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Basic Minimal Dominating Functions of Euler Totient Cayley Graphs

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Abstract: *Graph Theory is the fast growing area of research in Mathematics. The concepts of Number Theory, particularly, the "Theory of Congruence" in Graph Theory, introduced by Nathanson[8], paved the way for the emergence of a new class of graphs, namely,* ***"Arithmetic Graphs". Cayley graphs*** *are another class of graphs associated with the elements of a group. If this group is associated with some arithmetic function then the Cayley graph becomes an Arithmetic graph.*
The Cayley graph associated with Euler Totient function is called an ***Euler Totient Cayley graph*** *and in this paper we study the Basic Minimal Domination Functions of Euler Totient Cayley graphs.*
Keywords: *Euler Totient Cayley Graph, Minimal Dominating Functions, Basic Minimal Dominating Functions.*

I. Introduction

The concept of the domination number of a graph was first introduced by Berge [4] in his book on graph theory. Ore [9] published a book on graph theory, in which the words 'dominating set' and 'domination number' were introduced. Allan and Laskar [1], Cockayne and Hedetniemi [5], Arumugam [2], Sampath kumar [10] and others have contributed significantly to the theory of dominating sets and domination numbers. An introduction and an extensive overview on domination in graphs and related topics are given by Haynes et al. [6].

Here we consider Euler totient Cayley graph $G(Z_n,\phi)$. First we present some results on minimal dominating functions of $G(Z_n,\phi)$ and prove that these functions are basic minimal dominating functions in certain cases. First we define the Euler totient Cayley graph.

II. Euler Totient Cayley Graph And Its Properties

Definition 2.1: The **Euler totient Cayley** graph is defined as the graph whose vertex set V is given by $Z_n=\{0,1,2,......,n-1\}$ and the edge set is E = {(x, y) / x – y ∈ S or y – x ∈ S} and is denoted by $G(Z_n,\varphi)$. where S denote the set of all positive integers less than n and relatively prime to n. That is $S=\{r/1\le r<n \ and\ GCD(r,n)=1\}$, $|S|=\varphi(n)$.

Now we present some of the properties of Euler totient Cayley graphs studied by Madhavi [7].

1. The graph $G(Z_n,\varphi)$ is $\varphi(n)$ – regular and has $\frac{n\varphi(n)}{2}$ edges.
2. The graph $G(Z_n,\varphi)$ is Hamiltonian and hence it is connected.
3. The graph $G(Z_n,\varphi)$ is Eulerian for n ≥ 3.
4. The graph $G(Z_n,\varphi)$ is bipartite if n is even.
5. The graph $G(Z_n,\varphi)$ is complete if n is a prime.

Uma Maheswari [11] has studied the dominating sets of Euler totient Cayley graphs. We present the results without proofs.

Theorem 2.2: If n is a prime, then the domination number of $G(Z_n,\varphi)$ is 1.

Theorem 2.3: If n is power of a prime, then the domination number of $G(Z_n,\varphi)$ is 2.

Theorem 2.4: The domination number of $G(Z_n,\varphi)$ is 2, if n = 2p, where p is an odd prime.

Theorem 2.5: When n is neither a prime nor 2p nor power of a prime, then the domination number of $G(Z_n,\varphi)$ is $\lambda+1$, where λ is the length of the longest stretch of consecutive integers in V, each of which shares a prime factor with n.

III. Basic Minimal Dominating Functions

Definition 3.1: Let G(V, E) be a graph. A function $f: V \to [0,1]$ is called a **dominating function (DF)** of G if $f(N[v]) = \sum_{u\in N[v]} f(u) \geq 1$ for each $v \in V$.

A dominating function f of G is called **minimal dominating function (MDF)** if for all $g < f$, g is not a dominating function.

Definition 3.2: A MDF f of a graph is called **basic minimal dominating function (BMDF)** if f cannot be expressed as a proper convex combination of two distinct MDFs.

Let f be a DF of a graph G(V, E). The **boundary set** of f is defined by

$$B_f = \left\{ u \in V / f(N[u]) = \sum_{x\in N[u]} f(x) = 1 \right\}.$$

Let f be a DF of a graph G(V, E). The **positive set** of f is defined by

$$P'_f = \{ u \in V / 0 < f(u) < 1 \}.$$

We present the following theorem without proof which is useful for obtaining subsequent results and the proof can be found in Arumugam and Rejikumar [3].

Theorem 3.3: Let f be a MDF of a graph G(V, E) with $B_f = \{v_1, v_2, ..., v_m\}$ and $P'_f = \{ u \in V / 0 < f(u) < 1 \} = \{u_1, u_2, ..., u_n\}$. Let A = (a_{ij}) be a n × n matrix defined by

$$a_{ij} = \begin{cases} 1, \text{ if } v_i \text{ is adjacent with } u_j \text{ or } v_i = u_j, \\ 0, \text{otherwise}. \end{cases}$$

Consider the system of linear equations given by

$$\sum_{j=1}^{n} a_{ij}\ x_j = 0, \text{ where } i = 1, 2, ..., n \quad .... (1)$$

Then f is a BMDF if and only if (1) does not have a non-trivial solution.

IV. Results

Theorem 3.4: A function $f: V \to [0,1]$ defined by

$$f(v) = \frac{1}{m},\ \forall\ v \in V,\ m > 0$$

becomes a DF of $G(Z_n, \varphi)$. It is a MDF if m = n. Otherwise not a MDF when n is a prime.

Proof: Consider $G(Z_n, \varphi)$.

Let $f: V \to [0,1]$ be defined by

$$f(v) = \frac{1}{m},\ \forall\ v \in V,\ m > 0.$$

We now show that f is a MDF.

Case 1: Suppose m = n.

Then $f(v) = \frac{1}{n},\ \forall\ v \in V.$

And $\sum_{u\in N[v]} f(u) = \underbrace{\frac{1}{n} + \frac{1}{n} + ... + \frac{1}{n}}_{n-times} = \frac{n}{n} = 1,\ \forall\ v \in V.$

Therefore f is a DF.

We now check for the minimality of f.
Let us define $g : V \to [0,1]$ such that

$$g(v) = \begin{cases} r, & if \ v = v_k \in V, \\ \frac{1}{n}, & if \ v \in V - \{v_k\}. \end{cases}$$

where 0 < r < 1/n.
Since strict inequality holds at v_k, we have $g < f$.
Now for every $v \in V$,

$$\sum_{u \in N[v]} g(u) = \underbrace{\frac{1}{n} + \frac{1}{n} + \dots\dots\dots\dots + \frac{1}{n}}_{(n-1)-times} + r$$

$$< \underbrace{\frac{1}{n} + \frac{1}{n} + \dots\dots\dots + \frac{1}{n}}_{n\ -times} = \frac{n}{n} = 1.$$

That is $\sum_{u \in N[v]} g(u) < 1, \ \forall v \in V.$

$\Rightarrow g$ is not a DF.
Since g is defined arbitrarily, it follows that there exists no $g < f$ such that g is a DF.
Hence f is a MDF.

Case 2: Suppose 0 < m < n.

Then $\sum_{u \in N[v]} f(u) = \underbrace{\frac{1}{m} + \frac{1}{m} + \dots\dots\dots + \frac{1}{m}}_{n-times} = \frac{n}{m} > 1, \ \forall \ v \in V.$

i.e., f is a DF.
Now we check for the minimality of f.
Let us define $g : V \to [0,1]$ such that

$$g(v) = \begin{cases} r, & if \ v = v_k \in V, \\ \frac{1}{m}, & if \ v \in V - \{v_k\}. \end{cases}$$

where 0 < r < 1/m.
Clearly $g < f$, since strict inequality holds at the vertex v_k of V.

And $\sum_{u \in N[v]} g(u) = \underbrace{\frac{1}{m} + \frac{1}{m} + \dots\dots\dots + \frac{1}{m}}_{(n-1)-times} + r.$

$$< \underbrace{\frac{1}{m} + \frac{1}{m} + \dots\dots\dots\dots + \frac{1}{m}}_{n-times} = \frac{n}{m} > 1, \ \forall \ v \in V.$$

So g is a DF.
This implies that f is not a MDF. ■

Theorem 3.5: A function $f : V \to [0,1]$ defined by

$$f(v) = \frac{1}{m}, \ \forall \ v \in V, \text{ m} > 0$$

becomes a DF of $G(Z_n, \varphi)$. It becomes a MDF if m = $|S| + 1$. Otherwise it is not minimal when is non – prime.

Proof: Consider $G(Z_n, \varphi)$ with vertex set V={0, 1, 2,…..,n − 1}.
It is $\varphi(n)$ - regular for any n.
Every neighbourhood N[v] of $v \in V$ consists of $|S|+1$ vertices.
Let $|S|+1 = k$, say.
Let f be a function defined as in the hypothesis.
Case 1: Suppose m = k.
Then $f(v) = \frac{1}{k}$, $\forall\ v \in V$.
And $$\sum_{u \in N[v]} f(u) = \underbrace{\frac{1}{k} + \frac{1}{k} + \ldots\ldots\ldots\ldots + \frac{1}{k}}_{k-times} = \frac{k}{k} = 1, \forall\ v \in V.$$
$\Rightarrow f$ is a DF.
We now check for the minimality of f .

Suppose $g : V \to [0,1]$ is a function defined by
$$g(v) = \begin{cases} r, & \text{if } v = v_k, \\ \frac{1}{k}, & \text{if } v \in V - \{v_k\}. \end{cases}$$
where 0 < r < 1/k.
Since strict inequality holds at $v = v_k$, it follows that $g < f$.
If $v_k \notin N[v]$, then
$$\sum_{u \in N[v]} g(u) = \underbrace{\frac{1}{k} + \frac{1}{k} + \ldots\ldots\ldots\ldots + \frac{1}{k}}_{k-times} = \frac{k}{k} = 1.$$
If $v_k \in N[v]$, then
$$\sum_{u \in N[v]} g(u) = \underbrace{\frac{1}{k} + \frac{1}{k} + \ldots\ldots\ldots\ldots + \frac{1}{k}}_{(k-1)-times} + r$$
$$= \frac{k-1}{k} + r < \frac{k-1}{k} + \frac{1}{k} = 1.$$
Thus $\sum_{u \in N[v]} g(u) < 1$.
$\Rightarrow g$ is not a DF.
Since g is defined arbitrarily, it follows that there exists no g < f such that g is a DF.
Hence f is a MDF of $G(Z_n, \varphi)$.
Case 2: Suppose 0 < m < k.
We have $$\sum_{u \in N[v]} f(u) = \underbrace{\frac{1}{m} + \frac{1}{m} + \ldots\ldots\ldots\ldots\ldots + \frac{1}{m}}_{k-times} = \frac{k}{m} > 1.$$
i.e., $\sum_{u \in N[v]} f(u) > 1, \ \forall\ v \in V$.
Therefore f is a DF.
Now we check for the minimality of f .
Define $g : V \to [0,1]$ by

$$g(v) = \begin{cases} r, & if\ v = v_k, \\ \dfrac{1}{m}, & if\ v \in V - \{v_k\}. \end{cases}$$

where 0 < r < 1/m.

Clearly g < f , since strict inequality holds at $v = v_k$.

If $v_k \in$ N[v], then

$$\sum_{u \in N[v]} g(u) = \underbrace{\frac{1}{m} + \frac{1}{m} + ...+ \frac{1}{m}}_{(k-1)-times} + r$$

$$< \frac{k-1}{m} + \frac{1}{m} = \frac{k}{m} > 1.$$

If $v_k \notin N[v]$, then

$$\sum_{u \in N[v]} g(u) = \underbrace{\frac{1}{m} + \frac{1}{m} + ...+ \frac{1}{m}}_{k-times} = \frac{k}{m} > 1.$$

Therefore it follows that $\sum_{u \in N[v]} g(u) > 1, \ \forall\ v \in V.$

$\Rightarrow$ g is a DF.

So f is not a MDF. ■

Theorem 3.6: Let $f : V \to [0,1]$ be a function defined by

$$f(v) = \frac{1}{r+1}, \forall\ \ v \in V, \text{ where } r \text{ denotes the degree of } v \in V.$$

Then f is a BMDF of $G(Z_n, \varphi)$, if n is not a prime.

Proof: Consider $G(Z_n, \varphi)$, when n is not a prime.

We know that $G(Z_n, \varphi)$ is $|S|-$ regular.

Then $f(v) = \dfrac{1}{r+1} = \dfrac{1}{|S|+1} = \dfrac{1}{k} < 1, \ \forall\ v \in V$, where k = $|S|+1.$

Then by Theorem 3.5, Case 1, f is a MDF.

And $\sum_{u \in N[v]} f(u) = \underbrace{\frac{1}{k} + \frac{1}{k} + ...+ \frac{1}{k}}_{(k-times)} = \frac{k}{k} = 1.$

Then B_f = $\left\{ u \in V / f(N[u]) = \sum_{x \in N[u]} f(x) = 1 \right\} = \{u_1, u_2, ..., u_n\}$, as there are n vertices in V.

And $P'_f = \{ u \in V / \ 0 < f(u) < 1 \} = \{u_1, u_2, ..., u_n\}.$

Now we get B_f = P'_f = V.

Let A = (a_{ij}) be a n × n matrix defined by

$$a_{ij} = \begin{cases} 1, & if\ v_i\ is\ adjacent\ with\ u_j\ or\ v_i = u_j, \\ 0, & otherwise. \end{cases}$$

Then the system of linear equations associated with f is defined by

$$\sum_{j=1}^{n} a_{ij}\ x_j = 0, \text{ where i = 1, 2,……, n.}$$

Since every neighbourhood N[v] of $v \in B_f$ consists of (r + 1) vertices of V,

we have a_{ij} = 1 for (r + 1) values, where 1 ≤ i ≤ n, 1 ≤ j ≤ n, and a_{ij} = 0 for the remaining n – (r + 1) values.

For $v_1 \in B_f$, we get $1.x_1 + 1.x_2 + \underbrace{0 + 0 + ... + 0}_{[n-(r+1)]-times} + 1.x_n = 0$

Similarly for $v_2 \in B_f$, $1.x_1 + 1.x_2 + 1.x_3 + \underbrace{0 + 0 + ... + 0}_{[n-(r+1)]-times} = 0$

....

....

$v_n \in B_f$, $1.x_n + 1.x_{n-1} + \underbrace{0 + 0 + ... + 0}_{[n-(r+1)]-times} + 1.x_1 = 0$.

The above system of equations has a trivial solution.

Therefore f is a BMDF. ■

Theorem 3.7: Let $f : V \to [0,1]$ be a function defined by

$$f(v) = \frac{1}{r+1}, \forall\ v \in V, \text{ where } r \text{ denotes the degree of } v \in V.$$

Then f is not a BMDF of $G(Z_n, \varphi)$, if n is a prime.

Proof: Consider $G(Z_n, \varphi)$, when n is a prime.

Let $f(v) = \frac{1}{r+1} = \frac{1}{|S|+1} = \frac{1}{n} < 1, \forall\ v \in V.$

By Theorem 3.4, Case 1, f is a MDF of $G(Z_n, \varphi)$.

We claim that f is a BMDF.

Then $\sum_{u \in N[v]} f(u) = \underbrace{\frac{1}{n} + \frac{1}{n} + ... + \frac{1}{n}}_{(n-times)} = \frac{n}{n} = 1, \forall\ v \in V.$

Then $B_f = \left\{ u \in V / f(N[u]) = \sum_{x \in N[u]} f(x) = 1 \right\} = \{u_1, u_2, ..., u_n\}$, as there are n vertices in V.

And $P'_f = \{u \in V / 0 < f(u) < 1\} = \{u_1, u_2, ..., u_n\}$.

Now we get B_f = P'_f = V.

Let A = (a_{ij}) be a n × n matrix defined by

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \text{ is adjacent with } u_j \text{ or } v_i = u_j, \\ 0, & \text{otherwise.} \end{cases}$$

Then the system of linear equations associated with f is defined by

$$\sum_{j=1}^{n} a_{ij}\ x_j = 0 \text{ , where } i = 1, 2, ……, n.$$

Since every vertex $v \in B_f$ is adjacent with all the n vertices of V we have a_{ij} =1 where 1 ≤ j ≤ n.

For $v_1 \in B_f$, $1.x_1 + 1.x_2 + ... + 1.x_n = 0$

$v_2 \in B_f$, $1.x_1 + 1.x_2 + ... + 1.x_n = 0$

…………………………………………

$v_n \in B_f$, $1.x_1 + 1.x_2 + ... + 1.x_n = 0.$

This implies that $1.x_1 + 1.x_2 + ... + 1.x_n = 0$, which has a non-trivial solution.

By Theorem 2.4.2, it follows that f is not a BMDF. ■

Illustration 3.8: Consider the graph $G(Z_{14}, \varphi)$ and the graph is given below.

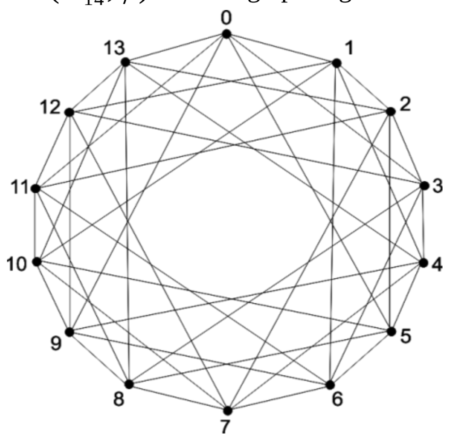

Figure 1: G(Z_{14}, φ)

It is a 6-regular graph. i.e., r = | S | = 6.

Define a function $f : V \to [0,1]$ by

$$f(v) = \frac{1}{r+1} = \frac{1}{7}, \forall\ v \in V.$$

We know that f is a MDF.

Then the summation values over the neighborhood N[v] of every vertex $v \in V$ is given below.

v :	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$f(v)$:	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$	$\frac{1}{7}$
$\sum_{u \in N[v]} f(u)$:	1	1	1	1	1	1	1	1	1	1	1	1	1	1

Therefore $\sum_{u \in N[v]} f(u) = 1, \forall\ v \in V.$

That is B_f = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13} = P'_f = V.

The system of linear equations are given by

$x_2 + x_4 + x_6 + x_{10} + x_{12} + x_{14} = 0$
$x_1 + x_3 + x_5 + x_7 + x_{11} + x_{13} = 0$
$x_2 + x_4 + x_6 + x_8 + x_{12} + x_{14} = 0$
$x_1 + x_3 + x_5 + x_7 + x_9 + x_{13} = 0$
$x_2 + x_4 + x_6 + x_8 + x_{10} + x_{14} = 0$
$x_1 + x_3 + x_5 + x_7 + x_9 + x_{11} = 0$
$x_2 + x_4 + x_6 + x_8 + x_{10} + x_{12} = 0$
$x_3 + x_5 + x_7 + x_9 + x_{11} + x_{13} = 0$
$x_4 + x_6 + x_8 + x_{10} + x_{12} + x_{14} = 0$
$x_1 + x_5 + x_7 + x_9 + x_{11} + x_{13} = 0$
$x_2 + x_6 + x_8 + x_{10} + x_{12} + x_{14} = 0$
$x_1 + x_3 + x_7 + x_9 + x_{11} + x_{13} = 0$
$x_2 + x_4 + x_8 + x_{10} + x_{12} + x_{14} = 0$
$x_1 + x_3 + x_5 + x_9 + x_{11} + x_{13} = 0.$

Solving these equations we get, $x_2 = x_4 = x_6 = x_8 = x_{10} = x_{12} = x_{14}$
$x_1 = x_3 = x_5 = x_7 = x_9 = x_{11} = x_{13}.$

Therefore the system has a trivial solution.

Hence f is a BMDF.

Illustration 3.9: Consider $G(Z_{19}, \varphi)$. The graph is shown below.

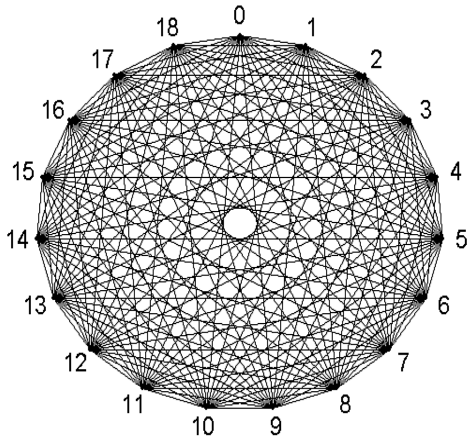

Figure 2: G(Z_{19}, φ)

It is 18-regular graph and so r = | S | = 18.

Let us define $f : V \rightarrow [0,1]$ such that

$$f(v) = \frac{1}{r+1} = \frac{1}{19}, \ \forall \ v \in V.$$

It is clearly a MDF.

Then the summation values over the neighborhood N[v] of every vertex $v \in V$ is given below.

v :	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$f(v)$:	$\frac{1}{19}$	$\frac{1}{19}$	$\frac{1}{19}$	$\frac{1}{19}$	$\frac{1}{19}$	$\frac{1}{19}$	$\frac{1}{19}$	$\frac{1}{19}$	$\frac{1}{19}$	$\frac{1}{19}$	$\frac{1}{19}$	$\frac{1}{19}$	$\frac{1}{19}$	$\frac{1}{19}$	$\frac{1}{19}$	$\frac{1}{19}$	$\frac{1}{19}$	$\frac{1}{19}$	$\frac{1}{19}$
$\sum_{u \in N[v]} f(u)$:	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1

$$\Rightarrow \sum_{u \in N[v]} f(u) = 1, \ \forall \ v \in V.$$

Then B_f = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18} = P_f' = V.

The system of linear equations are given by

For 0, 1, 2, 3, 4,..........,18 $\in B_f$,

1.x_1 +1.x_2 + 1.x_3 + 1.x_4 + 1.x_5 +.........+ 1.x_{18} +1. x_{19} = 0, which has a non-trivial solution.

That is f is not a BMDF.

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