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Resolution of Birch and Swinnerton –Dyre Conjecture, with respect to the solutions of the equation $x^2 + y^2 = z^2$ in whole numbers

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Abstract: *A general method to find complete solution of the equation:* $\boldsymbol{x}^2 + \boldsymbol{y}^2 = \boldsymbol{z}^2$ *in whole numbers, in the form of formulae, with proof and resolution of Birch and Swinnerton-Dyre conjecture with respect to the solutions of the equation* $\boldsymbol{x}^2 + \boldsymbol{y}^2 = \boldsymbol{z}^2$ *in whole numbers,.*

Content

Details of Birch and Swinnerton –Dyre conjecture

Birch and Swinnerton –Dyre conjecture, one of the millennium seven problems established, announced and published by Clay Mathematics Institute of Cambridge Massachusetts (CMT) is as follows:

Mathematicians have always been fascinated by the problem of all solutions in whole numbers x,y,z to algebraic equation like $\boldsymbol{x}^2 + \boldsymbol{y}^2 = \boldsymbol{z}^2$.

Euclid gave the complete solution for that equation, but for more complicated equations this becomes extremely difficult. Indeed in 1970 Yu. V. Matiyasevich showed that Hilbert's tenth problem is unsolvable, i.e. there is no general method for determining when such equations have a solution in whole numbers. But in special cases one can hope to say something when the solutions are point of an abelian variety, the Birch and Swinnerton-Dyre conjecture asserts that the size of the group of rational points is related to the behavior of an associated zeta function ζ(s) is equal to zero, then there are infinite number of rational points (solutions), and conversely, if ζ (1) is not equal to 0, then there is only finite number of such points.

I. Introduction and the results

1. Formula for finding Non-trivial primitive solutions of the equation $x^2 + y^2 = z^2$ in whole numbers, when an odd number x≤ 3, x ∈N and $when\; x = 2n + 1, y = \frac{(2n+1)^2-1}{2}, and\; z = \frac{(2n+1)^2+1}{2},$ $then\; (2n +$ *12+ 2n+12−122= 2n+12+122for all* n∈N=*{1,2,3,4,5,6,……}*

OR

When an odd number x ≥ 3, x ∈N={1,2,3,4,5,6 … … }

$$then\;\; x^2 + \left(\frac{x^2 - 1}{2}\right)^2 = \left(\frac{x^2 + 1}{2}\right)^2$$

2. Formula for finding non trivial primitive solutions of the equation $x^2 + y^2 = z^2$ in whole numbers, when an even number x≥ 4, x ∈N={1,2,3,4,5,6 … …} and *when* x=2n+2, $y = (n+1)^2 - 1$, and $z = (n+1)^2 + 1$, $then\ (2n+2)^2 + [(n+1)^\wedge 2 - 1]^2 = [(n+1)^\wedge 2 + 1]^2\ for\ all\ n \in N = \{1,2,3,4,5,6, \dots\dots\}$

OR

When an even number x ≥ 2 ∈N then $(2x)^2 + (x^2 - 1)^2 = (x^2 + 1)^2$

3. Formula for finding derived solutions of the equation $\boldsymbol{x}^2 + \boldsymbol{y}^2 = \boldsymbol{z}^2$ from its non –trivial primitive solutions, in whole numbers, when an odd number x≥ 3, n ∈N={1,2,3,4,5,6, … …} and x=2n+1, $y = \frac{(2n+1)^2-1}{2}, and\ z = \frac{(2n+1)^2+1}{2}$ then $[(n(2n+1)]^2 + [n\left(\frac{(2n+1)^2-1}{2}\right)]^2 = [n\left(\frac{(2n+1)^2+1}{2}\right)]^2\ for\ all\ n \in N = \{1,2,3,4,5,6, \dots\dots\}$

OR

When an odd number x≥ 3, x ∈N={1,2,3,4,5,6 … …}, then $\left[(nx)^2 + \left(\frac{n(x^2-1)}{2}\right)\right]^2 = \left[\frac{n(x^2+1)}{2}\right]^2$

4. Formula for finding derived solutions of the equation $\boldsymbol{x}^2 + \boldsymbol{y}^2 = \boldsymbol{z}^2$ in whole numbers, from its non-trivial primitive solutions, when an even number x≥ 4, x∈N={1,2,3,4,5,6, … …} and x=2n+2, y=$(n+1)^2 - 1$,

And $z = (n+1)^2 + 1$

$$then\big(n(2n+2)\big)^2 + (\ (n((n+1)^2 - 1))^2 = (n((n+1)^2 + 1))^\wedge 2$$

When an even number x≥4, x∈N={1,2,3,4,5,6, … …}

$then\big(n(2x)\big)^2 + (\ (nx)^2 - 1)^2 = (\ (nx)^2 + 1)^2$ For all n ∈N={1,2,3,4,5,6 … …}

Theorem 1: (Non- trivial primitive solutions of the equation $\boldsymbol{x}^2 + \boldsymbol{y}^2 = \boldsymbol{z}^2$ in whole numbers, when an odd number x≥ 3, x ∈N={1,2,3,4,5,6 … …})

Let, $x = 2n + 1, y = \frac{(2n+1)^2-1}{2}, and\ z = \frac{(2n+1)^2+1}{2}$,

$$then\ (2n+1)^{2+}\left(\frac{(2n+1)^2-1}{2}\right)^2 = \left(\frac{(2n+1)^2+1}{2}\right)^2 for\ all\ n \in N = \{1,2,3,4,5,6, \dots\dots\}$$

Proof :(Using Mathematical Induction)

Let be P_n the statement

$$(2n+1)^{2+}\left(\frac{(2n+1)^2-1}{2}\right)^2 = \left(\frac{(2n+1)^2+1}{2}\right)^2$$

Basic step: Show P_1 is true.

For n=1

The left hand side of P_1 is $(2*1+1)^2 + \left[\frac{(2*1+1)^2}{2}\right]^2 = 3^2 + 4^2 = 5^2$

The right hand side of P_1 is

$\left(\frac{(2*1+1)^2+1}{2}\right)^2 = \left(\frac{(3)^2+1}{2}\right)^2 = \left(\frac{10}{2}\right)^2 = 5^2$

$\therefore 3^2 + 4^2 = 5^2$

$\therefore$ L.H.S = R.H.S.

Induction step:

Let k≥ be an integer and assume

P_k is true

That is assume that

$$(2k+1)^2 + \left(\frac{(2k+1)^2-1}{2}\right)^2 = \left(\frac{(2k+1)^2+1}{2}\right)^2 for\ all\ k \in N = \{1,2,3,4,5,6, \dots\dots\}$$

Required to show is true P_{k+1} i.e. that

$$(2(k+1)+1)^2 + \left(\frac{(2(k+1)+1)^2-1}{2}\right)^2 = \left(\frac{(2(k+1)+1)^2+1}{2}\right)^2 for\ all\ (k+1) \in N = \{1,2,3,4,5,6, \dots\dots\}$$

L.H.S= $(2(k+1)+1)^2 + \left(\frac{(2(k+1)+1)^2-1}{2}\right)^2$

Consider

$(2(k+1)+1)^2 = (2k+2+1) = (2k+3)^2 = 4k^2 + 12k + 9$ (1)

Consider

$\left(\frac{(2(k+1)+1)^2-1}{2}\right)^2 = \left(\frac{((2k+3))^2-1}{2}\right)^2 = \left(\frac{4k^2+12k+8}{2}\right)^2 = (2k^2+6k+4)^2 = (4k^4+36k^2+16+12k^3+48k+$ 16k2=4k4+12k3+52k2+16++48k+16 (2)

From (1) & (2)

L.H.S. = $(4k^4+12k^3+56k^2+60k+25)$ (3)

R.H.S. = $\left(\frac{(2(k+1)+1)^2+1}{2}\right)^2 = \left(\frac{((2k+3)^2+1)^2}{2}\right)^2 = \left(\frac{4k^2+12k+10}{2}\right)^2 = (2k^2+6k+5)^2$

$=4k^4+24k^3+56k^2+60k+25$ (4)

From (3) & (4) we have

L.H.S. =R.H.S.

Hence P_{k+1} is true is P_k is true

By principle of mathematical induction P_n is true for all n∈N={1,2,3,4,5,6, … …} Hence the proof. QED.

Theorem 2: (Non-trivial primitive solutions of the equation $\boldsymbol{x}^2+\boldsymbol{y}^2=\boldsymbol{z}^2$ in whole numbers, when an even number x≥ 4 where x ∈N={1,2,3,4,5,6, … …})

$When\ x=2n+2, y=(n+1)^2-1, and\ z=(n+1)^2+1$

$then\ (2n+2)^2+((n+1)^2-1)^2=((n+1)^2+1)^2$ for all n ∈ N = {1,2,3,4,5,6 … …}

Proof: (Using Mathematical Induction)

Let P_n be the statement

$(2n+2)^2+((n+1)^2-1)^2=((n+1)^2+1)^2$

Basic step: Show P_1 is true

For n=1

L.H.S. = $(2*1+2)^2+((1+1)^2-1)^2 = 3^2+4^2=5^2$ (1)

R.H.S. = $((1+1)^2+1)^2=5^2$ (2)

From (1) & (2) we have

L.H.S. =R.H.S.

Induction step:

Let k≥ 1 be an integer and assume

P_k is true. That is, assume that

$$(2k+2)^2+((k+1)^2-1)^2=((k+1)^2+1)^2$$

Required to show P_{k+1} is true i.e. that

$$(2(k+1)+2)^2+(((k+1)+1)^2-1)^2=(((k+1)+1)^2+1)^2$$

L.H.S. =$(2(k+1)+2)^2+(((k+1)+1)^2-1)^2$

Consider

$(2(k+1)+2)^2=\left((2k+2)+2\right)^2=4k^2+16k+16$ (3)

Consider

$((k+1)+1)^2-1)^2=((k+2)-1)^2=((k^2+4k+3)^2=k^4+8k^3+22k^2+24k+9$ (4)

From (3) & (4) we have

L.H.S. = $k^4+8k^3+26k^2+40k+25$ (5)

R.H.S. $=\left(\left((k+1)+1\right)^2+1\right)^2=\left(\left((k+2)\right)^2+1\right)^2=(k^2+4k+4+1)^2=k^4+8k^3+26k^2+40k+25$ (6)

From (5) & (6) we have

L.H.S. =R.H.S.

Hence $P_{k+1}$are true if is $P_k$true. By the principle of Mathematical Induction Pn is true for all n∈ N = {1,2,3,4,5,6, … …}. Hence the proof. **QED**.

Theorem 3: (Derived solutions of the equation $\boldsymbol{x}^2+\boldsymbol{y}^2=\boldsymbol{z}^2$ in whole numbers, from its non-trivial primitive solutions when an odd number x≥3, x∈N={1,2,3,4,5,6, … …})

When x=2n+1,$y=\frac{(2n+1)^2-1}{2}$, and $z=\frac{(2n+1)^2+1}{2}$

$$then\ (\mathrm{n}(2\mathrm{n}+1))^2+\left(n\left(\frac{(2n+1)^2-1}{2}\right)\right)^2=\left(n\left(\frac{(2n+1)^2+1}{2}\right)\right)^2$$ for all n ∈ N = {1,2,3,4,5,6 … …}

Proof: (Using Mathematical Induction)

Let $P_n$be statement

$$(\mathrm{n}(2\mathrm{n}+1))^2+\left(n\left(\frac{(2n+1)^2-1}{2}\right)\right)^2=\left(n\left(\frac{(2n+1)^2+1}{2}\right)\right)^2$$

Basic step: Show P_1 is true.
For n=1

L.H.S. = $(1(2*1+1))^2+\left(1\left(\frac{(2*1+1)^2-1}{2}\right)\right)^2=3^2+4^2=5^2$ (1)

R.H.S. =$\left(1\left(\frac{(2*1+1)^2+1}{2}\right)\right)^2=\left(1\left(\frac{10}{2}\right)\right)^2=5^2$ (2)

$\therefore L.H.S.=R.H.S.$
Induction step:
Let k≥1 be an integer and assume P_k true, that is, assume that

$$(\mathrm{k}(2\mathrm{k}+1))^2+\left(k\left(\frac{(2k+1)^2-1}{2}\right)\right)^2=\left(k\left(\frac{(2k+1)^2+1}{2}\right)\right)^2$$

Required to show Pk+1 is true i.e. that $((\mathrm{k}+1)(2(\mathrm{k}+1)+1))^2+\left((k+1)\left(\frac{(2(k+1)+1)^2-1}{2}\right)\right)^2=\left((k+1)\left(\frac{(2(k+1)+1)^2+1}{2}\right)\right)^2$

L.H.S. = $((\mathrm{k}+1)(2(\mathrm{k}+1)+1))^2+\left((k+1)\left(\frac{(2(k+1)+1)^2-1}{2}\right)\right)^2$

Consider
$((\mathrm{k}+1)(2(\mathrm{k}+1)+1))^2=(\mathrm{k}+1)^2*(2\mathrm{k}+3)^2=(\mathrm{k}^2+2\mathrm{k}+1)(4\mathrm{k}^2+12\mathrm{k}+9)=4\mathrm{k}^4+20\mathrm{k}^3+37\mathrm{k}^2+30\mathrm{k}+9$ (3)

Consider
$\left((k+1)\left(\frac{(2(k+1)+1)^2-1}{2}\right)\right)^2=((\mathrm{k}+1)^2*(2\mathrm{k}^2+6\mathrm{k}+4)^2=(\mathrm{k}^2+2\mathrm{k}+1)(4\mathrm{k}^4+36\mathrm{k}^2+16+24\mathrm{k}^3+48\mathrm{k}+16\mathrm{k}^2=4\mathrm{k}^6+32\mathrm{k}^5+104\mathrm{k}^4+176\mathrm{k}\text{^}3+164\mathrm{k}^2+80\mathrm{k}+16$ (4)

From (3) and (4) we have
L.H.S. = $4\mathrm{k}^6+32\mathrm{k}^5+108\mathrm{k}\text{^}4+196\mathrm{k}\text{^}3+201\mathrm{k}^2+110\mathrm{k}+25$ (5)

R.H.S. = $\left((k+1)\left(\frac{(2(k+1)+1)^2+1}{2}\right)\right)^2=(\mathrm{k}+1)^2*(2\mathrm{k}^2+6\mathrm{k}+5)^2=4\mathrm{k}^6+32\mathrm{k}^5+108\mathrm{k}^4+196\mathrm{k}^3+201\mathrm{k}^2+110\mathrm{k}+25$
(6)

From (5) & (6) we have
L.H.S.=R.H.S.
Hence P_{k+1} are true if P_k is true. By the principle of Mathematical Induction P_k is true for all n ∈N={1,2,3,4,5,6, ...} Hence the proof.

QED

Theorem 4 :(Derived solutions of the equation $\boldsymbol{x}^2+\boldsymbol{y}^2=\boldsymbol{z}^2$ in whole numbers, from its non-trivial primitive solutions when an even number x≥4, x∈ $N=\{1,2,3,4,5,6,\ldots\ldots\}$)

$$When\ x=2n+2, y=(n+1)^2-1, and\ z=(n+1)^2,$$
$$then\ (n(2n+2))^2+\left(n((n+1)^2-1)\right)^2=\left(n((n+1)^2+1)\right)^2\ for\ all\ n\in N$$

Proof :(Using Mathematical Induction)
Let P_n be the statement

$$(n(2n+2))^2+\left(n((n+1)^2-1)\right)^2=\left(n((n+1)^2+1)\right)^2$$

Basic step: Show P_1 is true.
For n=1
L.H.S =
$(n(2n+2))^2+\left(n((n+1)^2-1)\right)^2=\left(1(2*1+2)\right)^2+\left(1((1+1)^2-1)\right)^2=4^2+3^2=5^2$ (1}

R.H.S. = $\left(n((n+1)^2+1)\right)^2=(1((1+1)^2+1)^2=5^2$ (2)

∴ L.H.S. =R.H.S.
Induction step:
Let k≥1 be an integer and assume P_k is true that is that

$$\left(k(2k+2)\right)^2+\left(k((k+1)^2-1)\right)^2=\left(k((k+1)^2+1)\right)^2$$

Required to show is P_{k+1}true i.e. that

$$\left((k+1)(2(k+1)+2)\right)^2+\left((k+1)(((k+1)+1)^2-1)\right)^2=\left((k+1)(((k+1)+1)^2+1)\right)^2$$

L.H.S. = $\left((k+1)(2(k+1)+2)\right)^2+\left((k+1)(((k+1)+1)^2-1)\right)^2$

Consider

$((k+1)(2(k+1)+2))^2=(k+1)^2*(2k+4)^2=(k^2+2k+1)(4k^2+16k+16)=4k^4+24k^3+52k^2+16k+16$ (3)

Consider

$((k+1)(((k+1)+1)^2-1)^2=(k+1)^2*(k^2+4k+3)^2=(k^2+2k+1)*(k^4+16k^2+9+8k^3+24k+6k^2=k^6+8k^5+39k^4+76k^3+79k^2+42k+9$ (4)

Adding (3) & (4) we have

$L.H.S.=\ k^6+10k^5+43k^4+100k^3+131k^2+90k+25$ (5)

R.H.S. = $\left((k+1)(((k+1)+1)^2+1)\right)^2=((k+1)^2*(k^2+4k+5)^2=(k^2+2k+1)*(k^4+8k^3+$ *26k2+40k+25=k6+10k5+43k4+100k3+131k2+90k+25* (6)

From (5) & (6) we have

L.H.S= R.H.S.

Hence P_{k+1} are true. By the principle of Mathematical Induction P_n is true for all n ∈N={1,2,3,4,5,6,… … }

Hence the proof. **QED**.

Resolution of Birch and Swinnerton –Dyre Conjecture, with respect to the equation $x^2+y^2=z^2$

Birch and Swinnerton –Dyre conjecture

When the solutions are points of an abelian variety the conjecture asserts that the size of the group of rational points is related to the behavior of an associated zeta function $\zeta(s)$ near the point s=1.

In particular this amazing conjecture asserts that if ζ(s) is equal to 0, then there are an infinite number of rational points(solutions), and conversely, if ζ(1) is not equal to 0, then there is only a finite number of such points.

Case-1

(When non-trivial primitive solutions of the equation $\boldsymbol{x}^2+\boldsymbol{y}^2=\boldsymbol{z}^2$ in whole numbers and an odd number x≥ 3, x ∈N={1,2,3,4,5,6,… … })

$$when\ x=2n+1, y=\frac{(2n+1)^2-1}{2}, and\ z=\frac{(2n+1)^2+1}{2},$$

$$then\ (2n+1)^2+\left(\frac{(2n+1)^2-1}{2}\right)^2=\left(\frac{(2n+1)^2+1}{2}\right)^2 for\ all\ n\in \mathrm{N}$$

Consider the equation

$$(2n+1)^2+\left(\frac{(2n+1)^2-1}{2}\right)^2=\left(\frac{(2n+1)^2+1}{2}\right)^2$$

$$\therefore\ (2n+1)^2+\left(\frac{(2n+1)^2-1}{2}\right)^2-\left(\frac{(2n+1)^2+1}{2}\right)^2=0$$

Using this weight value a series can be developed such that each term of series converges to 0 having one-to-one correspondence between the weight value of the term and the solution of the equation $\boldsymbol{x}^2+\boldsymbol{y}^2=\boldsymbol{z}^2$ where each term represents one and only one districts solution of the equation. i.e.

$$\sum_{n=1}^{\infty} a_n=\sum_{n=1}^{\infty}\left[\ (2n+1)^2+\left(\frac{(2n+1)^2-1}{2}\right)^2-\left(\frac{(2n+1)^2+1}{2}\right)^2\right]=0+0+0+0+0+\mathrm{o}+\cdots$$

Associating this with ζ(1) where Riemann zeta function $\zeta(1)=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$

$$\zeta(1)=\sum_{n=1}^{\infty}\frac{a_n}{\mathrm{n}}=\sum_{n=1}^{\infty}\frac{\left[(2n+1)^2+\left(\frac{(2n+1)^2-1}{2}\right)^2-\left(\frac{(2n+1)^2+1}{2}\right)^2\right]}{n}$$

$$=\frac{0}{1}+\frac{0}{2}+\frac{0}{3}+\frac{0}{4}+\frac{0}{5}+\frac{0}{6}+\cdots$$

Partial sum of the n^{th} term $=\frac{\left[(2n+1)^2+\left(\frac{(2n+1)^2-1}{2}\right)^2-\left(\frac{(2n+1)^2+1}{2}\right)^2\right]}{n}=\frac{[(4n^4+8n^3+8n^2+4n+1]-[4n^4+8n^3+8n^2+4n+1]}{n}=\frac{0}{n}$

$\lim_{n\to\infty}\frac{0}{n}$=0=ζ (1)

∴ ζ(1)=0 Hence the series is a trivial infinite series of which each term has one-to one correspondence with the distinct non-trivial primitive solution of the equation $\boldsymbol{x}^2+\boldsymbol{y}^2=\boldsymbol{z}^2$. ∴ It proves that there are infinite distinct non-trivial primitive solutions of the equation $\boldsymbol{x}^2+\boldsymbol{y}^2=\boldsymbol{z}^2$, in whole numbers when an odd number x≥ 3 , x ∈N={1,2,3,4,5,6,… … })

Conversely if ζ (1)≠ 0

Suppose ζ (1)≠ 0

Hence $\frac{\left[(2n+1)^2+\left(\frac{(2n+1)^2-1}{2}\right)^2-\left(\frac{(2n+1)^2+1}{2}\right)^2\right]}{n} \neq 0$ where n≠ 0

$$\therefore \left[\ (2n+1)^{2+}\left(\frac{(2n+1)^2-1}{2}\right)^2-\left(\frac{(2n+1)^2+1}{2}\right)^2 \neq 0\right]$$

$$\therefore\ (2n+1)^2+\left(\frac{(2n+1)^2-1}{2}\right)^2 \neq \left(\frac{(2n+1)^2+1}{2}\right)^2$$

A contradiction to the formula, for finding non -trivial primitive solutions of the equation $\boldsymbol{x}^2+\boldsymbol{y}^2=\boldsymbol{z}^2$ in whole numbers, when an odd number x≥ 3, x ∈N, i.e.

$$when\ x=2n+1, y=\frac{(2n+1)^2-1}{2}, and\ z=\frac{(2n+1)^2+1}{2},$$

$$then\ \ (2n+1)^{2+}\left(\frac{(2n+1)^2-1}{2}\right)^2=\left(\frac{(2n+1)^2+1}{2}\right)^2 for\ all\ n \in \mathrm{N}$$

And to our developed series: $\sum_{n=1}^{\infty} a_n$=$\sum_{n=1}^{\infty}\left[\ (2n+1)^{2+}\left(\frac{(2n+1)^2-1}{2}\right)^2-\left(\frac{(2n+1)^2+1}{2}\right)^2\right]$=$0+0+0+0+0+$ $\mathrm{o}+\cdots$

So at the most, we can say that if ζ (1)≠ 0 then there is a finite number of solutions, if we consider 0 solutions as finite.

Hence, the Birch and Swinnerton-Dyre conjecture holds true for case 1.

QED

Case 2: (When non-trivial primitive solutions of the equation $\boldsymbol{x}^2+\boldsymbol{y}^2=\boldsymbol{z}^2$ in whole numbers and an odd number x≥ 3, x∈N={1,2,3,4,5,6, ... })

Consider the formula

$$\text{When } x=2n+2, \mathrm{y}=(\mathrm{n}+1)^2-1, \text{and } \mathrm{z}=(\mathrm{n}+1)^2+1$$
$$\text{then } (2\mathrm{n}+2)^2+((\mathrm{n}+1)^2-1)^2=((\mathrm{n}+1)^2+1)^2 \text{ for all } \mathrm{n} \in \mathrm{N}$$

Take

$$(2\mathrm{n}+2)^2+((\mathrm{n}+1)^2-1)^2=((\mathrm{n}+1)^2+1)^2$$
$$\therefore \qquad (2\mathrm{n}+2)^2+((\mathrm{n}+1)^2-1)^2-((\mathrm{n}+1)^2+1)^2=\mathbf{0}$$

$\mathrm{L.H.S}=\ (2\mathrm{n}+2)^2+((\mathrm{n}+1)^2-1)^2-((\mathrm{n}+1)^2+1)^2=(\mathrm{n}^4+4\mathrm{n}^3+8\mathrm{n}^2+8\mathrm{n}+4)-(\mathrm{n}^4+4\mathrm{n}^3+$ $8\mathrm{n}^2+8\mathrm{n}+4=0$

$$\therefore \mathrm{L.H.S.}=\mathrm{R.H.S}$$

$$\therefore \qquad (2\mathrm{n}+2)^2+((\mathrm{n}+1)^2-1)^2-((\mathrm{n}+1)^2+1)^2=\mathbf{0} \qquad (1)$$

Using this weight value a series can be developed such that each term of series converges to 0 having one-to-one correspondence between the weight value of the term and each solution of the equation $\boldsymbol{x}^2+\boldsymbol{y}^2=\boldsymbol{z}^2$ where each term represents one and only one district solution of the equation. i.e.

$\sum_{n=1}^{\infty} a_n$=$\sum_{n=1}^{\infty}((2n+2)^2+((n+1)^2-1)^2)-((n+1)^2+1)^2=0+0+0+0+\cdots$

Associating $\zeta(1)$ with$\sum_{n=1}^{\infty} an$), where Riemann zeta function ζ(1)=$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$

We get

ζ(1)= $\sum_{n=1}^{\infty}\frac{a_n}{\mathrm{n}}=\sum_{n=1}^{\infty}\frac{\left((2n+2)^2+\left((n+1)^2-1\right)^2\right)-\left((n+1)^2+1\right)^2}{n}=\frac{0}{1}+\frac{0}{2}+\frac{0}{3}+\frac{0}{4}+\frac{0}{5}+\cdots$

Partial sum of $\boldsymbol{n}^{th}$ terms

$=\frac{\left((2n+2)^2+\left((n+1)^2-1\right)^2\right)-\left((n+1)^2+1\right)^2}{n}=\frac{(\mathrm{n}^4+4\mathrm{n}^3+8\mathrm{n}^2+8\mathrm{n}+4)-(\mathrm{n}^4+4\mathrm{n}^3+8\mathrm{n}^2+8\mathrm{n}+4}{n}=\frac{0}{n}$

$\lim_{n\to\infty}\frac{0}{n}=0$=$\zeta(1)$

$\therefore \qquad \zeta(1)=0$

1. Hence the series is a trivial infinite series of which each term has one-to one correspondence with the distinct non-trivial primitive solution of the equation $\boldsymbol{x}^2+\boldsymbol{y}^2=\boldsymbol{z}^2$. ∴ It proves that there are infinite distinct non-trivial primitive solutions of the equation $\boldsymbol{x}^2+\boldsymbol{y}^2=\boldsymbol{z}^2$, in whole numbers when an even number x≥ 4 , x∈N= {1,2,3,4,5,6, ... })

Conversely if ζ (1)≠ 0

Suppose ζ (1)≠ 0

$Hence\ \frac{\left((2n+2)^2+\left((n+1)^2-1\right)^2\right)-\left((n+1)^2+1\right)^2}{n}\neq 0$ Where n≠ 0

$$\therefore ((2n+2)^2 + ((n+1)^2 - 1)^2) - ((n+1)^2 + 1)^2 \neq 0$$
$$\therefore ((2n+2)^2 + ((n+1)^2 - 1)^2) \neq ((n+1)^2 + 1)^2$$

2. A contradiction to the formula, for finding, non-trivial primitive solutions of the equation $\boldsymbol{x}^2 + \boldsymbol{y}^2 = \boldsymbol{z}^2$ in whole numbers, when an even number x≥ 4, x ∈N= {1,2,3,4,5,6, … … }, i.e.

$$\text{When } x = 2n+2, \text{y} = (\text{n}+1)^2 - 1, \text{and z} = (\text{n}+1)^2 + 1$$
$$\text{then } (2\text{n}+2)^2 + ((\text{n}+1)^2 - 1)^2 = ((\text{n}+1)^2 + 1)^2 \text{ for all n} \in \text{N} = \{1,2,3,4,5,6, \dots\dots\}$$

and to our developed series i.e.
$\sum_{n=1}^{\infty} a\text{n}=\sum_{n=1}^{\infty}((2n+2)^2 + ((n+1)^2 - 1)^2) - ((n+1)^2 + 1)^2 = 0 + 0 + 0 + 0 + \cdots$
So at the most, we can say that if ζ (1)≠ 0 then there is a finite number of solutions, if we consider 0 solutions as finite.
Hence, the Birch and Swinnerton-Dyre conjecture holds true for case 2.

QED
Case 3:
When derived solutions of the equation $x^2 + y^2 = \text{z}^2$ in whole numbers, from its non-trivial primitive solutions, when an odd number x≥ 3 , x∈ $N = \{1,2,3,4,5,6, \dots\dots\}$.
Consider the formula

$$when\, x = 2n+1, y = \frac{(2n+1)^2 - 1}{2}, and\, z = \frac{(2n+1)^2 + 1}{2},$$
$$then\ \ (n(2n+1))^{2+}\left(n\left(\frac{(2n+1)^2 - 1}{2}\right)\right)^2 = \left(n\left(\frac{(2n+1)^2 + 1}{2}\right)\right)^2 for\ all\ n \in \text{N} = \{1,2,3,4,5,6 \dots\dots\}$$

Consider
$$(n(2n+1))^{2+}\left(n\left(\frac{(2n+1)^2 - 1}{2}\right)\right)^2 = \left(n\left(\frac{(2n+1)^2 + 1}{2}\right)\right)^2$$

$$\therefore\ \ (n(2n+1))^{2+}\left(n\left(\frac{(2n+1)^2 - 1}{2}\right)\right)^2 - \left(n\left(\frac{(2n+1)^2 + 1}{2}\right)\right)^2 = 0$$

L.H.S= $(n(2n+1))^{2+}\left(n\left(\frac{(2n+1)^2-1}{2}\right)\right)^2 - \left(n\left(\frac{(2n+1)^2+1}{2}\right)\right)^2 = (4n^6 + 8n^5 + 8n^4 + 4n^3 + n^2) -$
$(4n^6 + 8n^5 + 8n^4 + 4n^3 + n^2) = 0$

$$\therefore L.H.S = R.H.S$$
$$\therefore\ \ (n(2n+1))^{2+}\left(n\left(\frac{(2n+1)^2 - 1}{2}\right)\right)^2 - \left(n\left(\frac{(2n+1)^2 + 1}{2}\right)\right)^2 = 0$$

∴ Using this weight value a series can be developed such that each term of series converges to 0 having one-to-one correspondence between the weight value of the term and the solution of the equation $\boldsymbol{x}^2 + \boldsymbol{y}^2 = \boldsymbol{z}^2$ where each term represents one and only one districts solution of the equation. i.e.
$\sum_{n=1}^{\infty} a_n=\sum_{n=1}^{\infty}\left[\ (n(2n+1))^2 + \left(n\left(\frac{(2n+1)^2-1}{2}\right)\right)^2 - \left(n\left(\frac{(2n+1)^2+1}{2}\right)\right)^2\right]$=$0 + 0 + 0 + 0 + 0 + \text{o} + \cdots$
Associating this with ζ(1) where Riemann zeta function ζ(1)=$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$

ζ(1)= $\sum_{n=1}^{\infty} \frac{a_n}{\text{n}} = \sum_{n=1}^{\infty} \frac{\left[(n(2n+1))^{2+}\left(\frac{(n(2n+1))^2-1}{2}\right)^2 - \left(\frac{(n(2n+1))^2+1}{2}\right)^2\right]}{n}$=$\frac{0}{1} + \frac{0}{2} + \frac{0}{3} + \frac{0}{4} + \frac{0}{5} + \frac{0}{6} + \cdots$

Partial sum of the n^th^ term = $\frac{\left[(n(2n+1))^2 + \left(n\left(\frac{(2n+1)^2-1}{2}\right)\right)^2 - \left(n\left(\frac{(2n+1)^2+1)}{2}\right)^2\right]}{n} = \frac{[(4n^6+8n^5+8n^4+4n^3+n^2]-[4n^6+8n^5+8n^4+4n^3+n^2]}{n} = \frac{0}{n}$

$\lim_{n\to\infty} \frac{0}{n}$=0=ζ (1)
∴ ζ(1)=0 Hence the series is a trivial infinite series of which each term has one-to one correspondence with the distinct non-trivial primitive solution of the equation $\boldsymbol{x}^2 + \boldsymbol{y}^2 = \boldsymbol{z}^2$. ∴ It proves that there are infinite derived

solutions from each of distinct non-trivial primitive solutions of the equation $\boldsymbol{x}^2+\boldsymbol{y}^2=\boldsymbol{z}^2$, in whole numbers when an odd number x≥ 3 , x ∈N= {1,2,3,4,5,6, … … })
Conversely if ζ (1)≠ 0
Suppose ζ (1)≠ 0

Hence

$$\frac{\left[\left(n(2n+1)\right)^2+\left(n\left(\frac{(2n+1)^2-1}{2}\right)\right)^2-\left(n(\frac{(2n+1)^2+1)}{2}\right)^2\right]}{n}\neq 0 \quad \text{Where } n\neq 0$$

$$\therefore\ \left(n(2n+1)\right)^2+\left(n\left(\frac{(2n+1)^2-1}{2}\right)\right)^2-\left(n(\frac{(2n+1)^2+1)}{2}\right)^2\neq 0$$

$$\therefore\ \left(n(2n+1)\right)^2+\left(n\left(\frac{(2n+1)^2-1}{2}\right)\right)^2\neq\left(n(\frac{(2n+1)^2+1}{2})\right)^2$$

A contradiction to the formula, for finding derived solutions of the equation $x^2+y^2=\mathrm{z}^2$ in whole numbers, from its non-trivial primitive solutions, when an odd number x≥ 3 , x∈ N. i. e.

$$when\, x=2n+1, y=\frac{(2n+1)^2-1}{2}, and\ z=\frac{(2n+1)^2+1}{2},$$

$$then\ \ (n(2n+1))^{2+}\left(n\left(\frac{(2n+1)^2-1}{2}\right)\right)^2=\left(n\left(\frac{(2n+1)^2+1}{2}\right)\right)^2 for\ all\ n\in \mathrm{N}=\{1,2,3,4,5,6,\ldots\ldots\}$$

and to our developed series i.e.

$$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\left[\ \left(n(2n+1)\right)^2+\left(n\left(\frac{(2n+1)^2-1}{2}\right)\right)^2-\left(n\left(\frac{(2n+1)^2+1}{2}\right)\right)^2\right]=0+0+0+0+0+\mathrm{o}+\cdots$$

So at the most, we can say that if ζ (1)≠ 0 then there is a finite number of solutions, if we consider 0 solutions as finite.
Hence, the Birch and Swinnerton-Dyre conjecture holds true for case 3.

Case 4: (When derived solutions from non-trivial primitive solutions of the equation $\boldsymbol{x}^2+\boldsymbol{y}^2=\boldsymbol{z}^2$ in whole numbers when an odd number x≥ 3, x∈N={1,2,3,4,5,6, … … })
Consider the formula

$$\text{When } x=n(2n+2), \mathrm{y}=(\mathrm{n}(\mathrm{n}+1))^2-1, \text{and } \mathrm{z}=(\mathrm{n}(\mathrm{n}+1))^2+1$$

$$\text{then}(\mathrm{n}\ (2\mathrm{n}+2))^2+(\mathrm{n}((\mathrm{n}+1)^2-1))^2=(\mathrm{n}((\mathrm{n}+1)^2+1))^2 \text{ for all } \mathrm{n}\in \mathrm{N}=\{1,2,3,4,5,6\ldots\ldots\}$$

Take

$$(\mathrm{n}(2\mathrm{n}+2))^2+(\mathrm{n}((\mathrm{n}+1)^2-1))^2=(\mathrm{n}((\mathrm{n}+1)^2+1))^2$$

$$\therefore\qquad (\mathrm{n}\ (2\mathrm{n}+2))^2+(\mathrm{n}((\mathrm{n}+1)^2-1))^2-(\mathrm{n}((\mathrm{n}+1)^2+1))^2=\mathbf{0}$$

$\mathrm{L.H.S}=(\mathrm{n}\ (2\mathrm{n}+2))^2+(\mathrm{n}((\mathrm{n}+1)^2-1))^2-(\mathrm{n}((\mathrm{n}+1)^2+1))^2=(\mathrm{n}^6+4\mathrm{n}^5+8\mathrm{n}^4+8\mathrm{n}^3+4\mathrm{n}^2)-(\mathrm{n}^6+4\mathrm{n}^5+8\mathrm{n}^4+8\mathrm{n}\hat{}3+4\mathrm{n}\hat{}2=0$

$$\therefore \mathrm{L.H.S.}=\mathrm{R.H.S}$$

$$\therefore\qquad (\mathrm{n}\ (2\mathrm{n}+2))^2+(\mathrm{n}((\mathrm{n}+1)^2-1))^2-(\mathrm{n}((\mathrm{n}+1)^2+1))^2=\mathbf{0}$$
(1)

Using this weight value a series can is developed such that each term of series converges to 0 having one-to-one correspondence between the weight value of the term and each solution of the equation $\boldsymbol{x}^2+\boldsymbol{y}^2=\boldsymbol{z}^2$ where each term represents one and only one district solution of the equation. i. e.

$$\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}\left(\left(n(2n+2)\right)^2+\left(n((n+1)^2-1)\right)^2\right)-(n((n+1)^2+1))^2=0+0+0+0+\cdots$$

Associating $\zeta(1)$ with$\sum_{n=1}^{\infty}an$), where Riemann zeta function ζ(1)=1 $+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$
We get

$$\zeta(1)=\sum_{n=1}^{\infty}\frac{a_n}{\mathrm{n}}=\sum_{n=1}^{\infty}\frac{\left((n(2n+2))^2+\left(n\left((n+1)^2-1\right)\right)^2\right)-\left(n\left((n+1)^2+1\right)\right)^2}{n}=\frac{0}{1}+\frac{0}{2}+\frac{0}{3}+\frac{0}{4}+\frac{0}{5}+\cdots$$

Partial sum of $\boldsymbol{n}^{th}$ term

$$=\frac{\left((n(2n+2))^2+\left(n\left((n+1)^2-1\right)\right)^2\right)-\left(n\left((n+1)^2+1\right)\right)^2}{n}=\frac{(\mathrm{n}^6+4\mathrm{n}^5+8\mathrm{n}^4+8\mathrm{n}^3+4^2)-(\mathrm{n}^6+4\mathrm{n}^5+8\mathrm{n}^4+8\mathrm{nn}^3+4\mathrm{n}^2}{n}=\frac{0}{n}$$

$\lim_{n\to\infty}\frac{0}{n}=0$=$\zeta(1)$

$\therefore\qquad \zeta(1)=0$

Hence the series is a trivial infinite series of which each term has one-to one correspondence with the distinct derived solutions from non-trivial primitive solution of the equation $\boldsymbol{x}^2 + \boldsymbol{y}^2 = \boldsymbol{z}^2$. ∴ It proves that there are infinite distinct derived solutions from non-trivial primitive solutions of the equation $\boldsymbol{x}^2 + \boldsymbol{y}^2 = \boldsymbol{z}^2$, in whole numbers when an even number x≥ 4 , x∈N={1,2,3,4,5,6, … … })

Conversely if ζ (1)≠ 0

Suppose ζ (1)≠ 0

Hence

$$\frac{\left((n(2n+2))^2+\left(n((n+1)^2-1)\right)^2\right)-\left(n((n+1)^2+1)\right)^2}{n} \neq 0\,, where\; n \neq 0$$

$$\therefore (\mathrm{n}\ (2\mathrm{n}+2))^2 + (\mathrm{n}((\mathrm{n}+1)^2-1))^2 - (\mathrm{n}((\mathrm{n}+1)^2+1))^2 \neq 0$$

$$\therefore (\mathrm{n}\ (2\mathrm{n}+2))^2 + (\mathrm{n}((\mathrm{n}+1)^2-1))^2 \neq (\mathrm{n}((\mathrm{n}+1)^2+1))^2$$

A contradiction to the formula, for finding derived solutions of the equation $x^2 + y^2 = \mathrm{z}^2$ in whole numbers, from its non-trivial primitive solutions, when an even number x≥ 4 , x∈ $N = \{1,2,3,4,5,6, \dots \dots\}$. i. e.

$$\text{When } x = n(2n+2), \mathrm{y} = (\mathrm{n}(\mathrm{n}+1))^2 - 1, \text{and } \mathrm{z} = (\mathrm{n}(\mathrm{n}+1))^2 + 1$$

$$\text{then} (\mathrm{n}\ (2\mathrm{n}+2))^2 + (\mathrm{n}((\mathrm{n}+1)^2-1))^2 = (\mathrm{n}((\mathrm{n}+1)^2+1))^2 \text{ for all n} \in \mathrm{N}$$

and to our developed series i.e.

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\left(n(2n+2)\right)^2 + \left(n((n+1)^2-1)\right)^2\right) - \left(n((n+1)^2+1)\right)^2 = 0+0+0+0+\cdots$$

So at the most, we can say that if ζ (1)≠ 0 then there is a finite number of solutions, if we consider 0 solutions as finite.

Hence, the Birch and Swinnerton-Dyre conjecture holds true for case 4. **QED**

Proofs of cases 1-4 resolves the Birch and Swinnerton-Dyre conjecture with respect to the equation $\boldsymbol{x}^2 + \boldsymbol{y}^2 = \boldsymbol{z}^2$ whose solutions are in whole numbers, i.e. if ζ(1)=0, then there are infinite number of solutions to the equation $\boldsymbol{x}^2 + \boldsymbol{y}^2 = \boldsymbol{z}^2$ in whole numbers and conversely if ζ(1)≠ 0, there are finite number of solution to the equation $\boldsymbol{x}^2 + \boldsymbol{y}^2 = \boldsymbol{z}^2$in whole numbers. So the Birch and Swinnerton –Dyre conjecture holds true with respect to the equation $\boldsymbol{x}^2 + \boldsymbol{y}^2 = \boldsymbol{z}^2$.

SOLVED EXAMPLES

Formula for finding Non-trivial primitive solutions of the equation x^2+y^2=x^2 in whole numbers, when an odd number x≤ 3, x ∈N= {1,2,3,4,5,6, … … } and

$$when\; x = 2n+1, y = \frac{(2n+1)^2-1}{2}, and\; z = \frac{(2n+1)^2+1}{2}, \quad then\; (2n+1)^{2+}\left(\frac{(2n+1)^2-1}{2}\right)^2 = \left(\frac{(2n+1)^2+1}{2}\right)^2 for\; all\; n \in \mathrm{N} = \{1,2,3,4,5,6 \dots \dots\}$$

Examples

- When n=1 we have x=2*1+1=3, y=$\frac{(2*1+1)^2-1}{2}=\frac{8}{2}$=4, $z = \frac{(2*1+1)^2+1}{2} = \frac{10}{2} = 5$

$$\therefore 3^2 + 4^2 = 5^2$$

When n=2 we have x=2*2+1=5,$y = \frac{(2*2+1)^2-1}{2} = \frac{24}{2} = 12, z = \frac{(2*2+1)^2+1}{2} = \frac{26}{2} = 13$

$$\therefore 5^2 + 12^2 = 13^2$$

- When n=3 we have x=2*3+1=7, $y = \frac{(2*3+1)^2-1}{2} = \frac{48}{2} = 24, and\; z = \frac{(2*3+1)^2+1}{2} = \frac{50}{2} = 25$

$$\therefore 7^2 + 24^2 = 25^2$$

$When\; n = 4\; we\; have\; x = 2*4+1 = 9, y = \frac{(2*4+1)^2-1}{2} = \frac{80}{2} = 40, and\; z = \frac{(2*4+1)^2+1}{2} = \frac{82}{2} = 41$

$$\therefore 9^2 + 40^2 = 41^2$$

$When\; x = 5\; we\; have\; x = 2*5+1 = 11, y = \frac{(2*5+1)^2-1}{2} = \frac{120}{2} = 60, and\; z = \frac{(2*5+1)^2+1}{2} = \frac{122}{2} = 61$

$$\therefore \quad 11^2 + 60^2 = 61^2$$

OR

When an odd number x ≥ 3, x ∈N= {1,2,3,4,5,6, … … }

$$then\; x^2 + \left(\frac{x^2-1}{2}\right)^2 = \left(\frac{x^2+1}{2}\right)^2$$

Examples

When x=3,

$$then\; 3^2 + \left(\frac{3^2-1}{2}\right)^2 = \left(\frac{3^2+1}{2}\right)^2$$

$$\therefore 3^2 + 4^2 = 5\wedge 2$$

$When\ x = 5\ we\ have\ then\ \ 5^2 + \left(\frac{5^2-1}{2}\right)^2 = \left(\frac{5^2+1}{2}\right)^2$

$$\therefore 5^2 + 12^2 = 13\wedge 2$$

$When\ x = 7$ We have

$$then\ \ 7^2 + \left(\frac{7^2-1}{2}\right)^2 = \left(\frac{7^2+1}{2}\right)^2$$

$$\therefore 7^2 + 24^2 = 25^2$$

When x=9 we have

$$then\ \ 9^2 + \left(\frac{9^2-1}{2}\right)^2 = \left(\frac{9^2+1}{2}\right)^2$$

$$\therefore\ 9^2 + 40^2 = 41^2$$

$When\ x = 11\ we\ have\ then\ \ 11^2 + \left(\frac{11^2-1}{2}\right)^2 = \left(\frac{11^2+1}{2}\right)^2$

$$\therefore 11^2 + 60^2 = 61^2$$

Formula for finding non trivial primitive solutions of the equation $x^2 + y^2 = z^2$ in whole numbers, when an even number x≥ 4, x ∈N and *when* x=2n+2, $y = (n+1)^2 - 1$, and $z = (n+1)^2 + 1$, $then\ (2n+2)^2 + [(n+1)\wedge 2 - 1]^2 = [(n+1)\wedge 2 + 1]^2\ for\ all\ n \in N = \{1,2,3,4,5,6, \ldots \ldots\}$

Examples

When n=1, then we have $x = 2 * 1 + 2 = 4, y = (1+1)^2 - 1 = 3$, and $z = (1+1)^2 + 1 = 5$

$$\therefore 4^2 + 3^2 = 5^2$$

$When\ n = 2, then\ we\ have\ x = 2 * 2 + 2 = 6,\quad y = (2+1)^2 - 1 = 8$, and $z = (2+1)^2 + 1 = 10$

$$\therefore\ 6^2 + 8^2 = 10^2$$

$$When\ n = 3, then\ we\ have\ x = (2*3+2) = 8, y = ((3+1)^2 - 1) = 15, and\ z = ((3+1)^2 + 1) = 17$$

$$\therefore 8^2 + 15^2 = 17^2$$

$$When\ n = 4, then\ we\ have\ x = (2*4+2) = 10, y = ((4+1)^2 - 1) = 24, and\ z = ((4+1)^2 + 1) = 26$$

$$\therefore 10^2 + 24^2 = 26^2$$

$$When\ n = 5, then\ we\ have\ x = (2*5+2) = 12, y = ((5+1)^2 - 1) = 35, and\ z = ((5+1)^2 + 1) = 37$$

$$\therefore 12^2 + 35^2 = 37^2$$

OR

When an even number x ≥ 4 ∈N then $(2x)^2 + (x^2 - 1)^2 = (x^2 + 1)^2$

When x=4, then $(2*4)^2 + (4^2 - 1)^2 = (4^2 + 1)^2$

$$\therefore 8^2 + 15^2 = 17^2$$

$$When\ x = 6, then\ (2*6)^2 + (6^2 - 1)^2 = (6^2 + 1)^2$$

$$\therefore 12^2 + 35^2 = 37^2$$

$$When\ x = 8, then\ (2*8)^2 + (8^2 - 1)^2 = (8^2 + 1)^2$$

$$\therefore 16^2 + 63^2 = 65^2$$

$$When\ x = 10, then\ (2*10)^2 + (10^2 - 1)^2 = (10^2 + 1)^2$$

$$\therefore 20^2 + 99^2 = 101^2$$

Formula for finding derived solutions of the equation $\boldsymbol{x}^2 + \boldsymbol{y}^2 = \boldsymbol{z}^2$ from its non –trivial primitive solutions, in whole numbers, when an odd number x≥ 3, n ∈N={1,2,3,4,5,6, … …} and x=2n+1,$y = \frac{(2n+1)^2-1}{2}, and\ z = \frac{(2n+1)^2+1}{2}$ then $[(n(2n+1)]^2 + [n\left(\frac{(2n+1)^2-1}{2}\right)]^2 = [n\left(\frac{(2n+1)^2+1}{2}\right)]^2\ for\ all\ n \in N = \{1,2,3,4,5,6 \ldots \ldots\}$

When n=1, then

x=2*1+1=3, $y = \frac{(2*1+1)^2-1}{2} = 4, and\ z = \frac{(2*1+1)^2+1}{2} = 5$

$$\text{then } [(1(2*1+1)]^2 + [1\left(\frac{(2*1+1)^2-1}{2}\right)]^2 = [1\left(\frac{(2*1+1)^2+1}{2}\right)]^2$$

$$\therefore (1*3)^2 + (1*4)^2 = (1*5)^2$$

$$\therefore 3^2 + 4^2 = 5^2$$

When n=2, then x=2*2+1=5, $y = \frac{(2*2+1)^2-1}{2} = 12, and\ z = \frac{(2*2+1)^2+1}{2} = 13$

then

$$[(2(2*2+1)]^2 + [2\left(\frac{(2*2+1)^2-1}{2}\right)]^2 = [2\left(\frac{(2*2+1)^2+1}{2}\right)]^2$$

$$\therefore (2*5)^2 + (2*12)^2 = (2*13)^2$$
$$\therefore (10)^2 + (24)^2 = (26)^2$$
$When\, n = 3, then$ x=2*3+1=7, $y = \frac{(2*3+1)^2-1}{2} = 24, and$ z $= \frac{(2*3+1)^2+1}{2} = 25$
$$then(3*7)^2 + (3*24)^2 = (3*25)\text{^}2$$
$$\therefore 21^2 + 72^2 = 75^2$$
$When\, n = 4, then$ x=2*4+1=9, $y = \frac{(2*4+1)^2-1}{2} = 40, and$ z $= \frac{(2*4+1)^2+1}{2} = 41$
$$then\,(4*9)^2 + (4*40)^2 = (4*41)^2$$
$$\therefore 36^2 + 160^2 = 164^2$$
$$When\, n = 5, then\, x = 2*5+1 = 11, y = \frac{((2*5+1)^2-1)}{2} = 60, and\, z = \frac{(2*5+1)^2+1}{2} = 61$$
$$then\,(5*11)^2 + (5*60)^2 = (5*61)^2$$
$$\therefore 55^2 + 300^2 = 305^2$$

And so on

Take any non-trivial solution of the equation $\boldsymbol{x}^2 + \boldsymbol{y}^2 = \boldsymbol{z}^2$ in whole numbers and multiply both sides of the solution by n$\in N = \{1,2,3,4,5,6, \dots\dots\}$

Example-1
$$\therefore 3^2 + 4^2 = 5^2$$
When n=1, we have
$$\therefore (1*3)^2 + (1*4)^2 = (1*5)^2$$
$$\therefore 3^2 + 4^2 = 5^2$$
When n=2, we have
$$\therefore (2*3)^2 + (2*4)^2 = (2*5)^2$$
$$\therefore 6^2 + 8^2 = 10^2$$
When n=3, we have
$$\therefore (3*3)^2 + (3*4)^2 = (3*5)^2$$
$$\therefore 9^2 + 12^2 = 15^2$$
$$When\, n = 4, we\ have\ (4*3)^2 + (4*4)^2 = (4*5)^2$$
$$\therefore 12^2 + 16^2 = 20^2$$
$$When\, n = 5, we\ have\ (5*3)^2 + (5*4)^2 = (5*5)^2$$
$$\therefore 15^2 + 20^2 = 25^2$$

And so on

Example-2
$$5^2 + 12^2 = 13^2$$
$$When\, n = 1, then(1*5)^2 + (1*12)^2 = (1*13)^2$$
$$5^2 + 12^2 = 13^2$$
$$When\, n = 2, then(2*5)^2 + (2*12)^2 = (2*13)^2$$
$$10^2 + 24^2 = 26^2$$
$$When\, n = 3, then(3*5)^2 + (3*12)^2 = (3*13)^2$$
$$15^2 + 36^2 = 39^2$$
$$When\, n = 4, then(4*5)^2 + (4*12)^2 = (4*13)^2$$
$$20^2 + 48^2 = 52^2$$
$$When\, n = 5, then(5*5)^2 + (5*12)^2 = (5*13)^2$$
$$25^2 + 60^2 = 65^2$$
$$and\ so\ on$$

OR

When an odd number x$\geq$ 3, x$\in$N= $\{1,2,3,4,5,6, \dots\dots\}$, then $\left[(nx)^2 + \left(\frac{n(x^2-1)}{2}\right)\right]^2 = \left[\frac{n(x^2+1)}{2}\right]^2$ for all n$\in N$

When X=3 and n=1 we have $\left[(1*3)^2 + \left(\frac{1(3^2-1)}{2}\right)\right]^2 = \left[\frac{1(3^2+1)}{2}\right]^2$
$$\therefore 3^2 + 4^2 = 5^2$$
When X=3 and n=2 we have $\left[(2*3)^2 + \left(\frac{2(3^2-1)}{2}\right)\right]^2 = \left[\frac{2(3^2+1)}{2}\right]^2$
$$\therefore 6^2 + 8^2 = 10^2$$
When X=3 and n=3 we have $\left[(3*3)^2 + \left(\frac{3(3^2-1)}{2}\right)\right]^2 = \left[\frac{3(3^2+1)}{2}\right]^2$
$$\therefore 9^2 + 12^2 = 15^2$$

When X=3 and n=4 we have $\left[(4*3)^2+\left(\frac{4(3^2-1)}{2}\right)\right]^2=\left[\frac{4(3^2+1)}{2}\right]^2$

$$\therefore 12^2+16^2=20^2$$

When X=3 and n=5 we have $\left[(5*3)^2+\left(\frac{5(3^2-1)}{2}\right)\right]^2=\left[\frac{5(3^2+1)}{2}\right]^2$

$$\therefore 15^2+20^2=25^2$$

$$and\ so\ on$$

Formula for finding derived solutions of the equation $\boldsymbol{x}^2+\boldsymbol{y}^2=\boldsymbol{z}^2$ in whole numbers, from its non-trivial primitive solutions, when an even number x≥ 4, x∈N= {1,2,3,4,5,6 … …} and x=2n+2, y=$(n+1)^2-1$,
And z = $(n+1)^2+1$

$then\big(n(2n+2)\big)^2+(\ (n((n+1)^2-1))^2=(n((n+1)^2+1))$^2,

For all n ∈N={1,2,3,4,5,6 … … }.

Examples

$$When\ n=1, then\big(1(2*1+2)\big)^2+(\ (1((1+1)^2-1)^2=(\ 1(1+1)^2+1)^2$$

$$\therefore 4^2+3^2=5^2$$

$$When\ n=2, then, x=\big(2(2*2+2)\big), y=(2+1)^2-1, and\ z$$
$$=(2+1)^2+1, then, (2*6)^2+(2*(3^2)-1))\text{^}2=$$
$$12^2+16^2=20^2$$

$$When\ n=3, then, x=\big(3(2*3+2)\big), y=(3+1)^2-1, and\ z=(3+1)^2+1, then,$$
$$(3*8)^2+(3*15)^2=(3*17)^2$$
$$24^2+45^2=51^2$$

$$When\ n=4, then, x=\big(4(2*4+2)\big), y=(4+1)^2-1, and\ z=(4+1)^2+1, then,$$
$$(4*10)^2+(4*24)^2=(4*26)^2$$
$$40^2+96^2=104^2$$

$$When\ n=5, then, x=\big(5(2*5+2)\big), y=(5+1)^2-1, and\ z=(5+1)^2+1, then,$$
$$(5*12)^2+(5*35)^2=(5*37)^2$$
$$60^2+175^2=185^2$$

OR

When an even number x≥2, x∈N={1,2,3,4,5,6, … …}

$then\big(n(2x)\big)^2+(\ n(\mathrm{x}^2-1))^2=(\ n(\mathrm{x}^2+1))^2$ For all n ∈N={1,2,3,4,5,6 … … }

$$When\ x=2, and\ n=1, then\big(1(2*2)\big)^2+(\ (1*2)^2-1)^2=(\ (1*2)^2+1)^2$$
$$\therefore 4^2+3^2=5^2$$

$$When\ x=2, and\ n=2, then\big(2(2*2)\big)^2+(\ (2*2)^2-1)^2=(\ (2*2)^2+1)^2$$

$$\therefore 8^2+15^2=17^2$$

$$When\ x=2, and\ n=3, then\big(3(2*2)\big)^2+(\ (3*2)^2-1)^2=(\ (3*2)^2+1)^2$$
$$\therefore 12^2+35^2=37^2$$

$$When\ x=2, and\ n=4, then\big(4(2*2)\big)^2+(\ (4*2)^2-1)^2=(\ (4*2)^2+1)^2$$
$$\therefore 16^2+63^2=65^2$$

$$When\ x=2, and\ n=5, then\big(5(2*2)\big)^2+(\ (5*2)^2-1)^2=(\ (5*2)^2+1)^2$$

$$\therefore 20^2+99^2=101^2$$
$$and\ so\ on$$

$$When\ x=4, and\ n=1, then\big(1(2*4)\big)^2+(\ (1*4)^2-1)^2=(\ (1*4)^2+1)^2$$
$$\therefore 8^2+15^2=17^2$$

$$When\ x=4, and\ n=2, then\big(2(2*4)\big)^2+(2(\ (1*4)^2-1))^2=(2\ (1*4)^2+1))^2$$
$$\therefore 16^2+30^2=34^2$$

$$When\ x=4, and\ n=3, then\big(3(2*4)\big)^2+(3(\ (1*4)^2-1))^2=(3(1*4)^2+1))^2$$
$$\therefore 24^2+45^2=51^2$$

$$When\ x=4, and\ n=4, then\big(4(2*4)\big)^2+(4(\ (1*4)^2-1))^2=(4(1*4)^2+1))^2$$
$$\therefore 32^2+60^2=68^2$$

$$When\ x = 4, and\ n = 5, then\big(5(2*4)\big)^2 + (5((1*4)^2 - 1))^2 = (5(1*4)^2 + 1))^2$$
$$\therefore 40^2 + 75^2 = 85^2$$
$$and\ so\ on$$
$$When\ x = 6, and\ n = 1, then\big(1(2*6)\big)^2 + ((1*6)^2 - 1)^2 = ((1*6)^2 + 1)^2$$

$$\therefore 12^2 + 35^2 = 37^2$$
$$When\ x = 6, and\ n = 2, then\big(2(2*6)\big)^2 + ((2*6)^2 - 1)^2 = ((2*6)^2 + 1)^2$$
$$\therefore 24^2 + 143^2 = 145^2$$
$$When\ x = 6, and\ n = 3, then\big(3(2*6)\big)^2 + ((3*6)^2 - 1)^2 = ((3*6)^2 + 1)^2$$
$$\therefore 36^2 + 323^2 = 325^2$$
$$When\ x = 6, and\ n = 4, then\big(4(2*6)\big)^2 + ((4*6)^2 - 1)^2 = ((4*6)^2 + 1)^2$$
$$\therefore 48^2 + 575^2 = 577^2$$
$$When\ x = 6, and\ n = 5, then\big(5(2*6)\big)^2 + ((5*6)^2 - 1)^2 = ((5*6)^2 + 1)^2$$
$$\therefore 60^2 + 899^2 = 901^2$$
$$and\ so\ on$$

$$\boldsymbol{Developing\ chain\ solutions\ like\ n_1^2 + n_2^2 + n_3^2 + n_4^2 + n_5^2 + n^2{}_6 + \cdots = n^2, for\ all\quad n = \{1, 2, 3, 4, 5, 6 \ldots\ldots\} in\ whole\ numbers\ of\ the\ equation\ x^2 + y^2 = z^2, \text{using the above formulae}}$$

Examples

Examples-1

$$3^2 + 4^2 = 5^2$$
$$5^2 + 12^2 = 13^2$$
$$3^2 + 4^2 + 12^2 = 13^2$$
$$13^2 + 84^2 = 85^2$$
$$3^2 + 4^2 + 12^2 + 84^2 = 85^2$$
$$85^2 + 3612^2 = 3613^2$$
$$\therefore\ 3^2 + 4^2 + 12^2 + 84^2 + 3612^2 = 3613^2$$
$$3613^{\mathbf{2}} + 6526884^2 = 6526885\text{^}2$$
$$\therefore\ 3^2 + 4^2 + 12^2 + 84^2 + 3612^2 + 6526884^2 = 6526885^2$$
$$6526885^2 + 21300113901612^2 = 21300113901613^2$$
$$\therefore\ 3^2 + 4^2 + 12^2 + 84^2 + 3612^2 + 6526884^2 + 21300113901612^2 = 21300113901613^2$$
$$21300113901613^2 + 226847426110843688722000884^2 = 226847426110843688722000885\text{^}2$$
$$3^2 + 4^2 + 12^2 + 84^2 + 3612^2 + 6526884^2 + 21300113901612^2 + 2.5729877366557343481074291996722e + 52^2 = 2.5729877366557343481074291996722e + 52^2$$
$$and\ so\ on$$

Example-2

$$7^2 + 24^2 = 25^2$$
$$25^2 + 312^2 = 313^2$$
$$\therefore 7^2 + 24^2 + 312^2 = 313^2$$
$$313^2 + 48984^2 = 48985^2$$
$$\therefore 7^2 + 24^2 + 312^2 + 48984^2 = 48985^2$$
$$48985^2 + 1199765112^2 = 1199765113^2$$
$$\therefore 7^2 + 24^2 + 312^2 + 48984^2 + 1199765112^2 = 1199765113^2$$
$$1199765113^2 + 719718163185951384^2 = 719718163185951385^2$$
$$7^2 + 24^2 + 312^2 + 48984^2 + 1199765112^2 + 719718163185951384^2 + 2.5899711720987987395271016274749\text{e}^2 + 373.35397533615141267921001072334 05\text{e} + 74^2 = 3.7335397533615141267921001072334\text{e} + 744^2$$
$$and\ so\ on$$
$$6^2 + 8^2 = 10^2$$
$$10^2 + 24^2 = 26^2$$
$$\therefore 6^2 + 8^2 + 24^2 = 26^2$$
$$26^2 + 168^2 = 170^2$$
$$\therefore 6^2 + 8^2 + 24^2 + 168^2 = 170^2$$
$$170^2 + 7224^2 = 7226^2$$

$$\therefore 6^2 + 8^2 + 24^2 + 168^2 + 7224^2 = 7226^2$$
$$7226^2 + 13053768^2 = 13053770^2$$
$$\therefore 6^2 + 8^2 + 24^2 + 168^2 + 7224^2 + 13053768^2 = 13053770^2$$

And so on.

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