (X, μ) in $\mathcal{F}$.

Then by definition each (u_i ,α_i) is a monomorphism and hence $u_i : X_i \to X$ is injective [3].

Let $X' = \bigcap_{i\in I} u_i (X_i) \subseteq X$ [Since each $u_i(X_i)$ is a sub group of X isomorphic to the group X_i].

Suppose $X' = \phi$ (empty set). Then there is a unique morphism (to be also denoted as ϕ) from X' in to any another set.

Hence in this case it is clear that $(\phi,\phi) : (\phi,\phi) \to (X,\mu)$ is the intersection of the family.

So let us assume that $X' = \bigcap_{i\in I} u_i (X_i) \neq \emptyset$.

We define $\mu': X'\to[0,1]$ by $\mu'(x) = \mu(x)$ for all $x \in X'$.

In other words $\mu' = \frac{\mu}{X'}$.

Hence $(i_{X'}, i_{\mu'(X')}) : (X' , \mu') \to (X , \mu)$ is a fuzzy subgroup of (X, μ).

Moreover since each $u_i : X_i \to X$ is injective, if $y \in X' = \bigcap_{i\in I} u_i (X_i)$ then there exists a unique $x_i \in X_i$ such that $y = u_i(x_i)$ for all $i \in I$.

Hence we have a well defined function,

v_i: $X'(= \bigcap_{i \in I} u_i (X_i) \to X_i$ defined as $\vartheta_i (y)=x_i$ if $y = u_i(X_i)$ for each $i \in I$. Clearly each $\vartheta_i : X'\to X_i$ is a homomorphism of groups. Finally, we define $\beta_i : \mu'(X') \to \mu_i (X_i)$ as follows.

If $y \in X'$ and $y = u_i(x_i)$ for a unique $x_i \in X_i$, then define $\beta_i (\mu'(y))=\mu_i (x_i)$.

$$\begin{tikzcd}
\mu'(X') \arrow[r, "\beta_i"] & \mu_i(X_i) \arrow[r, "\alpha_i"] & \mu(X) \\
X' \arrow[u, "\mu'"] \arrow[r, "v_i"'] & X_i \arrow[u, "\mu_i"'] \arrow[r, "u_i"'] & X \arrow[u, "\mu"']
\end{tikzcd}$$

Then for each $i \in I$ and $y \in X'$, $\beta_i \mu'(y) = \mu_i(x_i)$ where $y = u_i(x_i) = \mu_i \vartheta_i(y)$, by definition , Hence $\beta_i \mu' = \mu_i \vartheta_i$ so that $(\vartheta_i, \beta_i):(X', \mu') \to (X_i, \mu_i)$ is a homomorphism of fuzzy groups.

Claim . $(i_{X'}, i_{\mu'(X')}) :(X', \mu') \to (X, \mu)$ is the intersection of the given set of sub objects.

Step 1. For all $y \in X'$, if $y = u_i(x_i)$, $x_i \in X_i$ for each $i \in I$, then $u_i \vartheta_i(y) = u_i(x_i) = y = i_{X'}(y)$ so that $u_i \vartheta_i = i_{X'}$. Further

$\alpha_i \beta_i \mu'(y) = \alpha_i \mu_i(x_i)$ (by definition of β_i if $y = u_i(x_i)$)

$= \mu u_i(x_i)$ [since $(u_i, \alpha_i): (X_i, \mu_i) \to (X, \mu)$ is a fuzzy group homomorphism]

$= \mu(y)$ [$y = u_i(x_i)$]

$= \mu'(y)$ [$\mu' = \mu / X'$]

and so $\alpha_i \beta_i = i_{\mu'(X')}$.

Thus $(i_{X'}, i_{\mu'(X')}):(X', \mu') \to (X, \mu)$ satisfies condition(i) of definition 1.1.

Step 2. Suppose $(h, \gamma):(Y, \eta) \to (X, \mu)$ is any fuzzy group homomorphism such that $(h, \gamma) = (u_i, \alpha_i)(h_i, \gamma_i)$ for fuzzy group homomorphisms $(h_i, \gamma_i):(Y, \eta) \to (X_i, \mu_i)$, define $(k, \delta):(Y, \eta) \to (X', \mu')$ as follows.

If $y \in Y$, then $h(y) = u_i h_i(y)$ [since $h = u_i h_i$]

$\in u_i(X_i)$ for all i [since $h_i(y) \in X_i$].

Hence $h(y) \in \bigcap_{i \in I} u_i(X_i) = X'$ ....(3)

So we define $k : Y \to X'$ as $k(y) = h(y)$.

Then for all $y \in Y$, $i_{X'} k(y) = i_{X'} h(y) = h(y)$.

Thus $i_{X'} k = h$ ....(A).

Again if $y \in Y$, then by (3) $h(y) \in X'$.

Hence $\mu' h(y) \in \mu'(X') \Rightarrow \mu h(y) \in \mu'(X')$, since $\mu' = \mu / X'$.

So we define $\delta : \eta(Y) \to \mu'(X')$ as $\delta\eta(y) = \mu h(y)$.

Clearly δ is well defined for

$\eta(y_1) = \eta(y_2) \Rightarrow \gamma \eta(y_1) = \gamma \eta(y_2)$

$\Rightarrow \mu h(y_1) = \mu h(y_2)$ [since $\gamma \eta = \mu h$]

$\Rightarrow \delta\eta(y_1) = \delta\eta(y_2)$.

Now for all $y \in Y, \mu' k(y) = \mu k(y)$ [since $\mu' = \mu / X'$]

$= \mu h(y)$ [by definition of k]

$= \delta\eta(y)$ [by definition of δ].

Then $\mu' k = \delta\eta$, so that $(k, \delta): (Y, \eta) \to (X', \mu')$ is a fuzzy group homomorphism.

Finally for all $y \in Y$, $i_{\mu'(X')} \delta\eta(y) = \delta\eta(y)$

$= \mu h(y) = \gamma\eta(y)$ (since $\mu h = \gamma \eta$)

$$\Rightarrow i_{\mu'(X')}\delta = \gamma \ .$$

Further since $i_{X'}$ and $i_{\mu'(X')}$ are injective, we conclude that (k, δ) is unique.
Thus we have proved (ii) of definition 1.1.
Hence $(i_{X'}, i_{\mu'(X')}) : (X', \mu') \to (X, \mu)$ is an intersection of the given set of sub objects.

Remark 1.4: As in any category we can prove that any two intersections of a given family of fuzzy subgroups are isomorphic fuzzy subgroups. Hence we can talk of " the " intersection of a family of fuzzy subgroups. Moreover if $(i_{X_i}, i_{\mu_i(X_i)}) : (X_i, \mu_i) \to (X, \mu)$ are fuzzy subgroups where i_{X_i} and $i_{\mu_i(X_i)}$ are the respective inclusions , then the intersection of the family will be taken as $(i_{X'}, i_{\mu'(X')}) : (X', \mu') \to (X, \mu)$ where $X' = \bigcap_i X_i$ [intersection of sets] and μ' is the restriction of μ.
Next we define pullbacks in the category of fuzzy groups and investigate some of its properties

II. Pullbacks

Definition 2.1: Given two fuzzy group homomorphisms $(f_1, \alpha_1) : (X_1, \mu_1) \to (X, \mu)$ and $(f_2, \alpha_2) : (X_2, \mu_2) \to (X, \mu)$ a fuzzy group (P, θ) is called a ***pull back*** (cartesian square/fibre product) of (f_1, α_1) and (f_2, α_2) if

(i) there exist fuzzy group homomorphism $(g_1, \beta_1) : (P, \theta) \to (X_1, \mu_1)$ and $(g_2, \beta_2) : (P, \theta) \to (X_2, \mu_2)$ such that $(f_1, \alpha_1)(g_1, \beta_1) = (f_2, \alpha_2)(g_2, \beta_2)$ and

(ii) if there exist fuzzy group homomorphism $(k_1, \delta_1) : (Q, \xi) \to (X_1, \mu_1)$ and $(k_2, \delta_2) : (Q, \xi) \to (X_2, \mu_2)$ such that $(f_1, \alpha_1)(k_1, \delta_1) = (f_2, \alpha_2)(k_2, \delta_2)$ then there exists a unique fuzzy group homomorphism $(h, \gamma) : (Q, \xi) \to (P, \theta)$ such that $(g_1, \beta_1)(h, \gamma) = (k_1, \delta_1)$ and $(g_2, \beta_2)(h, \gamma) = (k_2, \delta_2)$

$$\begin{tikzcd}
(Q,\xi) \arrow[dr, "{(h,\gamma)}"] \arrow[drr, bend left, "{(k_2,\delta_2)}"] \arrow[ddr, bend right, "{(k_1,\delta_1)}"'] & & \\
& (P,\theta) \arrow[r, "{(g_2,\beta_2)}"] \arrow[d, "{(g_1,\beta_1)}"'] & (X_2,\mu_2) \arrow[d, "{(f_2,\alpha_2)}"] \\
& (X_1,\mu_1) \arrow[r, "{(f_1,\alpha_1)}"'] & (X,\mu)
\end{tikzcd}$$

Remark 2.2: As usual, we can prove that if the pull back of a pair of fuzzy group homomorphisms exists, then any two such pull backs are isomorphic fuzzy groups.

Example 2.3: Let $S_3 = \{\rho_0, \rho_1, \rho_2, \tau_1, \tau_2, \tau_3\}$ be the symmetric group of degree three where $\rho_0 = id, \rho_1 = (1\ 2\ 3), \rho_2 = (1\ 3\ 2), \tau_1 = (2\ 3), \tau_2 = (1\ 3)$ *and* $\tau_3 = (1, 2)$

Let $(X, \mu) = (S_3, \mu)$ be the fuzzy group with $\mu(\rho_0) = 1$, $\mu(\rho_1) = \mu(\rho_2) = \epsilon_1$ and $\mu(\tau_1) = \mu(\tau_2) = \mu(\tau_3) = \epsilon_2$ with $0 \leq \epsilon_2 < \epsilon_1 < 1$

Let $(X_1, \mu_1) = (Z_4, \mu_1)$ where $\mu_1(0) = 1, \mu_1(2) = t_1, \mu_1(1) = \mu_1(3) = t_2$ with $0 \leq t_2 < t_1 < 1$:

Let $H = \{\rho_0, \tau_1\}$ be the subgroup of S_3 and $(X_2, \mu_2) = (H, \mu_2)$ where $\mu_2 = \mu / H$.

Consider the following diagram, where

$$\begin{tikzcd}
(P,\theta) \arrow[r] \arrow[d] & (X_2,\mu_2) \arrow[d, "(f_2,\alpha_2)"] \\
(X_1,\mu_1) \arrow[r, "(f_1,\alpha_1)"'] & (X,\mu)
\end{tikzcd}$$

(f_2,α_2) is a strong monomorphism. Then the above square is a pull back if and only if $(P,\theta)=(f_1,\alpha_1)^{-1}(X_2,\mu_2)$

Proof. Since (f_2,α_2) is a strong monomorphism $(f_1,\alpha_1)^{-1}(X_2,\mu_2)$ exists. Then from the definitions of inverse images and pull backs, it immediately follows that (P,θ) is a pull back if and only if $(P,\theta)=(f_1,\alpha_1)^{-1}(X_2,\mu_2)$

$$\begin{tikzcd}
(P.\theta) \arrow[r] \arrow[d] & (H,\mu_2) \arrow[d, "(i_H,i_{\mu_2(H)})"] \\
(Z_4,\mu_1) \arrow[r, "(f_1,\alpha_1)"] & (S_3,\mu)
\end{tikzcd}$$

Let $(f_1,\alpha_1):(Z_4,\mu_1)\to(S_3,\mu)$ be the fuzzy group homomorphism defined as
$f_1(0) = f_1(2) = \rho_0$ $f_1(1) = f_1(3) = \tau_1$; $\alpha_1(1) = 1 = \alpha_1(t_1), \alpha_1(t_2) = \varepsilon_2$
Let $P=f_1^{-1}(H)=Z_4$. Then $(P,\theta)=(Z_4,\mu_1)$ is the inverse image of (f_1,α_1) and hence the pull back of (f_1,α_1) and and $(i_H,i_{\mu_z(H)})$. The next proof shows that finite intersection can be characterized via pull backs.

Proposition 2.4: Consider the following commutative diagram

$$\begin{tikzcd}
(P,\theta) \arrow[r] \arrow[d] & (X_2,\mu_2) \arrow[d, "(f_2,\alpha_2)"] \\
(X_1,\mu_1) \arrow[r, "(f_1,\alpha_1)"'] & (X,\mu)
\end{tikzcd}$$

Where (f_1,α_1) and (f_2,α_2) are strong monomorphism. Then the above square is a pullback if and only if (P,θ) is the intersections of (X_1,μ_1) and (X_2,μ_2).

Proof. Since (f_1,α_1) and (f_2,α_2) are strong monomorphism (X_1,μ_1) and (X_2,μ_2) are fuzzy subgroups of (X,μ), Therefore from the definitions of pullbacks and intersections it follows that (P,θ) is a pull back iff (P,θ) is an intersection of (X_1,μ_1) and (X_2,μ_2).

Example 2.5: Let (S_3,μ) be the fuzzy group. Let $H_1=\{\rho_0,\tau_1\}$ and $H_2=\{\rho_0,\tau_2\}$ be two groups of S_3. Then (H_1,μ_1) and (H_2,μ_2) are fuzzy subgroups of (S_3,μ) where μ_1 and μ_2 are restrictions of μ to H_1 and H_2. Let (H_0,μ_0) be the fuzzy subgroup of (S_3,μ) where $H_0=\{\rho_0\}$ and μ_0 is the restriction of μ. Then (H_0,μ_0) is the intersection of (H_1,μ_1) and (H_2,μ_2) [up to isomorphism] so that the following diagram is a pullback, with respect to the inclusion fuzzy group homomorphism.

$$\begin{tikzcd}
(H_0,\mu_0) \arrow[r] \arrow[d] & (H_2,\mu_2) \arrow[d] \\
(H_1,\mu_1) \arrow[r] & (s_3,\mu)
\end{tikzcd}$$

Properties 2.6: Let us see some properties of Pullbacks. Consider the pullback diagram

$$\begin{tikzcd}
(p,\theta) \arrow[r, "{(g_2,\beta_2)}"] \arrow[d, "{(g_1,\beta_1)}"'] & (X_2,\mu_2) \arrow[d, "{(f_2,\alpha_2)}"] \\
(X_1,\mu_1) \arrow[r, "{(f_1,\alpha_1)}"'] & (X,\mu)
\end{tikzcd}$$

(i). If (f_1,α_1) is a monomorphism then (g_2,β_2) is also a monomorphism

(ii). If (f_1,α_1) is a retraction then (g_2,β_2) is also a retraction

(iii). If (f_1,α_1) is an isomorphism then (g_2,β_2) is also an isomorphism

(iv). If (f_1,α_1) is an equalizer then so is (g_2,β_2)

Proof (i).

$$\begin{tikzcd}
(Q,\xi) \arrow[dr, "{(k_1,\delta_1)\ (k_2,\delta_2)}"] \arrow[drr, bend left, "{(g_2,\beta_2)\,(k_2,\delta_2)}"] \arrow[ddr, bend right, "{(g_1,\beta_1)\,(k_1,\delta_1)}"'] & & \\
& (P,\theta) \arrow[r, "{(g_2,\beta_2)}"] \arrow[d, "{(g_1,\beta_1)}"'] & (X_2,\mu_2) \arrow[d, "{(f_2,\alpha_2)}"] \\
& (X_1,\mu_1) \arrow[r, "{(f_1,\alpha_1)}"'] & (X,\mu)
\end{tikzcd}$$

Suppose that (f_1,α_1) is a monomorphism…………………………….. (1)

Since the given square is a pull back, we have

$$(f_1,\alpha_1)(g_1,\beta_1)=(f_2,\alpha_2)(g_2,\beta_2) \text{ ……….. (2)}$$

Suppose there exists fuzzy groups homomorphism (k_1,δ_1) , (k_2,δ_2): $(Q,\xi) \to (P,\theta)$

Such that $(g_2,\beta_2)(k_1,\delta_1) = (g_2,\beta_2)(k_2,\delta_2)$ ……………………………….......(3)

Then

$(f_1,\alpha_1)[(g_1,\beta_1)(k_1,\delta_1)] = (f_2,\alpha_2)(g_2,\beta_2)(k_1,\delta_1)$ using (2)

$= (f_2, \alpha_2)[(g_2, \beta_2)(k_2, \delta_2)]$ using (3)........(A)

$= (f_1, \alpha_1)[(g_1, \beta_1)(k_2, \delta_2)]$ using (2)

This implies $(g_1, \beta_1)(k_1, \delta_1) = (g_1, \beta_1)(k_2, \delta_2)$ ....(4)

[Since (f_1, α_1) is a monomorphism]

Since the given square is a pull back, and (A) is true there exists a unique fuzzy group homomorphism $(h, \gamma):(Q, \xi) \to (P, \theta)$ such that $(g_1, \beta_1)(h, \gamma) = (g_1, \beta_1)(k_1, \delta_1)$ and $(g_2, \beta_2)(h, \gamma) = (g_2, \beta_2)(k_2, \delta_2)$...............................(5)

However from (3) and (4), it follows that $(h, \gamma) = (k_1, \delta_1)$ and $(h, \gamma) = (k_2, \delta_2)$ both satisfy (5). Therefore by uniqueness $(k_1, \delta_1) = (k_2, \delta_2)$

Thus (g_2, β_2) is a monomorphism.

Proof (ii): Let $(f_1, \alpha_1) : (X_1, \mu_1) \to (X, \mu)$ be a retraction. Then there exists a fuzzy group homomorphism $(k, \delta):(X, \mu) \to (X_1, \mu_1)$ such that $(f_1, \alpha_1)(k, \delta) = I_{(X, \mu)}$ .... (6) .

Now $(k, \delta)(f_2, \alpha_2):(X_2, \mu_2) \to (X_1, \mu_1)$ is such that

$$(f_1, \alpha_1)[(k, \delta)(f_2, \alpha_2)] = [(f_1, \alpha_1)(k, \delta)](f_2, \alpha_2)$$
$$= 1_{(X, \mu)}(f_2, \alpha_2)$$
$$= (f_2, \alpha_2)$$
$$= (f_2, \alpha_2) I_{(X_2, \mu_2)}$$

Since the given square is a pullback, there exists a unique fuzzy group homomorphism $(h, \gamma) : (X_2, \mu_2) \to (P, \theta)$ such that $(g_2, \beta_2)(h, \gamma) = I_{(X_2, \mu_2)}$.

Therefore (g_2, β_2) is a retraction with right inverse (h, γ)

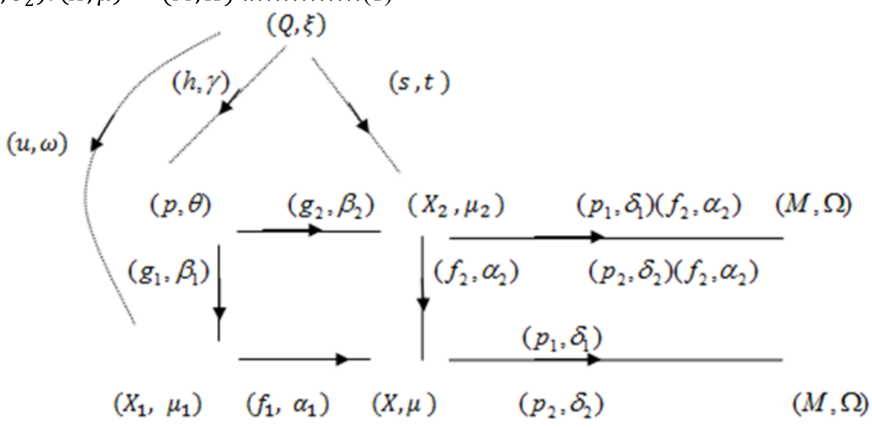
Proof(iii): (f_1, α_1) is an isomorphism $\Rightarrow (f_1, \alpha_1)$ is both a coretraction and retraction.

$\Rightarrow (f_1, \alpha_1)$ is a monomorphism and a retraction [since every coretraction is a monomorphism]

$\Rightarrow (g_2, \beta_2)$ is a monomorphism and a retraction by (i) & (ii)

$\Rightarrow (g_2, \beta_2)$ is an isomorphism

Proof (iv): Let (f_1, α_1) be an equalizer of (p_1, δ_1) and $(p_2, \delta_2): (X, \mu) \to (M, \Omega)$ ....(1)



Since the given square is a pullback, we have

$(f_1, \alpha_1)(g_1, \beta_1) = (f_2, \alpha_2)(g_2, \beta_2)$ .... (2)

From the definitions of an equalizer,

we have $(p_1, \delta_1)(f_1, \alpha_1) = (p_2, \delta_2)(f_1, \alpha_1)$ .... (3)

Claim. (g_2, β_2) is the equalizer for $(p_1, \delta_1)(f_2, \alpha_2)$ and $(p_2, \delta_2)(f_2, \alpha_2)$

By repeated application of the associative law, we have

$[(p_1,\delta_1)(f_2,\ \alpha_2)]\ (g_2,\beta_2) = (p_1,\delta_1)(f_1,\alpha_1)(g_1,\beta_1)$ by (2)
$= (p_2,\delta_2)(f_1,\alpha_1)(g_1,\beta_1)$ by (3)
$= [(p_2,\delta_2)(f_2,\ \alpha_2)]\ (g_2,\beta_2)$ by (2)……………… (A)

This shows condition (1) of the definition of an equalizer [4] is satisfied

Suppose there exists $(s,t)\colon (Q,\xi)\to(X_2,\mu_2)$ such that

$(p_1,\delta_1)(f_2,\alpha_2)(s,t)=(p_2,\delta_2)(f_2,\alpha_2)(s,t)$ ……………………… (4)

By the Universal Mapping Property of the equalizer [4] (f_1,α_1) there exists a unique fuzzy group homomorphism

$(u,\omega)\colon (Q,\xi)\to(X_1,\mu_1)$ such that $(f_1,\alpha_1)(u,\omega)=(f_2,\alpha_2)(s,t)$ ……………… (5)

Since the square is a pull back (5) implies that there exists a unique Fuzzy group homomorphism $(h,\gamma)\colon(Q,\xi)\to(P,\theta)$ such that $(g_2,\beta_2)(h,\gamma)=(s,t)$ ………………… (B)

From (A) or (B) we conclude that (g_2,β_2) is the equalizer for $(p_1,\delta_1)\ (f_2,\alpha_2)$ and $(p_2,\delta_2)\ (f_2,\alpha_2)$.

This completes the proof.

Remark 2.7: By symmetry it follows that if (f_2,α_2) is a monomorphism (retraction/isomorphism) then so is (g_1,β_1). The next proposition shows that monomorphisms can be characterized via pullbacks.

Proposition 2.8: A fuzzy group homomorphism $(f,\alpha)\colon(X,\mu)\to(Y,\eta)$ is a monomorphism if and only if the diagram below is a pullback diagram.

$$\begin{tikzcd}
(X,\mu) \arrow[r, "I_{(X,\mu)}"] \arrow[d, "I_{(X,\mu)}"'] & (X,\mu) \arrow[d, "(f,\alpha)"] \\
(X,\mu) \arrow[r, "(f,\alpha)"'] & (Y,\eta)
\end{tikzcd}$$

Proof. Let (f,α) be a monomorphism. Clearly condition (i) in the definition of pullback is true. Suppose there exists (g_1,β_1) and $(g_2,\beta_2)\colon(P,\theta)\to(X,\mu)$ such that $(f,\alpha)(g_1,\beta_1)=(f,\alpha)(g_2,\beta_2)$.

Then $(g_1,\beta_1)=(g_2,\beta_2)$[since (f,α) is a monomorphism]……..(1)

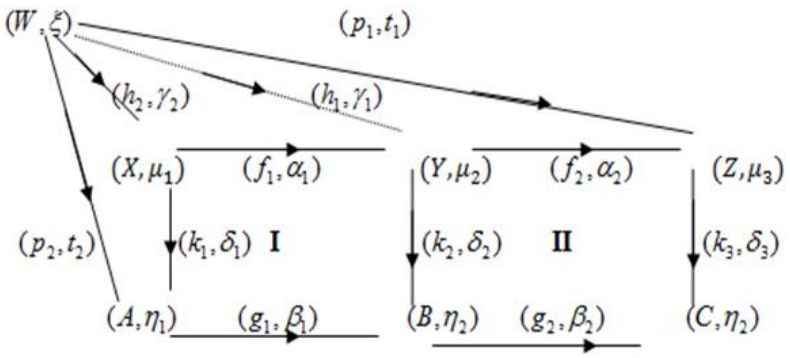
Thus we have $(g_1,\beta_1)=(g_2,\beta_2)\colon(P,\theta)\to(X,\mu)$ such that $I_{(X,\mu)}\ (g_1,\beta_1)=(g_1,\beta_1)$ and $I_{(X,\mu)}\ (g_2,\beta_2)=(g_2,\beta_2)$.

Moreover this (g_1,β_1) is unique, since $I_{(X,\mu)}$ = identity. Thus the above diagram is a pullback.

Conversely,

if the above diagram is a pullback and $(g_1,\beta_1),(g_2,\beta_2)\colon(P,\theta)\to(X,\mu)$ such that $(f,\alpha)(g_1,\beta_1)=(f,\alpha)(g_2,\beta_2)$. Then there exists a unique $(h,\gamma)\colon(P,\theta)\to(X,\mu)$ such that $1_{(X,\mu)}(h,\gamma)=(g_1,\beta_1)$ and $1_{(X,\mu)}(h,\gamma)=(g_2,\beta_2)$. Hence $(g_1,\beta_1)=(h,\gamma)=(g_2,\beta_2)$ so that (f,α) is a monomorphism.

Proposition 2.9: Consider the following commutative diagram, where the right rectangle II is a pullback.



Then the outer rectangle is a pullback if and only if the left rectangle I is a pullback.

Proof. Assume that I is a pullback. Now

$(k_3,\delta_3)(f_2,\alpha_2)(f_1,\alpha_1) = (g_2,\beta_2)(k_2,\delta_2)(f_1,\alpha_1)$ [since II is a pullback]

$=(g_2,\beta_2)(g_1,\beta_1)(k_1,\delta_1)$ [since I is a pullback]

and hence the outer rectangle is commutative.

Suppose there exists fuzzy group homomorphisms such that $(p_1,t_1):(W,\xi)\rightarrow(Z,\mu_3)$ and $(p_2,t_2):(W,\xi)\rightarrow(A,\eta_1)$ such that

$(k_3,\delta_3)(p_1,t_1)=(g_2,\beta_2)(g_1,\beta_1)(p_2,t_2)$ ....(1)

$= (g_2,\beta_2)[(g_1,\beta_1)(p_2,t_2)]$

Then there exists a unique $(h_1,\gamma_1):(W,\xi)\rightarrow(Y,\mu_2)$ such that $(f_2,\alpha_2)(h_1,\gamma_1)=(p_1,t_1)$ and $(k_2,\delta_2)(h_1,\gamma_1)=(g_1,\beta_1)(p_2,t_2)$ ....(2), Since rectangle II is a pullback.

Now rectangle I is a pullback, equation (2) implies that there exists a unique fuzzy group homomorphism $(h_2,\gamma_2):(W,\xi)\rightarrow(X,\mu_1)$ such that $(f_1,\alpha_1)(h_2,\gamma_2)=(h_1,\gamma_1)$ and $(k_1,\delta_1)(h_2,\gamma_2)=(p_2,t_2)$ ....(3)

Now $(f_2,\alpha_2)(f_1,\alpha_1)(h_2,\gamma_2)=(f_2,\alpha_2)(h_1,\gamma_1)$ [by (3)]

$= (p_1,t_1)$ [by (2)]

Therefore the outer rectangle is a pullback.

Conversely,

Let us assume that the outer rectangle is a pullback. By hypothesis the right hand triangle II is a pullback. We have to prove that the I is a pullback. By hypothesis

$(g_1,\beta_1)(k_1,\delta_1)= (k_2,\delta_2)(f_1,\alpha_1)$ ....(4)

Suppose there exists a fuzzy group homomorphism $(q_1,\psi_1):(M,\Omega)\rightarrow(Y,\mu_2)$ and $(q_2,\psi_2):(M,\Omega)\rightarrow(A,\eta_1)$ such that $(k_2,\delta_2)(q_1,\psi_1)=(g_1,\beta_1)(q_2,\psi_2)$ ....(5)

Then $(k_3,\delta_3)[(f_2,\alpha_2)(q_1,\psi_1)]= (g_2,\beta_2)(k_2,\delta_2)(q_1,\psi_1)$

[since II is a pullback]

$= (g_2,\beta_2)(g_1,\beta_1)(g_2,\psi_2)$ [by (5)]

$= [(g_2,\beta_2)(g_1,\beta_1)](g_2,\psi_2)$ ....(6)

Since outer rectangle is a pull back, there exists a unique fuzzy group homomorphism $(h,\gamma):(M,\Omega)\rightarrow(X,\mu)$ such that $(f_2,\alpha_2)(f_1,\alpha_1)(h,\gamma) = (f_2,\alpha_2)(q_1,\psi_1)$ ....(A)

$(k_1,\delta_1)(h,\gamma)=(q_2,\psi_2)$ ....(B)

Since the rectangle II is a pullback and both $(f_1,\alpha_1)(h,\gamma)$ and $(q_1,\psi_1):(M,\Omega)\rightarrow(Y,\eta)$ satisfy the condition $(f_2,\alpha_2)[(f_1,\alpha_1)(h,\gamma)]= (f_2,\alpha_2)(q_1,\psi_1)$ by the Universal Mapping Property we have $(f_1,\alpha_1)(h,\gamma)= (q_1,\psi_1)$ ....(C)

From (B) and (C) it follows that the rectangle I is also a pullback.

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