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Stability of a Prey-Predator Model with SIS Epidemic Disease in Predator Involving Holling Type II Functional Response

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Abstract: *In this paper, a prey-predator model with infectious disease in predator population involving Holling type II functional response is proposed and studied. The existence, uniqueness and boundedness of the solution of the system are studied. The existence of all possible equilibrium points is discussed. The local stability analysis of each equilibrium point is investigated. Finally further investigations for the global dynamics of the proposed system are carried out with the help of numerical simulations.*

Keywords: *eco-epidemiological model, SIS epidemic disease, prey-predator model, stability analysis, Holling type II functional response.*

I. Introduction

Mathematical models are divided into two main sections called the first ecology models where interested in studying the interactions between members of a particular community, for example, (humans or animals, etc.) and known model (lotka - volterra) [1,2] the basis for many of the studies in this field. The second type is called epidemiological models that care about the study and analysis of the spread of infectious diseases between humans and animals and the SIR is the basis model for this type of models has been by Kermack and McKendric in 1927 [3]. In more studied showed that combines both these types where named eco-epidemiological models, for example, in 1986 Anderson and May [4] were the first who merged the above two fields, ecological system and epidemiology system, they formulated a prey-predator model with infectious disease spread among prey by contact between them. In the subsequent time many researchers proposed and studied different prey-predator models with disease spread in prey population [5-8]. In addition to the above there are many investigations about prey-predator model with disease in the predator population. Haque [9] proposed a prey-predator model includes a Susceptible-Infected-Susceptible (SIS) parasitic infection in the predator population with linear functional response and nonlinear disease incidence rate. Haque and Venturino [10] considered a prey-predator model with SI epidemic disease spread in predators involving linear functional response. Das [11] studied a prey-predator model with SI epidemic disease in predators included Holling type-II as a functional response. Venturino [12] proposed and analyzed prey-predator model with SIS disease in predators included linear functional response and linear disease incidence. Haque and Venturino [13] considered a prey-predator model with SI epidemic disease spread in predators included ratio-dependent functional response and linear disease's incidence rate. Dahlia [14] studied a prey-predator model with SIS epidemic disease in prey. In this paper we proposed and analyzed a mathematical model describing prey-predator model having SIS epidemic disease in the predator population involving Holling type-II functional response with the disease transmitted between the predator species by contact.

II. The Mathematical Model

Consider an eco-epidemiological model consisting of two preys and two predators is proposed for study. Let $X(T)$ denotes to the density of the first prey population at time T , and $Y(T)$ is the density of the second prey population at time T . However, the predator is divided in to two classes namely, infected $Z(T)$ and susceptible $W(T)$, here $Z(T)$ and $W(T)$ represent the population density at time T for the susceptible and infected predator respectively. Now in order to formulate the above model mathematically the following assumptions are considered:

1. The preys $X(T)$ and $Y(T)$ grow logistically in the absence of predation with intrinsic growth rates $r_i > 0; i = 1,2$ with carrying capacity $K > 0$ and $L > 0$ respectively.
2. The predators $Z(T)$ and $W(T)$ consume the prey $X(T)$ according to Lotka-Vollterra functional responses and Holling type II functional response with attack rates $m > 0$, $p > 0$ and conversion rates $m_1 > 0$, $p_1 > 0$ respectively, While $b > 0$ is the half saturation constant. Further, it is assumed that the predator

$Z(T)$ consume the prey $Y(T)$ according to Lotka-Vollterra functional responses with attack rate $n > 0$ and conversion rate $n_1 > 0$ respectively.

3. The disease transmitted within the population of predator only by contact between the predator individuals, according to simple mass action law with contact infected rate $\beta > 0$.
4. The infected predator $W(T)$ is recovered and they become susceptible $Z(T)$ again with recover rate constant $\alpha > 0$.
5. The predator populations (susceptible and infected) decrease due to the natural death rates $\mu_i > 0; i = 1,2$.
6. The disease in infected predator $W(T)$ may causes mortality with a constant mortality rate represented by $\mu_3 > 0$.

Moreover the dynamics of the above model can be represented by the following set of nonlinear first order differential equations:

$$\begin{aligned}
\frac{dX}{dT} &= r_1 X(1-\tfrac{X}{K}) - mXZ - \frac{pXW}{b+X} \\
\frac{dX}{dT} &= r_2 Y(1-\tfrac{Y}{L}) - nYZ \\
\frac{dZ}{dT} &= (m_1 X + n_1 Y)Z - \beta WZ + \alpha W - \mu_1 Z \\
\frac{dW}{dT} &= \frac{p_1 XW}{b+X} + \beta WZ - \alpha W - (\mu_2 + \mu_3)W
\end{aligned} \tag{1}$$

Note that the above model contains (16) positive parameters in all, which makes mathematical analysis of the system very difficult. So in order to reduce the number of parameters and determined which parameter represents the control parameter, the following dimensionless variable are used:

$t = r_1 T$, $x = \frac{X}{K}$, $y = \frac{Y}{L}$, $z = \frac{mZ}{r_1}$, $w = \frac{pW}{r_1 K}$

Accordingly, system (1) can be rewritten in the following non dimensional form:

$$\begin{aligned}
\frac{dx}{dt} &= x(1-x) - xz - \frac{xw}{w_1 + x} = xf_1(x,y,z,w) \\
\frac{dy}{dt} &= w_2 y(1-y) - w_3 yz = yf_2(x,y,z,w) \\
\frac{dz}{dt} &= w_4 xz + w_5 yz - w_6 wz + w_7 w - w_8 z = zf_3(x,y,z,w) \\
\frac{dw}{dt} &= \frac{w_9 wx}{w_1 + x} + w_{10} wz - (w_{11} + w_{12} + w_{13})w = wf_4(x,y,z,w)
\end{aligned} \tag{2}$$

Here $x(0) \geq 0$, $y(0) \geq 0$, $z(0) \geq 0$ and $w(0) \geq 0$ with the following constants represent the non dimensional parameters

$w_1 = \frac{b}{K}$, $w_2 = \frac{r_2}{r_1}$, $w_3 = \frac{n}{m}$, $w_4 = \frac{m_1 K}{r_1}$, $w_5 = \frac{n_1 L}{r_1}$, $w_6 = \frac{\beta k}{p}$, $w_7 = \frac{\alpha m K}{r_1 p}$, $w_8 = \frac{\mu_1}{r_1}$, $w_9 = \frac{p_1}{r_1}$, $w_{10} = \frac{\beta}{m}$, $w_{11} = \frac{\alpha}{r_1}$, $w_{12} = \frac{\mu_2}{r_1}$, $w_{13} = \frac{\mu_3}{r_1}$.

It has been observed that the non dimensional system (2) contains (13) parameters only, while the original system (1) contains (16) parameters. Obviously the interaction functions f_1, f_2, f_3 and f_4 of the system (2) are continuous and have continuous partial derivatives on the state space R_+^4, therefore these functions are Lipschizian on its domain R_+^4 and then the solution of system (2) with non negative initial condition exists and is unique. In addition, all the solutions of system (2) which initiate in R_+^4 are uniformly bounded as shown in the following theorem.

Theorem 1: All the solutions of the system (2), which initiate in R_+^4 are uniformly bounded provided that

$$B > \frac{w_9 w_7}{w_4} \tag{3a}$$

Where

$$B = w_{11} + w_{12} + w_{13} \quad (3b)$$

Proof: Let $(x(t), y(t), z(t), w(t))$ be any solution of the system (2). Since

$$\frac{dx}{dt} \le x(1-x), \quad \frac{dy}{dt} \le w_2 y(1-y)$$

Thus by solving these differential inequalities:

$$\lim_{t\to\infty} Sup\, x(t) \le 1 \Rightarrow x(t) \le 1, \forall t > 0$$

$$\lim_{t\to\infty} Sup\, y(t) \le 1 \Rightarrow y(t) \le 1, \forall t > 0$$

Now, consider the function:

$$U(x,y,z,w) = w_4 x + \frac{w_5}{w_3} y + z + \frac{w_4}{w_9} w$$

Then the time derivative of $U(\cdot)$ along the solution of the system (2) is:

$$\frac{dU}{dt} \le s - \gamma U$$

Where

$$s = 2\left[w_4 + \frac{w_2 w_5}{w_3}\right]$$

and $\gamma = \min.\left\{1, w_2, w_8, \left(B - \frac{w_9 w_7}{w_4}\right)\right\}$

Clearly, γ is positive constant under the sufficient condition (3a). By comparing the above differential inequality with the associated linear differential equation, we obtain:

$$\lim_{t\to\infty} Sup.U(t) \le \frac{s}{\gamma} \Rightarrow U(t) \le \frac{s}{\gamma}\ \forall t > 0$$

hence, all the solutions of system (2) that initiate in R_+^4 are confined in the region $\Omega = \{(x,y,z,w) \in R_+^4 : U = w_4 x + \frac{w_5}{w_3} y + z + \frac{w_4}{w_9} w \le \frac{s}{\gamma}\}$ under the given condition, thus these solutions are uniformly bounded, and then the proof is complete. ■

III. The Existence Of Equilibrium Points

In this section, the existence of all possible equilibrium points of system (2) is discussed. It is observed that, system (2) has at most ten equilibrium points, namely $E_0 = (0,0,0,0)$, $E_x = (1,0,0,0)$, $E_y = (0,1,0,0)$, $E_{xy} = (1,1,0,0)$ always exist. While the existence of other equilibrium points are shown in the following:

The equilibrium point $E_{xz} = (\bar{x}, 0, \bar{z}, 0)$, where

$$\bar{x} = \frac{w_8}{w_4}, \ \bar{z} = 1 - \frac{w_8}{w_4} \quad (4)$$

exists uniquely in the $Int.R_+^2$ of $xz-$plane provided that:

$$\frac{w_8}{w_4} < 1 \quad (5)$$

The equilibrium point $E_{yz} = (0, \tilde{y}, \tilde{z}, 0)$, where

$$\tilde{y} = \frac{w_8}{w_5}, \ \tilde{z} = \frac{w_2(w_5 - w_8)}{w_3 w_5} \quad (6)$$

exists uniquely in the $Int.R_+^2$ of $yz-$plane provided that:

$$w_5 > w_8 \quad (7)$$

The equilibrium point $E_{xyz} = (\hat{x}, \hat{y}, \hat{z}, 0)$ exists in $Int.R_+^3$ of xyz-plane, where

$$\hat{x} = \frac{w_3 w_5 - w_2(w_5 - w_8)}{w_2 w_4 + w_3 w_5} \quad (8a)$$

$$\hat{y} = \frac{w_2 w_4 - w_3[w_4 - w_8]}{w_2 w_4 + w_3 w_5} \quad (8b)$$

$$\hat{z} = \frac{w_2[w_4 + w_5 - w_8]}{w_2 w_4 + w_3 w_5} \tag{8c}$$

exists uniquely in the $Int.R_+^3$ of xyz – plane provided that:

$$w_8 > \max.\{w_4, w_5\} \tag{9a}$$

$$w_4 + w_5 > w_8 \tag{9b}$$

The equilibrium point $E_{xzw} = (\hat{x}, 0, \hat{z}, \hat{w})$ exists in $Int.R_+^3$ of xzw - plane, where

$\hat{z} = \frac{Bw_1 + (B - w_9)\hat{x}}{(w_1 + \hat{x})w_{10}}$ and

$$\hat{w} = \frac{(w_8 - w_4\hat{x})[Bw_1 + (B - w_9)\hat{x}]}{w_1[w_7 w_{10} - w_6 B] + [w_7 w_{10} - w_6(B - w_9)]\hat{x}} \tag{10}$$

While, $\hat{x}$ represents the positive root of each of the following equation.

$$q_1 x^3 + q_2 x^2 + q_3 x + q_4 = 0 \tag{11}$$

where:

$$q_1 = w_6(B - w_9) - w_7 w_{10}$$

$$q_2 = [w_7 w_{10} - w_6(B - w_9)]\left[1 - w_1 - \left(\frac{B - w_9}{w_{10}}\right)\right] - w_1[w_7 w_{10} - w_6 B] + w_4[B - w_9]$$

$$q_3 = w_1[(w_7 w_{10} - w_6[B - w_9])\left[1 - \frac{B}{w_{10}}\right] + [w_7 w_{10} - w_6 B]\left(1 - w_1 - \left[\frac{B - w_9}{w_{10}}\right]\right) + w_4 B] - w_8[B - w_9]$$

$$q_4 = w_1[w_1(w_7 w_{10} - w_6 B)\left(1 - \frac{B}{w_{10}}\right) - w_8 B]$$

So by using Descartes rule of signs, Eq. (11) has a unique positive root say $\hat{x}$ provided that one set of the following sets of conditions hold:

$$q_1 > 0, q_2 > 0 \text{ and } q_4 < 0 \tag{12a}$$

$$q_1 > 0, q_3 < 0 \text{ and } q_4 < 0 \tag{12b}$$

$$q_1 < 0, q_2 < 0 \text{ and } q_4 > 0 \tag{12c}$$

$$q_1 < 0, q_3 > 0 \text{ and } q_4 > 0 \tag{12d}$$

Therefore, by substituting $\hat{x}$ in Equation (10), system (2) has a unique equilibrium point in the $Int.R_+^3$ of xzw - plane given by $E_{xzw} = (\hat{x}, 0, \hat{z}, \hat{w})$, provided that

$$w_8 > w_4\hat{x},\ w_7 w_{10} > w_6 \max.\{B, B - w_9\} \tag{13a}$$

Or,

$$w_8 < w_4\hat{x},\ w_7 w_{10} < w_6 \min.\{B, B - w_9\} \tag{13b}$$

Finally the coexistence equilibrium point $E_{xyzw} = (x^*, y^*, z^*, w^*)$ exists in $Int.R_+^4$, where

$$y^* = \frac{w_1[w_2 w_{10} - w_3 B] + [w_2 w_{10} - w_3(B - w_9)]x^*}{(w_1 + x^*)w_2 w_{10}} \tag{14a}$$

$$z^* = \frac{Bw_1 + (B - w_9)x^*}{(w_1 + x^*)w_{10}} \tag{14b}$$

$$w^* = \frac{B_2[Bw_1 + (B - w_9)x^*)]}{w_2 w_{10}(w_1 + x^*)B_3} \tag{14c}$$

Where

$$B_2 = w_2 w_{10}(w_1 + x^*)(w_8 - w_4 x^* - w_5) + w_3 w_5(Bw_1 + [B - w_9]x^*)^2$$

$B_3 = w_1[w_7 w_{10} - w_6 B] + [w_7 w_{10} - w_6(B - w_9)]x^*$

While, x^* represents the positive root of each of the following equation:

$$Q_1 x^4 + Q_2 x^3 + Q_3 x^2 + Q_4 x + Q_5 = 0 \qquad (15)$$

where:

$$Q_1 = -w_2 w_{10}[w_7 w_{10} - w_6(B - w_9)]$$

$$\begin{aligned} Q_2 = {} & w_2[w_7 w_{10} - w_6(B - w_9)] \\ & \times\left[(1 - 2w_1)w_{10} - (B - w_9)\right] \\ & - w_2 w_{10}[w_1(w_7 w_{10} - w_6 B) \\ & - (B - w_9)w_4] \end{aligned}$$

$$\begin{aligned} Q_3 = {} & w_1 w_2[(w_7 w_{10} - w_6[B - w_9]) \\ & \times (2w_{10}(1 - w_1) - 2B + w_9) \\ & + [w_7 w_{10} - w_6 B] \\ & \times [w_{10}(1 - w_1) - (B - w_9)] \\ & + (B - w_9) \\ & \times [w_1 w_2(w_5 - w_8) - (B - w_9)w_3 w_5] \end{aligned}$$

$$\begin{aligned} Q_4 = {} & w_1^2 w_2[(w_7 w_{10} - w_6[B - w_9])[w_{10} - B] \\ & + w_1^2 w_2[w_7 w_{10} - w_6 B] \\ & \times [w_{10}(2 - w_1) - 2B + w_9)] \\ & - 2w_1 w_3 w_5 B(B - w_9) \end{aligned}$$

$$\begin{aligned} Q_5 = {} & w_1^3 w_2[w_7 w_{10} - w_6 B][w_{10} - B] \\ & - w_1^2 B[w_2 w_{10}(w_8 - w_5) + w_3 w_5 B] \end{aligned}$$

So by using Descartes rule of signs, Equation (15) has a unique positive root say x^* provided that one set of the following sets of conditions hold:

$Q_1 > 0, Q_2 > 0, Q_3 > 0$ and $Q_5 < 0$ (16a)

$Q_1 > 0, Q_2 > 0, Q_4 < 0$ and $Q_5 < 0$ (16b)

$Q_1 > 0, Q_3 < 0, Q_4 < 0$ and $Q_5 < 0$ (16c)

$Q_1 < 0, Q_3 > 0, Q_4 > 0$ and $Q_5 > 0$ (16d)

$Q_1 < 0, Q_2 < 0, Q_4 > 0$ and $Q_5 > 0$ (16e)

$Q_1 < 0, Q_2 < 0, Q_3 < 0$ and $Q_5 > 0$ (16f)

Therefore, by substituting x^* in Equations (14a), (14b) and (14c), system (2) has a unique equilibrium point in the $Int.R_+^4$ by $E_{xyzw} = (x^*, y^*, z^*, w^*)$, provided that

$w_2 w_{10} > w_3 \max.\{B, B - w_9\}$ (17a)

$B_2 > 0$ *and* $B_3 > 0$ (17b)

Or,

$B_2 < 0$ *and* $B_3 < 0$ (17c)

IV. The Stability Analysis

In this section the stability locally analysis of the above mentioned equilibrium points of system (2) are investigated analytically. The Jacobian matrix of system (2) at each of these points is determined and then the eigenvalues for the resulting matrix are computed, finally the obtained results are summarized in the following: The Jacobian matrix of system (2) at the equilibrium point $E_0 = (0,0,0,0)$ can be written as $J_0 = J(E_0) = [c_{ij}]_{4\times 4}; i, j = 1,2,3,4$, where $c_{11} = 1$, $c_{22} = w_2$, $c_{33} = -w_8$, $c_{34} = w_7$, $c_{44} = -B$ and zero otherwise. Then the eigenvalues of J_0 are:

$\lambda_{01} = 1 > 0$

$\lambda_{02} = w_2 > 0$ (18)

$\lambda_{03} = -w_8 < 0$ $\lambda_{04} = -B < 0$

Therefore, the equilibrium point E_0 is a saddle point.

The Jacobian matrix of system (2) at the equilibrium point $E_x = (1,0,0,0)$ can be written as $J_x = J(E_x) = [d_{ij}]_{4\times4}; i,j = 1,2,3,4$, where $d_{11} = d_{13} = -1$, $d_{14} = \frac{-1}{w_1+1}$ $d_{22} = w_2$, $d_{33} = w_4 - w_8$, $d_{34} = w_7$, $d_{44} = \frac{w_9}{w_1+1} - B$ and zero otherwise. Hence, the eigenvalues of J_x are:

$\lambda_{x_1} = -1 < 0$, $\lambda_{x_2} = w_2 > 0$, $\lambda_{x_3} = w_4 - w_8$ and $\lambda_{x_4} = \frac{w_9}{w_1+1} - B$ (19)

Clearly, E_x is a saddle point.

Now, the Jacobian matrix of system (2) at the equilibrium point $E_y = (0,1,0,0)$ can be written as $J_y = J(E_y) = [e_{ij}]_{4\times4}; i,j = 1,2,3,4$, where $e_{11} = 1$, $e_{22} = -w_2$, $e_{23} = -w_3$, $e_{33} = w_5 - w_8$, $e_{34} = w_7$, $e_{44} = -B$ and zero otherwise. The eigenvalues of J_y are:

$\lambda_{y_1} = 1$, $\lambda_{y_2} = -w_2 < 0$, $\lambda_{y_3} = w_5 - w_8$,

$\lambda_{y_4} = -B$ (20)

Hence, E_y is a saddle point.

Now, the Jacobian matrix of system (2) at the equilibrium point $E_{xy} = (1,1,0,0)$ can be written as $J_{xy} = J(E_{xy}) = [f_{ij}]_{4\times4}; i,j = 1,2,3,4$, where

$f_{11} = f_{13} = -1$, $f_{14} = \frac{-1}{w_1+1}$, $f_{22} = -w_2$, $f_{23} = -w_3$, $f_{33} = w_4 + w_5 - w_8$, $f_{34} = w_7$ and $f_{44} = \frac{w_9}{w_1+1} - B$ and zero otherwise. Therefore, the eigenvalues of J_{xy} are given by:

$\lambda_{x_1y_1} = -1$, $\lambda_{x_2y_2} = -w_2 < 0$, $\lambda_{x_3y_3} = w_4 + w_5 - w_8$, $\lambda_{x_4y_4} = \frac{w_9}{w_1+1} - B$ (21)

Consequently, E_{xy} is locally asymptotically stable in the R_+^4 if the following condition is satisfied.

$w_4 + w_5 < w_8$ (22a)

$\frac{w_9}{w_1+1} < B$ (22b)

However, E_{xy} will be saddle point in the R_+^4 if we reversed of the above condition.

Theorem 3: Assume that the equilibrium point $E_{xz} = (\bar{x}, 0, \bar{z}, 0)$ of system (2) exists. Then it is locally asymptotically stable in R_+^4 if one of the following sets of conditions hold

$\frac{w_9\bar{x}}{w_1+\bar{x}} + w_{10}\bar{z} < B$ (23a)

$w_2 < w_3\bar{z}$ (23b)

Proof: Note that, it is easy to check that the Jacobian matrix of system (2) at the equilibrium point E_{xz} can be written $J_{xz} = J(E_{xz}) = [g_{ij}]_{4\times4}; i,j = 1,2,3,4$

$g_{11} = g_{13} = -\bar{x}$, $g_{14} = \frac{-\bar{x}}{w_1+\bar{x}}$, $g_{22} = w_2 - w_3\bar{z}$, $g_{31} = w_4\bar{z}$, $g_{32} = w_5\bar{z}$, $g_{34} = w_7 - w_6\bar{z}$, $g_{44} = \frac{w_9\bar{x}}{w_1+\bar{x}} + w_{10}\bar{z} - B$ and zero otherwise. Therefore the characteristic equation can be written in the form:

$$(g_{44} - \bar{\lambda})[\bar{\lambda}^3 + A_1\bar{\lambda}^2 + A_2\bar{\lambda} + A_3] = 0$$

Consequently, either

$$\bar{\lambda}_{x_1z_1} = \frac{w_9\bar{x}}{w_1+\bar{x}} + w_{10}\bar{z} - B$$

Or $\bar{\lambda}^3 + A_1\bar{\lambda}^2 + A_2\bar{\lambda} + A_3 = 0$ (24a)

Here

$$A_1 = -(g_{11} + g_{22})$$
$$A_2 = g_{11}a_{22} - g_{13}g_{31} \quad (24b)$$
$$A_3 = g_{13}g_{22}g_{31}$$

and

$$\Delta = A_1A_2 - A_3$$
$$= -(g_{11} + g_{22})g_{11}g_{22} + g_{11}g_{13}g_{31} \quad (25)$$

So, by substituting the values of g_{ij}, and then simplifying the resulting terms we obtain:

$A_1 = \bar{x} - (w_2 - w_3\bar{z})$, $A_3 = -w_4\bar{x}\bar{z}(w_2 - w_3\bar{z})$ and $\Delta = \bar{x}[(w_2 - w_3\bar{z})(w_2 - w_3\bar{z} - \bar{x}) + w_4\bar{x}\bar{z}]$

According to the Routh-Hurwitz criterion all the roots (eigenvalues) of third order equations have real parts provided that $A_i > 0; i = 1,3$ and $\Delta > 0$. Note that, $A_i > 0; i = 1,3$ and $\Delta > 0$ provided that condition (23b) holds. Further $\bar{\lambda}_{x_1z_1} < 0$ provided that condition (23a) holds. Therefore all the eigenvalues of J_{xz} have negative real parts under the give conditions and hence the equilibrium point E_{xz} is locally asymptotically stable, which completes the proof. ■

Theorem 4: Assume that the equilibrium point $E_{yz} = (0, \tilde{y}, \tilde{z}, 0)$ of system (2) exists. Then it is locally asymptotically stable in R_+^4 if one of the following sets of conditions hold

$w_{10}\tilde{z} < B$ (26a)

$\tilde{z} > 1$ (26b)

Proof: Note that, it is easy to check that the Jacobian matrix of system (2) at the equilibrium point E_{yz} can be written:

$J_{yz} = J(E_{yz}) = [h_{ij}]_{4\times4}; i, j = 1,2,3,4$

$h_{11} = 1 - \tilde{z}$, $h_{22} = -w_2\tilde{y}$, $h_{23} = -w_3\tilde{y}$, $h_{31} = w_4\tilde{z}$, $h_{32} = w_5\tilde{z}$, $h_{34} = w_7 - w_6\tilde{z}$, $h_{44} = w_{10}\tilde{z} - B$ and zero otherwise. Therefore the characteristic equation can be written in the form:

$$(h_{44} - \tilde{\lambda})[\tilde{\lambda}^3 + C_1\tilde{\lambda}^2 + C_2\tilde{\lambda} + C_3] = 0$$

Consequently, either

$$\tilde{\lambda}_{y_1z_1} = w_{10}\tilde{z} - B$$

Or $\tilde{\lambda}^3 + C_1\tilde{\lambda}^2 + C_2\tilde{\lambda} + C_3 = 0$ (27a)

here

$$C_1 = -(h_{11} + h_{22})$$
$$C_2 = h_{11}h_{22} - h_{23}h_{32} \quad (27b)$$
$$C_3 = h_{11}h_{23}h_{32}$$

and

$$\Delta = C_1C_2 - C_3$$
$$= -(h_{11} + h_{22})h_{11}h_{22} + h_{22}h_{23}h_{32} \quad (28)$$

So, by substituting the values of h_{ij}, and then simplifying the resulting terms we obtain:

$C_1 = w_2\tilde{y} - (1 - \tilde{z})$, $C_3 = -w_3w_5\tilde{y}\tilde{z}(1 - \tilde{z})$ and $\Delta = w_2\tilde{y}[(1 - \tilde{z})(1 - \tilde{z} - w_2\tilde{y}) + w_3w_5\tilde{y}\tilde{z}]$

Again due the Routh-Hurwitz criterion all the roots (eigenvalues) of third order equations have real parts provided that $C_i > 0; i = 1,3$ and $\Delta > 0$. Note that, $C_i > 0; i = 1,3$ and $\Delta > 0$ provided that condition (26b) holds. Further $\tilde{\lambda}_{y_1z_1} < 0$ provided that condition (26a) holds. Therefore all the eigenvalues of J_{yz} have negative real parts under the give conditions and hence the equilibrium point E_{yz} is locally asymptotically stable, which completes the proof. ■

Theorem 5: Assume that the equilibrium point $E_{xyz} = (\hat{x}, \hat{y}, \hat{z}, 0)$ of system (2) exists. Then it is locally asymptotically stable in R_+^4 if the following condition hold:

$$\frac{w_9\hat{x}}{w_1+\hat{x}} + w_{10}\hat{z} < B \qquad (29)$$

Proof: Note that, it is easy to check that the Jacobian matrix of system (2) at the equilibrium point E_{xyz} can be written:

$$J_{xyz} = J(E_{xyz}) = [k_{ij}]_{4\times 4}; i, j = 1,2,3,4$$

$k_{11} = k_{13} = -\hat{x}$, $k_{14} = \frac{-\hat{x}}{w_1+\hat{x}}$, $k_{22} = -w_2\hat{y}$, $k_{23} = -w_3\hat{y}$, $h_{31} = w_4\hat{z}$, $k_{32} = w_5\hat{z}$, $k_{34} = w_7 - w_6\hat{z}$, $k_{44} = \frac{w_9\hat{x}}{w_1+\hat{x}} + w_{10}\hat{z} - B$ and zero otherwise. Therefore the characteristic equation can be written in the form:

$$(k_{44} - \hat{\lambda})[\hat{\lambda}^3 + K_1\hat{\lambda}^2 + K_2\hat{\lambda} + K_3] = 0$$

Consequently, either

$$\hat{\lambda}_{x_1y_1z_1} = \frac{w_9\hat{x}}{w_1+\hat{x}} + w_{10}\hat{z} - B$$

Or $\hat{\lambda}^3 + K_1\hat{\lambda}^2 + K_2\hat{\lambda} + K_3 = 0$ (30a)

here

$$K_1 = -(k_{11} + k_{22})$$
$$K_2 = k_{11}k_{22} - k_{23}k_{32} - k_{13}k_{31} \qquad (30b)$$
$$K_3 = k_{11}k_{23}k_{32} + k_{13}k_{22}k_{31}$$

$$\Delta = K_1K_2 - K_3$$

and

$$= -(k_{11} + k_{22})k_{11}k_{22} + k_{11}k_{13}k_{31} + k_{22}k_{23}k_{32} \qquad (31)$$

So, by substituting the values of h_{ij}, and then simplifying the resulting terms we obtain:

$K_1 = \hat{x} + w_2\hat{y}$, $K_3 = \hat{x}\hat{y}\hat{z}(w_3w_5 + w_2w_4)$ and $\Delta = \hat{x}^2[w_2\hat{y} + w_4\hat{z}] + w_2\hat{y}^2[w_2\hat{x} + w_3w_5\hat{z}]$

According to the Routh-Hurwitz criterion all the roots (eigenvalues) of third order equations have real parts provided that $K_i > 0; i = 1,3$ and $\Delta > 0$. Note that, $K_i > 0; i = 1,3$ and $\Delta > 0$ always as shown above. Further $\hat{\lambda}_{x_1y_1z_1} < 0$ provided that condition (29) holds. Therefore all the eigenvalues of J_{xyz} have negative real parts under the give condition and hence the equilibrium point E_{xyz} is locally asymptotically stable, which completes the proof. ■

Theorem 6: Assume that the equilibrium point $E_{xzw} = (\hat{x}, 0, \hat{z}, \hat{w})$ of system (2) exists. Then it is locally asymptotically stable in R_+^4 if one of the following sets of conditions hold

$$w_2 < w_3\hat{z} \qquad (32a)$$

$$\min.\left\{\hat{w}, \frac{w_4\hat{x}\hat{z}^2(w_1+\hat{x})}{-(w_7-w_6\hat{z})w_7\hat{w}}\right\} < (w_1+\hat{x})^2 < \frac{w_1w_9 + w_{10}\hat{w}}{w_{10}} \qquad (32b)$$

$$w_7 < w_6\hat{z} \qquad (32c)$$

Proof: Note that, it is easy to check that the Jacobian matrix of system (2) at the equilibrium point E_{xzw} can be written:

$$J_{xzw} = J(E_{xzw}) = [l_{ij}]_{4\times 4}; i, j = 1,2,3,4$$

$l_{11} = \hat{x}\left(-1 + \frac{\hat{w}}{(w_1+\hat{x})^2}\right)$, $l_{13} = -\hat{x}$, $l_{14} = \frac{-\hat{x}}{w_1+\hat{x}}$, $l_{22} = w_2 - w_3\hat{z}$, $l_{31} = w_4\hat{z}$, $l_{32} = w_5\hat{z}$, $l_{33} = \frac{-w_7\hat{w}}{\hat{z}}$, $l_{34} = w_7 - w_6\hat{z}$, $l_{41} = \frac{w_1w_9\hat{w}}{(w_1+\hat{x})^2}$, $l_{43} = w_{10}\hat{w}$ and zero otherwise. Therefore the characteristic equation can be written in the form:

$$(l_{22} - \hat{\lambda})[\hat{\lambda}^3 + L_1\hat{\lambda}^2 + L_2\hat{\lambda} + L_3] = 0$$

Consequently, either

$$\hat{\lambda}_{x_1 z_1 w_1} = w_2 - w_3\hat{z}$$

Or $\hat{\lambda}^3 + L_1\hat{\lambda}^2 + L_2\hat{\lambda} + L_3 = 0$ (33a)

here
$$L_1 = -(l_{11} + l_{33})$$
$$L_2 = l_{11}l_{33} - l_{13}l_{31} - l_{14}l_{41} - l_{34}l_{43} \quad (33b)$$
$$L_3 = l_{34}(l_{11}l_{43} - l_{13}l_{41}) - l_{14}(l_{31}l_{43} - l_{33}l_{41})$$

and
$$\Delta = L_1L_2 - L_3$$
$$= (l_{11} + l_{22})(l_{13}l_{31} - l_{11}l_{33}) + l_{41}(l_{11}l_{14} + l_{34}l_{13}) + l_{43}(l_{33}l_{34} + l_{14}l_{31}) \quad (34)$$

So, by substituting the values of l_{ij}, and then simplifying the resulting terms we obtain:

$$L_1 = -\hat{x}\left(-1 + \frac{\hat{w}}{(w_1+\hat{x})^2}\right) + \frac{w_7\hat{w}}{\hat{z}}$$

$$L_3 = \hat{x}\hat{w}(w_7 - w_6\hat{z})\left[\left(-1 + \frac{\hat{w}}{(w_1+\hat{x})^2}\right)w_{10} + \frac{w_1w_9}{(w_1+\hat{x})^2}\right] + \frac{\hat{x}}{w_1+\hat{x}}\left[w_4w_{10}\hat{z}\hat{w} + \frac{w_1w_7w_9\hat{w}^2}{(w_1+\hat{x})^2\hat{z}}\right]$$

and

$$\Delta = \left[\hat{x}\left(-1 + \frac{\hat{w}}{(w_1+\hat{x})^2}\right) - \frac{w_7\hat{w}}{\hat{z}}\right] \times \left[-w_4\hat{x}\hat{z} + \frac{w_7\hat{x}\hat{w}}{\hat{z}}\left(-1 + \frac{\hat{w}}{(w_1+\hat{x})^2}\right)\right] + \frac{w_1w_9\hat{x}\hat{w}}{(w_1+\hat{x})^2}\left[\frac{-\hat{x}}{w_1+\hat{x}}\left(-1 + \frac{\hat{w}}{(w_1+\hat{x})^2}\right) - (w_7 - w_6\hat{z})\right] + w_{10}\hat{w}\left[\frac{-(w_7 - w_6\hat{z})w_7\hat{w}}{\hat{z}} - \frac{w_4\hat{x}\hat{z}}{w_1+\hat{x}}\right]$$

According to the Routh-Hurwitz criterion all the roots (eigenvalues) of third order equations have real parts provided that $L_i > 0; i = 1,3$ and $\Delta > 0$. Note that, $L_1 > 0$ under the condition (32b), while conditions (32a)-(32c) ensure the positivity of L_3 (i.e. $L_3 > 0$) and $\Delta > 0$. Further $\hat{\lambda}_{x_1 z_1 w_1} < 0$ provided that condition (32a) holds. Therefore all the eigenvalues of J_{xzw} have negative real parts under the give condition and hence the equilibrium point E_{xzw} is locally asymptotically stable, which completes the proof.

Theorem 7: Assume that the positive equilibrium point $E_{xyzw} = (x^*, y^*, z^*, w^*)$ of system (2) exists and let the following inequalities hold:

$$w^* < (w_1 + x^*)^2 \quad (35a)$$
$$w_1w_7w_9 + (w_1 + x^*)w_{10}z^* < w_1w_9w_6z^* \quad (35b)$$
$$(w_4 - 1)^2 < 2w_7w^*\left(1 - \frac{w^*}{(w_1+x^*)^2}\right) \quad (35c)$$
$$(w_5 - w_3)^2 < 2w_2w_7w^* \quad (35d)$$

Then it is locally asymptotically stable in the $Int.R_+^4$.

Proof. It is easy to verify that, the linearized system of system (2) can be written as

$$\frac{dX}{dt} = \frac{dU}{dt} = J(E_{xyzw})U$$

here $X = (x, y, z, w)^t$ and $U = (u_1, u_2, u_3, u_4)^t$ where $u_1 = x - x^*$, $u_2 = y - y^*$, $u_3 = z - z^*$ and $u_4 = w - w^*$. Moreover,

$a_{11} = x^*\left(-1+\frac{w^*}{(w_1+x^*)^2}\right)$, $a_{13} = -x^*$, $a_{14} = \frac{-x^*}{w_1+x^*}$, $a_{22} = -w_2 y^*$, $a_{23} = -w_3 y^*$, $a_{31} = w_4 z^*$, $a_{32} = w_5 z^*$, $a_{33} = \frac{-w_7 w^*}{z^*}$, $a_{34} = w_7 - w_6 z^*$, $a_{41} = \frac{w_1 w_9 w^*}{(w_1+x^*)^2}$, $a_{43} = w_{10} w^*$ and zero otherwise. Now consider the following positive definite function

$$V = \frac{u_1^2}{2x^*} + \frac{u_2^2}{2y^*} + \frac{u_3^2}{2z^*} + \frac{u_4^2(w_1+x^*)}{2w_1 w_9 w^*}$$

It is clearly that $V : R_+^4 \to R_+^4$ and is a continuously differentiable function so that $V(x^*, y^*, z^*, w^*) = 0$ and $V(x, y, z, w) > 0$ otherwise. So by differentiating V with respect to
time t, gives

$$\frac{dV}{dt} = \frac{u_1}{x^*}\cdot\frac{du_1}{dt} + \frac{u_2}{y^*}\cdot\frac{du_2}{dt} + \frac{u_3}{z^*}\cdot\frac{du_3}{dt} + \frac{u_4(w_1+x^*)}{w_1 w_9 w^*}\cdot\frac{du_4}{dt}$$

Substituting the values of $\frac{du_i}{dt}; i = 1,2,3,4$ in the above equation, and after doing some algebraic manipulation; we get that:

$$\frac{dV}{dt} = -\left(1-\frac{w^*}{(w_1+x^*)^2}\right)u_1^2 + (w_4-1)u_1u_3 - w_2u_2^2 + (w_5-w_3)u_2u_3 - \frac{w_7 w^* u_3^2}{z^{*2}} + \left[\frac{w_7-w_6z^*}{z^*} + \frac{(w_1+x^*)w_{10}}{w_1w_9}\right]u_3u_4$$

Obviously, due to conditions, we get that

$$\frac{dV}{dt} < -\left[\sqrt{1-\frac{w^*}{(w_1+x^*)^2}}\,u_1 - \sqrt{\frac{w_7w^*}{2}}\frac{u_3}{z^*}\right]^2 - \left[\sqrt{w_2}\,u_2 - \sqrt{\frac{w_7w^*}{2}}\frac{u_3}{z^*}\right]^2 + \left[\frac{(w_7-w_6z^*)w_1w_9 + (w_1+x^*)w_{10}z^*}{w_1w_9z^*}\right]u_3u_4$$

Clearly $\frac{dV}{dt} < 0$, therefore the origin and then E_{xyzw} is locally asymptotically stable point in the $Int.R_+^4$ and hence the proof is complete. ■

V. Numerical Simulation

In this section the global dynamics of system (2) is investigated numerically. The objectives are confirming our analytical results and discuss the role of the existence of disease in the intermediate predator population on the dynamical behavior of the system. For the following set of hypothetical, biologically feasible, set of parameters, definitely different set of hypothetical parameters can be chosen also, system (2) is solved numerically starting at different initial points as illustrated in Figure (2).

$$\begin{aligned} &w_1 = 0.5\,;\ w_2 = 0.6\,;\ w_3 = 0.6\,;\ w_4 = 0.2\,;\ w_5 = 0.3\,;\ w_6 = 0.3; w_7 = 0.1; w_8 = 0.1 \\ &w_9 = 0.1\,;\ w_{10} = 0.3\,; w_{11} = 0.1\,;\ w_{12} = 0.1\,;\ w_{13} = 0.0001 \end{aligned} \qquad (36)$$

Note that from now onward we will use solid line to describes the trajectory of the prey x; the dashed line to describes the trajectory of prey y; the doted line to describes the trajectory of infected predator z; the dash dot to describes the trajectory of susceptible top predator w.

The effect of varying the half saturation constant rate of the susceptible first prey w_1 on the dynamics of system (2) is studied and then the trajectories of the system (2) are drawn in figures (3a)-(3d), for the values of $w_1 = 0.5$, 0.7, 0.9 with the other parameter fixed as given in equation (36).

According to the above figures, it is clear that as the half saturation constant rate of susceptible of first prey w_1 increase the solution of system (2) approaches asymptotically to the coexistence equilibrium point. However decreasing w_1.

Now the effect of varying the growth rate of the susceptible second prey w_2 on the dynamics of system (2) is studied and then the trajectories of the system (2) are drawn in figures (4a)-(4d), for the values of $w_2 = 0.6,\ 0.8,\ 1,\ 0.2$ with the other parameter fixed as given in equation (36).

According to the above figures, it is clear that as the growth rate of susceptible of second prey w_2 increase the solution of system (2) approaches asymptotically to the coexistence equilibrium point. However decreasing w_2, say $(0 \le w_2 \le 0.2)$, the solution of system (2) approaches asymptotically to E_{xzw} equilibrium point.

The effect of varying the attack of second prey by the susceptible of predator w_3 on the dynamical behavior of system (2) is studied. The system (2) is solved numerically different values of the attack rate $w_3 = 0.6,\ 0.7,\ 0.9$ keeping other parameters fixed as given in equation (36), and then the solutions of system (2) are drawn in figures (5a)-(5c).

Obviously, as the attack rate of second prey by the susceptible of predator w_3 increase the value the first prey species started increase while the value of other species are decrease and the system (2) still has asymptotically stable coexistence equilibrium point.

The effect of varying the conversion rate from first prey by susceptible predator w_4 on the dynamical behavior of system (2) is investigated by choosing $w_4 = 0.3,\ 0.5,\ 0.8$ keeping other parameter fixed as given in equation (36) and then the solutions of the system (2) are drawn in figures (6a)-(6b).

From the above figures, it is observed that, as the conversion rate between first prey and susceptible predator w_4 increases the all prey species starting decreases and all predator species increases but the system still has an asymptotically stable coexistence point.

Now, the dynamical behavior of system (2) under the effect of varying the contact infection rate w_6, w_{10} is investigated. The system (2) is solved numerically for the set of parameters values given by equation (36) with $w_6 = w_{10} = 0.3,\ 0.4,\ 0.7$ and then the trajectories of the system (2) are drawn in figures (7a)-(7c).

Again, the system (2) has an asymptotically stable coexistence equilibrium point. In addition it is observed that, as the contact infection rate increases the second prey species and infected predator species started increase while the value of the first prey species and susceptible predator species decrease.

Now, the effect of varying recovery rate $w_7 = w_{11}$ on the dynamical behavior of parameters values given in equation (36) with $w_7 = w_{11} = 0.1,\ 0.3,\ 0.5$ and then the results are shown in figures (8a)-(8c).

Clearly, from the above figures, it is observed that increasing the value of recovery rate causes decreasing in the value of the second prey species and infected predator species while the values of the first prey species and susceptible predator species increasing. And then the system (2) has approaches asymptotically to the E_{xyz} equilibrium point.

VI. Conclusions And Discussion

In this paper, an eco-epidemiological model has been proposed and analyzed to study the dynamical behavior of a Holling-type II prey–predator model with the disease in predator species. The model consists of fore non-linear autonomous differential equations that describe the dynamics of fore different population's namely first prey (X), second prey (Y), susceptible predator (Z), infected predator (W). In order to confirm our analytical results and understand the effect of varying the infection rate $(w_i, i = 6{,}10)$ and recovery rate $(w_i, i = 7{,}11)$, on the dynamical behavior of the system (2), system (2) has been solved numerically for different sets of initial points and different sets of parameters and the following observations are made:

1. For the set of hypothetical parameters values given by Equation (36), the system (2) approaches asymptotically to globally stable point $E_{xyzw} = (0.06, 0.37, 0.62, 0.17)$.
2. For the value of the half saturation constant rate of susceptible of first prey w_1 increase the solution of system (2) approaches asymptotically to the coexistence equilibrium point and increase in values of x, z and w while decrease value of y.
3. For the values of the growth rate w_2 increase then the system (2) still approaches to coexistence

equilibrium point and the values of y, z and w increase while the value of x decrease.

4. The value of attack rate parameter w_3 increase then the system (2) still approaches to E_{xyzw} equilibrium point. And the value of x increase but the values of y, z and w decrease.
5. For increasing the value of conversion rate w_4 leads to increase in the values of z, w while decreasing in the values of x,y species.
6. In addition it is observed that, the system (2) has an asymptotically stable coexistence equilibrium point, as the contact infection rate increases the values of y, w species started increase while the value of x, z species decrease.
7. It is observed that increasing the value of recovery rate causes decreasing in the value of y, w species while the values of x, z species increasing. And then the system (2) has approaches asymptotically to the E_{xyz} equilibrium point.

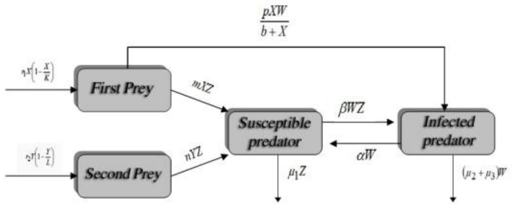


Figure (1): Bloke diagram of our proposed model.

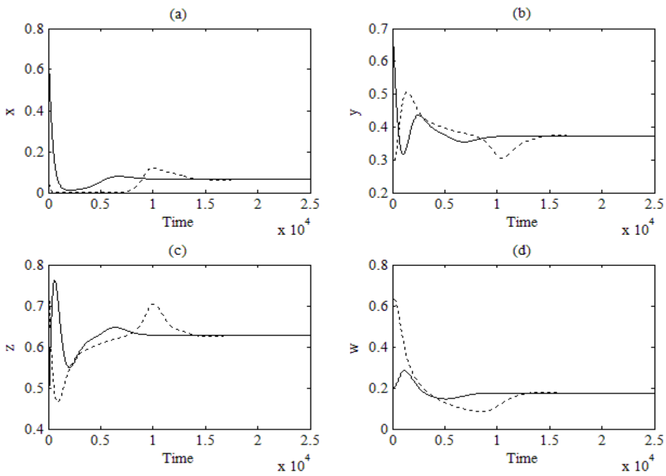
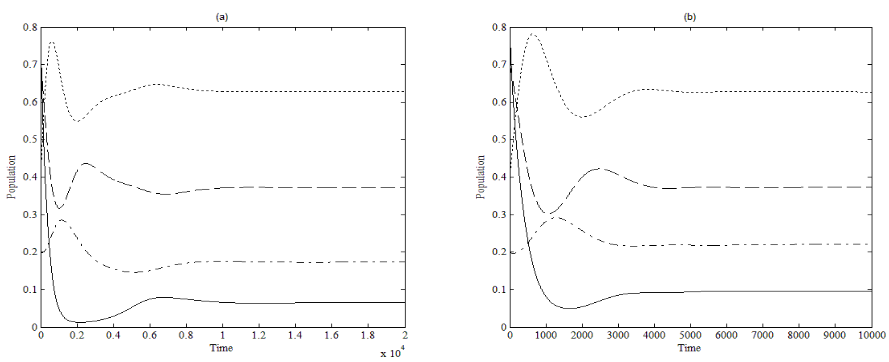


Figure (2): The solution of system (2) approaches asymptotically to the positive equilibrium point $E_{xyzw} = (0.06, 0.37, 0.62, 0.17)$ for that data given by Eq. (36) starting from two different initial points (0.8, 0.7, 0.4, 0.2) and (0.1, 0.3, 0.8, 0.6) for sold line and dashed line respectively. (a) Trajectories of x . (b) Trajectories of y . (c) Trajectories of z . (d) Trajectories of w .



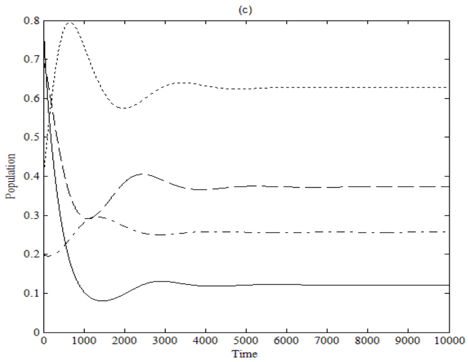


Figure (3): Time series of solutions of the system (2). (a) *for* $w_1 = 0.5$, (b) *for* $w_1 = 0.7$, (c) *for* $w_1 = 0.9$.

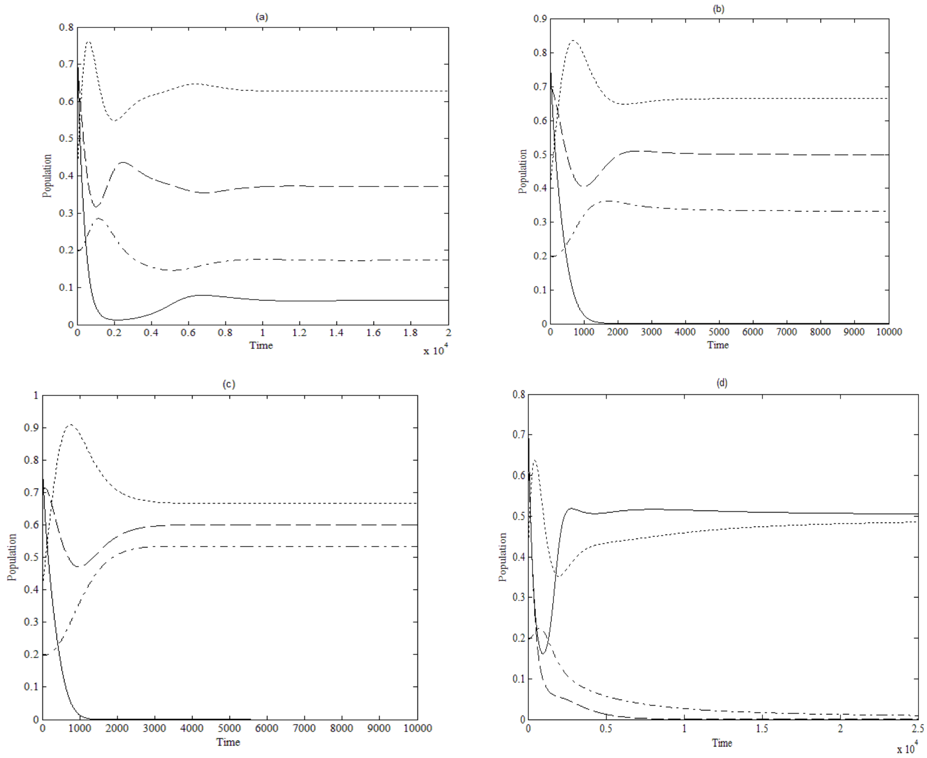
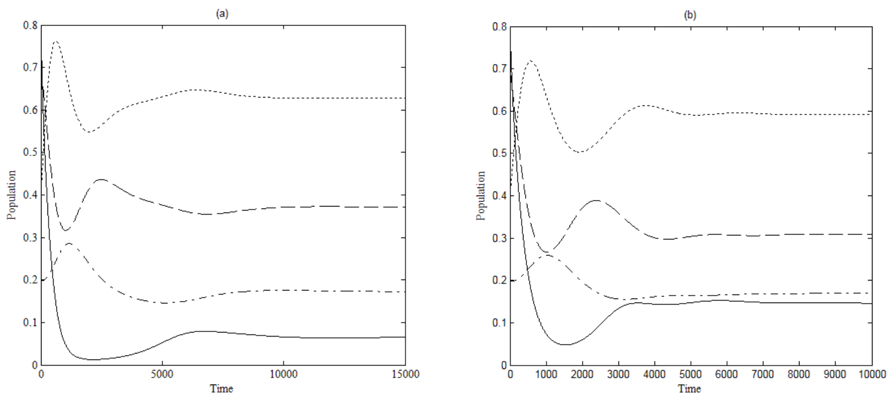


Figure (4): Time series of solutions of the system (2). (a) *for* $w_2 = 0.6$, (b) *for* $w_2 = 0.8$, (c) *for* $w_2 = 1$, (d) *for* $w_2 = 0.2$.



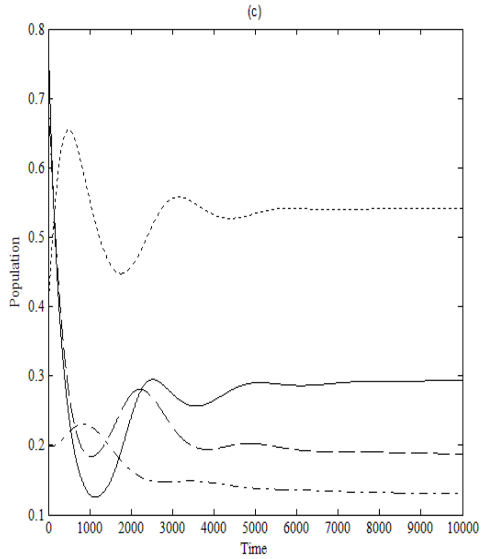


Figure (5): Time series of solutions of the system (2). (a) *for* $w_3 = 0.6$, (b) *for* $w_3 = 0.7$, (c) *for* $w_3 = 0.9$.

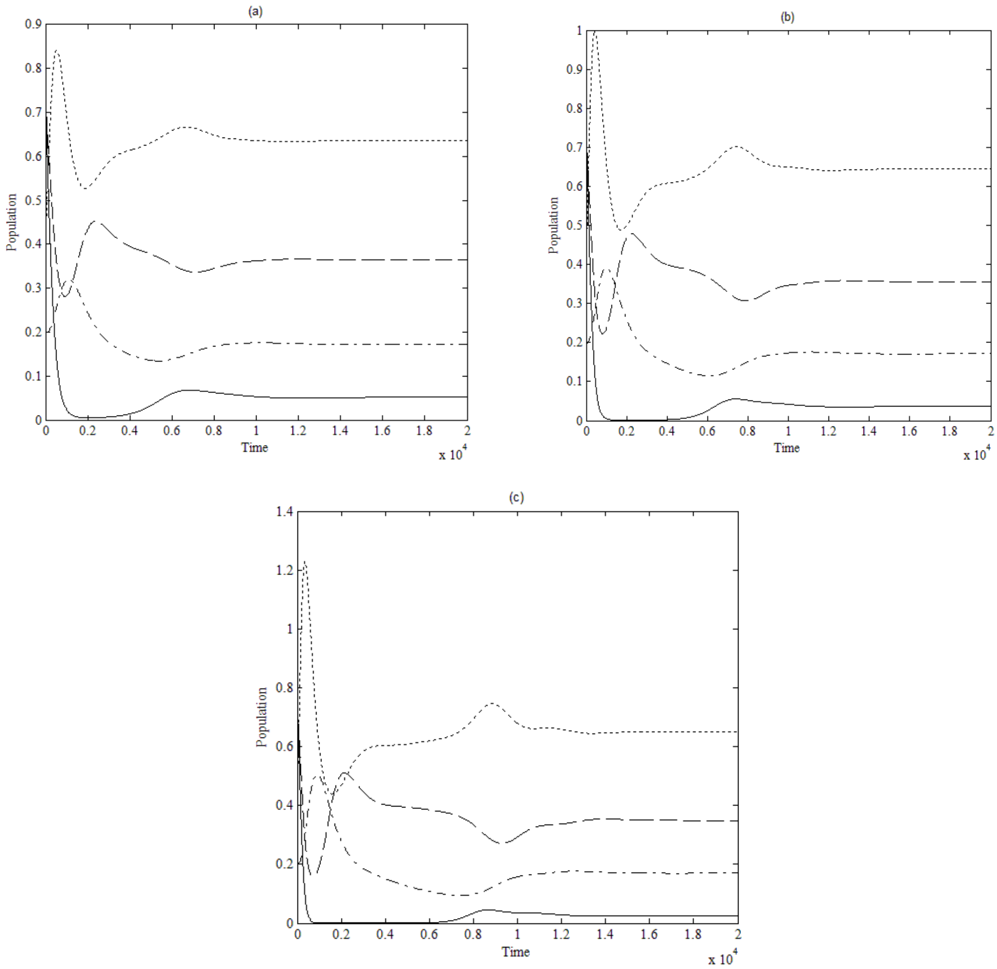


Figure (6): Time series of solutions of the system (2). (a) *for* $w_4 = 0.3$, (b) *for* $w_4 = 0.5$, (c) *for* $w_4 = 0.8$.

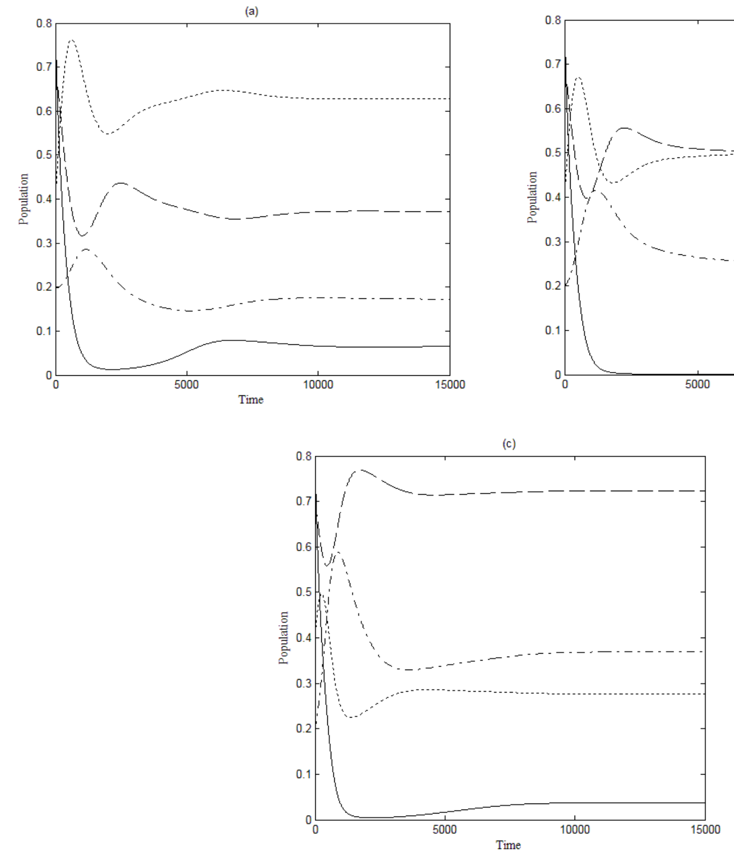


Figure (7): Time series of solutions of the system (2). (a) *for* $w_6 = w_{10} = 0.3$, (b) *for* $w_6 = w_{10} = 0.4$, (c) *for* $w_6 = w_{10} = 0.7$.

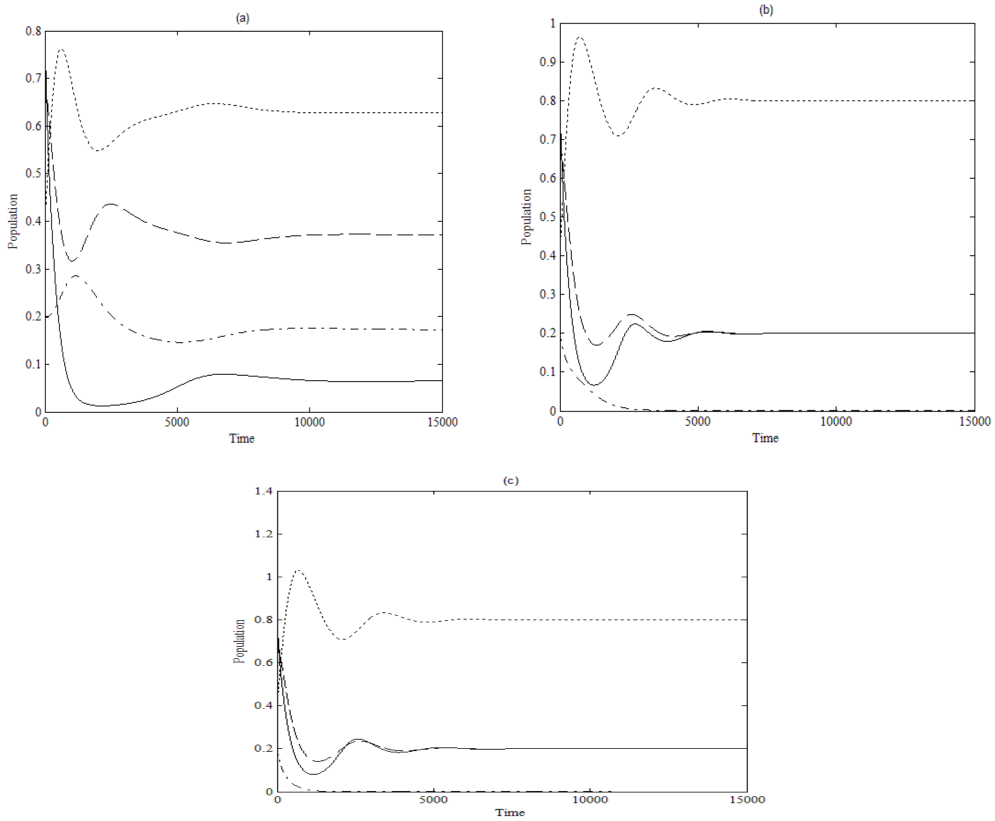


Figure (8): Time series of solutions of the system (2). (a) $for\ w_7 = w_{11} = 0.1$, (b) $for\ w_7 = w_{11} = 0.3$, (c) $for\ w_7 = w_{11} = 0.5$.

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