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Elzaki Transform and Integro-Differential Equation with a Bulge Function

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Abstract: *The aim of this paper, is to study the integro-differential equations with a bulge function, to find the exact solution we use Elzaki transform, inverse Elzaki transform and the convolution theorem. This method is more efficient and easy to handle such partial differential equations and integro-differential equations with a bulge function in comparison to other methods. The result showed the efficiency, accuracy and validation of Elzaki transform method.*
Keywords: *Elzaki transform, Integro-differential equations, convolution theorem*

I. Introduction

Nonlinear equations are of great importance to our contemporary world. Nonlinear phenomena have important applications in applied mathematics, physics, and issues related to engineering. Despite the importance of obtaining the exact solution of nonlinear partial differential equations in physics and applied mathematics there is still the daunting problem of finding new methods to discover new exact or approximate solutions.

In the recent years, many authors have devoted their attention to study solutions of nonlinear partial differential equations using various methods. Among these attempts are the Adomian decomposition method, homotopy perturbation method, variational iteration method [1-5], Laplace variational iteration method [6-8] differential transform method, Elzaki transform[14-17] and projected differential transform method.

Many analytical and numerical methods have been proposed to obtain solutions for nonlinear PDEs with fractional derivatives, such as local fractional variational iteration method [9], local fractional Fourier method, Yang-Fourier transform and Yang-Laplace transform and other methods. Two Laplace variational iteration methods are currently suggested by Wu in [10-13].

The main purpose of their work is to provide a new numerical approach based on the use of continuous collocation Taylor polynomials for the numerical solution of delay integro-differential equations. In this paper, we study the integro-differential equations with a bulge function. The solution is derived by using Elzaki transform, inverse Elzaki transform, the convolution theorem and the Taylor series expansion.

Definition 1.
The Elzaki Transform [2]. Given a function $f(t)$ defined for all $t \geq 0$, the Laplace transform of f is the function F defined as follow:

$$E[f(t),v] = T(v) = v\int_0^t f(t)e^{-\frac{t}{v}}dt \quad , \quad v \in (k_1, k_2) \qquad (1)$$

for all values of s for which the improper integral converges

Theorem 1. The Convolution Theorem [3].
Let $f(t)$ and $g(t)$ be defined in A . having Elzaki transform $M(v)$ and $N(v)$ then the Elzaki transform of the convolution of $f(t)$ and $g(t)$ is,

$$E[(f*g)(t)] = \frac{1}{v} M(v)N(v) \qquad (2)$$

Where

$$(f*g)(t) = \int_0^t f(x-t)g(t)dt$$

whenever the integral is defined.
The Integro-differential equation is an equation that involves both integrals and derivatives of an unknown function of the form:

$$\frac{d}{dx}y(t) + \int_{x_0}^{x} f(t, y(t))dt = g(x, y(x)) \qquad y(x_0) = x_0 \qquad (3)$$

The Trapezoidal rule can be used for the numerical solution of the integro-differential equation as follows:

$$\int_{x_k}^{x_k+h} f(x,y(x))dx = h[f(x_k,y_k) + f(x_{k+1},y_k + hy_k')] + O(h^3) \qquad (4)$$

And,

$$\int_{x_k}^{x_k+h}\int_{x_0}^{x} F(x,s,y(s))dsdx = \frac{h}{2}\left[\int_{x_0}^{x} f(x_k,s,y(s)) + \int_{x_0}^{x} f(x_{k+1},y_k + hy_k')\right] \ (5)$$

II. Solution of the Integro-Differential Equation With a Bulge Function by Using Elzaki Transform

Lemma 1.

The Elzaki transform of the bulge function $e^{-\frac{(t-l)^2}{2}}$ is expressed by.

$$E\left\{e^{-\frac{(t-l)^2}{2}}\right\} = e^{-\frac{l^2}{2}}[v^2 + lv^3 + (-1+l^2)v^4 + (-3l+l^3)v^5] \qquad (6)$$

Proof.

The Taylor series expansion e^x is of the form

$$e^{-\frac{(t-l)^2}{2}} = \sum_{n=0}^{\infty}\frac{x_n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots \qquad (7)$$

Therefore, by substituting equation (7) with $x = -\frac{(t-l)^2}{2}$, we obtain

$$e^{-\frac{(t-l)^2}{2}} = e^{-\frac{l^2}{2}} + e^{-\frac{l^2}{2}}lt + e^{-\frac{l^2}{2}}\left(-\frac{1}{2}+\frac{l^2}{2}\right)t^2 + e^{-\frac{l^2}{2}}\left(-\frac{l}{2}+\frac{l^3}{6}\right)t^3 \qquad (8)$$

By taking the Elzaki transform to equation (8) and using the fact that the Elzaki transform is linear, we derived,

$$E\left\{e^{-\frac{(t-l)^2}{2}}\right\} = e^{-\frac{l^2}{2}}[v^2 + lv^3 + (-1+l^2)v^4 + (-3l+l^3)v^5] \qquad (9)$$

Lemma 2.

The solution of the integro differential equation with a bulge function,

$$\frac{dy}{dt} = e^{-\frac{(t-l)^2}{2}} + \int_0^t y(t-u)\cos u\, du \quad , \ y(0) = \delta$$

can be expressed by:

$$y(t) = \frac{e^{-\frac{l^2}{2}}}{720}t[720 + 360tl + 120t^2l^2 + 30(-3l+l^3)t^3 + 6(-1+l^2)t^4 + (-3l+l^3)t^5] + \frac{\delta(2+t^2)}{2} (10)$$

Proof.

By taking the Elzaki transform to the above equation, we have,

$$E\left\{\frac{dy}{dt}\right\} = E\left\{e^{-\frac{(t-l)^2}{2}}\right\} + E\left\{\int_0^t y(t-u)\cos u\, du\right\} \qquad (11)$$

Applying the convolution theorem, it yields,

$$E\left\{\frac{dy}{dt}\right\} = E\left\{e^{-\frac{(t-l)^2}{2}}\right\} + E\{y(t)\}E\{\cos t\} \qquad (12)$$

And again by applying the convolution theorem and Lemma 1 to equation (12), we obtain,

$$\frac{T(v)}{v} - v f(0) = e^{-\frac{l^2}{2}}[v^2 + lv^3 + (-1+l^2)v^4 + (-3l+l^3)v^5] + \frac{1}{v}\left[T(v)\frac{v^2}{1+v^2}\right] \qquad (13)$$

Or,

$$T(v) = e^{-\frac{l^2}{2}}[v^2 + lv^3 + (-1+l^2)v^4 + (-3l+l^3)v^5]\frac{v+v^3}{1} + \delta\frac{v^2+v^4}{1} \qquad (14)$$

We can next use the partial fraction method to equation (14), we have,

$$E[f(t)] = e^{-\frac{l^2}{2}}[v^3 + lv^4 + l^2v^5 + (-3l+l^3)v^6 + (-1+l^2)v^7 + l(-3l+l^3)v^8] + \delta(v^2+v^4) \quad (15)$$

Then, the inverse Elzaki transform can be employed to equation (14) to Obtain,

$$y(t) = \frac{e^{-\frac{l^2}{2}}}{720} t[720 + 360tl + 120t^2 l^2 + 30(-3l + l^3)t^3 + 6(-1 + l^2)t^4 + (-3l + l^3)t^5] + \frac{\delta(2+t^2)}{2} \quad (16)$$

Example

we consider the integro-differential equation with a bulge function from lemma 2. which is,

$$\frac{dy}{dt} = e^{-\frac{(t-l)^2}{2}} + \int_0^t y(t-u)\cos u \, du \quad , \; y(0) = 1$$

by fixing $l = 2, 6$, $\delta = 1$ and $h = 0.1$ in the trapezoidal rule, we compare the exact solution from equation (16) and the approximate solution obtained by the trapezoidal rule [4] as shown graphically.

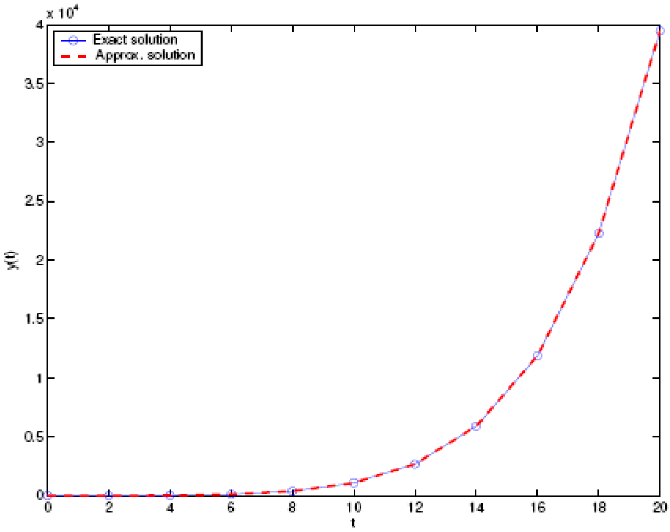

Figure 1: Exact solution and numerical solution of example for l = 2 and h = 0:1.

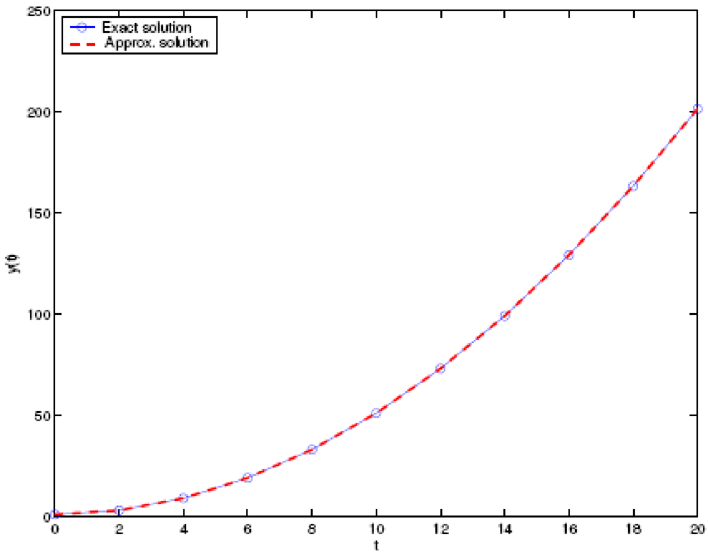

Figure 2: Exact solution and numerical solution of example for l = 6 and h = 0:1.

III. Conclusion

In this work, we studied the integro-differential equations with a bulge function. We applied the trapezoidal rule for solving the numerical solutions. To approach the exact solution, we employed Elzaki transform, inverse Elzaki transform, Taylor series expansion and the convolution theorem. We can conclude, according to our examples, that the approximate solutions obtained by the trapezoidal rule in good agreement with the exact solution.

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