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Radon Measure on Compact Topological Measurable Space

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Abstract: *In this paper we study Radon measure on compact topological measurable space, particularly on measurable regular space, second countable measurable space and e-normal space.*
Key words: *Regular measurable space, Regular measure space, second countable measurable space, e-normal space.*

I. Introduction

A concept of measure manifold has been introduced by the author S. C. P. Halakatti [6]. In this paper, we are studying Radon measure on measurable regular space, second countable measure space and e-normal space. This study is a base-work for deriving some properties of measures and extending Radon measure on Measure Manifolds.

II. Preliminaries

Definition 2.1: σ–algebra: A $\sigma -$algebra Σ on a topological space (R^n,τ) is a collection of subsets of (R^n, τ) such that

(i) $\emptyset$,$R^n \in\Sigma$

(ii) $A \in \Sigma$, then $A^c \in\Sigma$

(iii) If $A_i \in\Sigma$, for $i \in N$, then $\bigcup_{i=1}^{\infty} A_i \in\Sigma$, $\bigcap_{i=1}^{\infty} A_i \in\Sigma$.

The triplet (R^n, τ, Σ) is called a measurable space.

Definition 2.2: Borel σ–algebra: The Borel $\sigma -$algebra $B(R^n)$ on (R^n, τ) is the smallest $\sigma -$algebra generated by the open sets belonging to τ such that $B(R^n) = \Sigma(\tau(R^n))$. A set that belongs to the $\sigma -$algebra is called a Borel set.

Definition 2.3: Topological Measurable Space: The space (R^n, τ, Σ) is called a topological measurable space if the space (R^n, τ) is a topological space equipped with $\sigma -$algebra where the members of τ which belongs to $\sigma -$algebra Σ are the Borel sets in (R^n, τ, Σ).

Definition 2.4: Measurable Hausdorff Space: The space (R^n, τ, Σ) is called a measurable Hausdorff space provided that if x and y are distinct members of (R^n, τ, Σ) then, there exists disjoint Borel open sets A and B such that $x \in A$ and $y \in B$.

Definition 2.5: Measure space: A measure μ on a measurable space (R^n, τ, Σ) is a function
$\mu : \Sigma \longrightarrow [0,\infty]$ such that

a) $\mu(\emptyset)=0$

b) If $\{A_i \in \Sigma : i \in N\}$ is a countable disjoint collection of sets in Σ, then

$\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i)$ ………. $\sigma -$additivity property.

Therefore the space (R^n, τ, Σ, μ) is called a measure topological space.

Definition 2.6: Locally compact topological space: A topological space (R^n, τ) is locally compact if for all $x \in R^n$ there exists an open neighborhood $V \subset R^n$ of x such that $\overline{V}$ is compact. (Alternatively, this is equivalent to requiring that to each $x \in R^n$ there exists a compact neighborhood N_x of x.

Definition 2.7: Compact Support of a function: Compact support of a function f is the closure of the set $\{ x \in R^n : f(x) \neq 0\}$. It is denoted by suppf.

Definition 2.8: Locally Finite Measure on (R^n, τ, Σ, μ): The measure μ is locally finite if every point x of (R^n, τ, Σ, μ) has a neighborhood A of x for which $\mu(A) < \infty$.

Definition 2.9: Regular Measure on (R^n, τ, Σ, μ): A regular measure on (R^n, τ, Σ, μ) is a measure for which every Borel set can be approximated from above by an open measurable set and from below by a compact measurable set. i.e., a measure is called regular if it is outer regular and inner regular.

Definition 2.10: Radon Measure on (R^n, τ, Σ , μ): A Radon Measure on a topological measure space (R^n, τ, Σ, μ) is a positive Borel measure

$$\mu : B \longrightarrow [0,\infty] \text{ such that}$$

1. Locally finite on compact sets i.e., the measure μ is locally finite if every point $p \in (R^n, \tau, \Sigma, \mu)$ has a neighborhood A of p for which $\mu(A) < \infty$;
2. Inner regular with respect to compact sets i.e., the measure μ is called inner regular if, for any Borel set A,

$\mu(A) = \sup\{\mu(K_i) : i \in I; K_i \subseteq A ; K_i \text{ Compact}\}$. We designate Radon measure as μ_R.

Example 2.1: Lebesgue measure on R^n is a Radon measure.

Definition 2.11: **Locally compact Hausdorff topological space**

A topological space is locally compact Hausdorff, if for all p≠q ∃ open neighborhoods A and B belonging to (R^n, τ), $A \subset \bar{A}$ and $B \subset \bar{B}$ where $\bar{A}$ and $\bar{B}$ compact subsets belonging to (R^n, τ), such that $p \in \bar{A}$, $q \in \bar{B}$ and $\bar{A} \cap \bar{B} = \emptyset$.

Definition 2.12: Locally compact Hausdorff regular measurable space

A topological space is locally compact Hausdorff regular measurable space, if for all $p \in \bar{A}$, ∃ open Borel neighborhoods A and B belonging to (R^n, τ, Σ) such that $A \subset \bar{A}$ and $B \subset \bar{B}$ where $\bar{A}$ and $\bar{B}$ compact subsets belonging to (R^n, τ), and ∃ closed Borel subset $F \subseteq \bar{B}$ such that $\bar{A} \cap \bar{B} = \emptyset$.

Definition 2.13: Second countable measure space (e-second countable)
A measure space (R^n, τ, Σ, μ) is second countable provided there is a countable base for all q∈ (R^n, τ, Σ, μ) satisfying the following conditions:
(i) for each $q_i \in (R^n, \tau, \Sigma, \mu)$ there exist $B_i \in \mathcal{B}_q$ and $A_i \in \Sigma$ such that $q_i \in B \subseteq A_i$, for i=1,2,….
(ii) for $B_i \subseteq A_i$, $\mu(B_i) \leq \mu(A_i)$.

Definition 2.14: e-Normal Space (e-Normal) [7]
A measure space(R^n, τ, Σ, μ) is said to be e-normal measure space (or e-normal) if each pair of disjoint F_σ-sets A and B in (R^n, τ, Σ, μ), ∃ a pair of disjoint G_δ-sets U and V such that
$A \subset U, B \subset V$.

Proposition 2.1: Suppose that (R^n, τ) is a Hausdorff topological space, $K \subset (R^n, \tau)$ and $x \in K^c$. Then there exists $U, V \in (R^n, \tau)$ such that $U \cap V = \emptyset$, $x \in U$ and $K \subset V$.
In particular, K is closed (so compact subsets of Hausdorff topological spaces are closed).
More generally if K and F are two disjoint compact subsets of (R^n, τ), there exist disjoint open sets $U, V \in (R^n, \tau)$ such that $K \subset V$ and $F \subset U$.

Theorem 2.1: [7]: An e-compact Hausdorff measure space (R^n, τ, Σ, μ) is e-normal.

III. Main Results

Definition 3.1: Regular measurable space
The space (R^n, τ, Σ) is called measurable regular if given any point $p \in (R^n, \tau)$ and closed set $F \in \Sigma$ in (R^n, τ, Σ) such that $p \notin F$, ∃ Borel open sets A and $B \in \Sigma$ such that for all $p \in A$, $F \subset B$ and $A \cap B = \emptyset$.

Definition 3.2: Regular measure space (e-regular)
A measure μ on a measurable regular space (R^n, τ, Σ) is a function $\mu : \Sigma \longrightarrow [0,\infty]$ given any point p and Borel closed set $F \in \Sigma$ and $p \notin F$, ∃ Borel open sets A and B ∈ Σ such that $p \in A$,

$A \subset (R^n, \tau, \Sigma)$ satisfying the measure conditions:

(i) $\mu(\emptyset) = 0$

(ii) $\mu(A) > 0$, ∀ p ∈ A and μ(F)≤ μ(B) such that $\mu(A \cap$B) = 0.

Definition 3.3: Radon measure on regular measurable space
Let (R^n, τ, Σ) be a locally compact Hausdorff regular measurable space. A Radon measure μ_R on locally compact Hausdorff regular measurable space is defined as follows:

∀ p ∈ $\bar{A}$ and Borel closed set F in (R^n, τ, Σ), ∃ Borel open set A ⊂ $\bar{A}$ and B ⊂ $\bar{B}$ ∈ Σ , ∋ p ∈ A , F ⊂ $\bar{B}$, p ∉ F and

$\bar{A} \cap \bar{B}$, = ∅ satisfying the Radon measure conditions:

I. For p ∈ $\bar{A}$, (i) ∀ p ∈ $\bar{A}$ ⊂ (R^n, τ, Σ), $\mu_R(\bar{A}) < \infty$;

(ii) For any Borel open set A ⊂ $\bar{A}$

$\mu_R(\bar{A})$ = sup {$\mu_R(E_i)$, i ∈ I: $E_i \subseteq \bar{A}$: E_i compact}

II. For F ⊂ B, (i) ∀ q ∈ F⊂ (R^n, τ, Σ), μ_R(B)< ∞;

(ii) For any Borel open set B,

$\mu_R(\bar{B})$ = sup {$\mu_R(F_i)$, i ∈ I: $F_i \subseteq \bar{B}$: F_i compact}.

Theorem 3.1: A second countable regular measure space (R^n, τ, Σ, μ) is e-normal.
Proof: Let (R^n, τ, Σ) be a loc ally compact Hausdorff second countable regular measurable space.

To prove (R^n, τ, Σ, μ) is e-normal. Let $\mathfrak{B}_q$ be a countable base of open Borel sets.

Let $\bar{A}$, $\bar{B}$ ⊆ (R^n, τ, Σ, μ) be non-empty disjoint F_σ - sets.

For each q ∈ A, ∃ an open Borel set containing q whose closure is disjoint from B.

A closed Borel set $\bar{A}$ is covered by a countable family {$\bar{A}_1$, $\bar{A}_2$,….} of open Borel sets which belongs to $\mathfrak{B}_q$ such that $A_i \subset \bar{A}_i$, for 1,2,… such that $\overline{A_n} \cap \bar{B} = \emptyset$ satisfying the measure conditions:

(i) $\mu(\bar{A}) > 0$ and $\mu(\overline{A_n} \cap \bar{B}) = \mu(\emptyset) = 0$

(ii) $\mu(\bar{A}) \leq \mu(\bigcup_{n=1}^{\infty} \bar{A}_n)$

$= \sum_{n=1}^{\infty} \mu(\bar{A}_n)$ ……. σ −additivity property

Also, the closed Borel set B is covered by a countable family {B_1, B_2,….} of open Borel sets which belongs to $\mathfrak{B}_q$ such that $\overline{B_n} \cap \bar{A} = \emptyset$ satisfying the measure conditions:

(i) $\mu(\bar{B}) > 0$ and $\mu(\overline{B_n} \cap \bar{A}) = \mu(\emptyset) = 0$

(ii) $\mu(\bar{B}) \leq \mu(\bigcup_{n=1}^{\infty} \bar{B}_n)$

$= \sum_{n=1}^{\infty} \mu(\bar{B}_n)$ ……. σ −additivity property.

Now, we define open Borel sets U_1, U_2,.... and V_1, V_2,.... as follows:

U_n=A_n \ ($\overline{B_1}$ ∪…∪ $\overline{B_n}$), V_n=B_n \ ($\overline{A_1}$ ∪…∪ $\overline{A_n}$) and let $U = \bigcup_{n=1}^{\infty} U_n$ and $V = \bigcup_{n=1}^{\infty} V_n$ satisfying the measure conditions:

$\mu(U) = \mu(\cup_{n=1}^{\infty} U_n) = \sum_{n=1}^{\infty} \mu(U_n)$ and $\mu(V) = \mu(\cup_{n=1}^{\infty} V_n) = \sum_{n=1}^{\infty} \mu(V_n)$.

Since, $A \subset \bar{A}$, $B \subset \bar{B}$. then, $A \subseteq U, B \subseteq V$ and $U \cap V = \emptyset$ satisfying the measure conditions:

$\mu(A) \leq \mu(U)$ and $\mu(B) \leq \mu(V)$ and $\mu(U \cap V) = \mu(\emptyset) = 0$.

Hence, (R^n, τ, Σ, μ) is a e-normal space for arbitrary measure μ.

Therefore, a second countable regular measure space is e-normal.

In the following theorem, we study a second countable regular measure space as e-normal with Radon measure as a special case.

Theorem 3.2: A second countable measure space is e-normal with Radon measure.

Proof: Let (R^n, τ, Σ, μ) be a locally compact Hausdorff regular measure space. Let $\bar{A}, \bar{B} \subseteq (R^n, \tau, \Sigma, \mu)$ be non-empty disjoint F_σ – sets.

For each $q \in \bar{A}$,, $\exists$ an open Borel set containing q whose closure is disjoint from $\bar{B}$.

Each closed Borel set $\bar{A}$ is covered by a countable family $\{A_1, A_2, \ldots\}$ of open Borel sets which belongs to $\mathfrak{B}_q$ such that $A_i \subset \bar{A}_i$, for 1,2,… such that $\overline{A_n} \cap \bar{B} = \emptyset$ satisfying the Radon measure conditions:

(i) μ_R is locally finite, that is if $q \in \bar{A} \subseteq (R^n, \tau, \Sigma, \mu)$ then $\mu(\bar{A}_q) < \infty$,

(ii) μ_R is inner regular with respect to the compact Borel sets

$\mu(\bar{V}) = sup\{\mu_R(B_i) : i \in I; B_i \subseteq \bar{V} ;\ B_i\ compact\}$

Since the above theorem shows that $\mu(A) \leq \mu(U)$ and $\mu(B) \leq \mu(V)$ and $\mu(U \cap V) = \mu(\emptyset) = 0$ is for arbitrary measure μ where U and V are open subsets belonging to τ.

Now, we are applying Radon measure to e-normal space.

Let U_1, U_2,.... and V_1, V_2,.... are open Borel sets such that

$U_n = A_n \setminus (\overline{B_1} \cup \ldots \cup \overline{B_n})$, $V_n = B_n \setminus (\overline{A_1} \cup \ldots \cup \overline{A_n})$ and let $U = \cup_{n=1}^{\infty} U_n$ and $V = \cup_{n=1}^{\infty} V_n$

Such that

(i) $\mu(U) < \infty$,

(ii) $\mu(U) = sup\{\mu_R(\bar{A}_i): i \in I; \bar{A}_i \subseteq U ;\ \bar{A}_i\ compact\}$ and $\mu(A) \leq \mu(U)$ and

(i) $\mu(V) < \infty$,

(ii) $\mu(V) = sup\{\mu_R(\bar{B}_i): i \in I; \bar{B}_i \subseteq U ;\ \bar{B}_i\ compact\}$ and $\mu(A) \leq \mu(U)$

and $(U \cap V) = \mu(\emptyset) = 0$.

Hence, $A \subseteq U, B \subseteq V$ and $U \cap V = \emptyset$.

Therefore a locally compact Hausdorff regular second countable measure space, (R^n, τ, Σ, μ) is a e-normal space satisfying Radon measure.

IV. Conclusion

We study Radon measure on other topological properties which are locally compact and extend it on measure manifold which has lot of applications in the field of engineering science, brain and neural networks.

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