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On a Finsler space with Binomial (α,β) - metrics

Ghanashyam Kr. Prajapati
(DST-Centre for Interdisciplinary Mathematical Sciences, Faculty of Science, Banaras Hindu University, Varanasi-221005, India)

Abstract: *In this paper, we study a class of (α,β)-Finsler metrics called Binomial (α,β)-metrics on an n-dimensional differential manifold M and get the conditions for such metrics to be Berwald, Douglas and Projectively flat. Further, we prove that a Binomial (α,β)-metric is of scalar flag curvature and isotropic S-curvature if and only if it is isotropic Berwald metric with almost isotropic flag curvature.*
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I. Introduction

Let (M,F) is a Finsler manifold, where M is an n-dimensional differential manifold and F is a Finsler metric on M. The (α,β)-metrics are interesting examples of Finsler metric introduced by M. Matsumoto as a generalization of Randers metric $F=\alpha+\beta$, where $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta=b_iy^i$ is a 1-form. A Finsler metrics $F(\alpha,\beta)$ on a differential manifold M is called an (α,β)-metric, if F is a positively homogeneous of degree one in α and β. In the present paper we study a Finsler metric $F=\alpha\phi(s)$, where $s=\dfrac{\beta}{\alpha}$ and $\phi(s)=(1+s)^{m+1}$ that is,

$$F=\frac{(\alpha+\beta)^{m+1}}{\alpha^m}, \tag{1.1}$$

where m is an arbitrary real number and called Binomial (α,β)-metrics. This class of (α,β)-metrics contains Randers metric $F=\alpha+\beta$ for $m=0$; Riemannian metric $F=\alpha$ for $m=-1$; Matsumoto metric $F=\dfrac{\alpha^2}{(\alpha-\beta)}$, if we replace β by $-\beta$ and take $m=-2$ and Z. Shen's square metric $F=\dfrac{(\alpha+\beta)^2}{\alpha}$ for $m=1$. Z. Shen's square metric is interesting in the sense that the metric

$$F(x,y)=\frac{(\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}+\langle x,y\rangle)^2}{(1-|x|^2)^2\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}},(x,y)\in TR^n;$$

constructed by L. Berwald in 1929, which is projectively flat on unit ball B^n with constant flag curvature $K=0$; can be written in form $F=\dfrac{(\alpha+\beta)^2}{\alpha}$ for some suitable α and β. Here $|\cdot|$ and $\langle,\rangle$ denote the standard Euclidean norm and inner product respectively on R^n and TR^n is tangent space on R^n.

The flag curvature in Finsler geometry is a natural extension of the sectional curvature in Riemannian geometry, which is first introduced by L. Berwald. In general, for a tangent plane $P=span(y,u)$, y and u are linearly independent vectors of tangent space T_xM of M at point $x\in M$, the flag curvature $K=K(P,u)$ depends on plane P as well as vector $u\in P$. A Finsler metric F is of scalar flag curvature if for any non-zero vector $y\in T_xM$, $K=K(x,y)$ is independent of P containing $y\in T_xM$. F is called of almost isotropic flag curvature if

$$K=\frac{3c_{x^m}y^m}{F}+\sigma, \tag{1.2}$$

where $c=c(x)$ and $\sigma=\sigma(x)$ are some scalar functions on M.

The S-curvature $S=S(x,y)$ in Finsler geometry is introduced by Shen [1] as a non-Riemannian quantity, defined as:

$$S(x,y)=\frac{d}{dt}[\tau(\sigma(t),\dot{\sigma}(t))]_{|t=0} \tag{1.3}$$

where $\tau=\tau(x,y)$ is a scalar function on $T_xM\setminus\{0\}$, called distortion of F and $\sigma=\sigma(t)$ be the geodesic with $\sigma(0)=x$ and $\dot{\sigma}(0)=y$.

A Finsler metric F is called of isotropic S-curvature if

$$S=(n+1)cF, \tag{1.4}$$

for some scalar function $c=c(x)$ on M. One of the fundamental problems in Riemann-Finsler geometry is to study and characterize Finsler metrics of scalar flag curvature with isotropic S-curvature. In [2], it is proved that if a Finsler metric F, of scalar flag curvature is of isotropic S-curvature, then it has almost isotropic flag curvature. A geodesic curve $c=c(t)$ of a Finsler metric $F=F(x,y)$ on a smooth manifold M is given by $\ddot{c}^i(t)+2G^i(c(t),\dot{c}(t))=0$, where the local functions $G^i=G^i(x,y)$ are called the spray coefficients given by $G^i=\frac{1}{4}g^{il}\{[F^2]_{x^k y^l}y^k-[F^2]_{x^l}\}$. A Finsler metric is called Berwald metric, if G^i are quadratic in $y\in T_xM$ for any $x\in M$. The Berwald curvature tensor of a Finsler metric F is defined as $B:=B^i_{jkl}dx^j\otimes\partial_i\otimes dx^k\otimes dx^l$, where $B^i_{jkl}=[G^i]_{y^j y^k y^l}$. A Finsler metric F is said to be isotropic Berwald metric if its Berwald curvature is in the following form

$$B^i_{jkl}=c(F_{y^j y^k}\delta^i_l+F_{y^k y^l}\delta^i_j+F_{y^l y^j}\delta^i_k+F_{y^j y^k y^l}y^i), \tag{1.5}$$

where $c=c(x)$ is a scalar function on M.

The E-curvature or mean Berwald curvature in Finsler geometry is defined as $E:=E_{ij}dx^i\otimes dx^j$, where $E_{ij}=\frac{1}{2}B^m_{mij}=\frac{1}{2}S_{y^i y^j}(x,y)=\frac{1}{2}\frac{\partial^2}{\partial y^i\partial y^j}\left[\frac{\partial G^m}{\partial y^m}\right]$. A Finsler metric F is said to be isotropic mean Berwald metric if its mean curvature is in the following form $E_{ij}=\frac{n+1}{2F}ch_{ij}$, where $c=c(x)$ is a scalar function on M and h_{ij} is the angular metric tensor. The metric tensor g_{ij} and Cartan tensor C_{ijk} of a Finsler metric F is defined as $g_{ij}=\frac{1}{2}\frac{\partial^2 F^2}{\partial y^i\partial y^j}$ and $C_{ijk}=\frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$. Also mean Cartan torsion I_y define by $I_y(u):=I_i(y)u^i$, where $I_i=g^{jk}C_{ijk}$. The horizontal covariant derivative of I along a vector $u\in T_xM$ gives rise to the mean Landsberg curvature $J_y(u):=J_i(y)u^i$, where $J_i=I_{i|s}y^s$.

In the present paper we prove the following theorems:

Theorem 1.1 *A Finsler space with Binomial* (α,β)*-metric* $F=\frac{(\alpha+\beta)^{m+1}}{\alpha^m}$ *is a Berwald space if and only if* $b_{i;j}=0$.

Theorem 1.2 *A Finsler space with Binomial* (α,β)*-metric* $F=\frac{(\alpha+\beta)^{m+1}}{\alpha^m}$ *is a Douglas space if and only if*

$b_{i;j}=0$, *provided* $m\neq 0,\pm 1$.

Remark: For $m=0$, the Bionomial (α,β)-metric (1.1) is a Randers metric $F=\alpha+\beta$. A Randers metric is Douglas if and only if β is closed [3] . For $m=1$, the Bionomial (α,β)-metric (1.1) reduces to a square metric $F=\frac{(\alpha+\beta)^2}{\alpha}$. The condition, for a square metric to be Douglas, has been studied in [4]. Finally for $m=-1$, the Bionomial (α,β)-metric (1.1) reduces to a Riemannian metric, which is trivially Douglas.

Theorem 1.3 *A Finsler space with Binomial* (α,β) *-metric* $F=\frac{(\alpha+\beta)^{m+1}}{\alpha^m}$ *is locally projectively flat if and only if* β *is parallel with respect to* α *and* α *is locally projectively flat.*

Theorem 1.4 *Let* $F=\frac{(\alpha+\beta)^{m+1}}{\alpha^m}$ *be a Binomial* (α,β) *-metric on* n *-dimensional Finsler manifold* M *. Then* F *is of scalar flag curvature with isotropic* S *-curvature if and only if it has isotropic Berwald curvature with almost isotropic flag curvature. In this case,* F *must be locally Minkowskian.*

II. Preliminaries

Let $\alpha=\sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric, $\beta=b_iy^i$ is a 1-form and let $F=\alpha\phi(s)$, $s=\frac{\beta}{\alpha}$, where $\phi=\phi(s)$ is a positive C^∞ function defined in a neighbourhood of the origin $s=0$. It is well known that $F=\alpha\phi(s)$ is a Finsler metric for any α and β with $b=\|\beta\|_\alpha<b_0$ if and only if

$\phi(s)>0$, $\phi(s)-s\phi'(s)+(b^2-s^2)\phi''(s)>0$, $(|s|\leq b<b_0)$.

Let G^i and G^i_α denote the spray coefficients of F and α respectively, given by

$$G^i=\frac{1}{4}g^{il}\left\{[F^2]_{x^ky^l}y^k-[F^2]_{x^l}\right\} \tag{2.1}$$

and

$$G^i_\alpha=\frac{1}{4}a^{il}\left\{[\alpha^2]_{x^ky^l}y^k-[\alpha^2]_{x^l}\right\} \tag{2.2}$$

where $(a^{ij})=(a_{ij})^{-1}$, $F_{x^k}=\frac{\partial F}{\partial x^k}$ and $F_{y^k}=\frac{\partial F}{\partial y^k}$.

Consider the following notations [1]

$$r_{ij}=\frac{1}{2}\{b_{i;j}+b_{j;i}\},\quad r^i_j=a^{ih}r_{hj},\quad r_j=b_ir^i_j,$$

$$s_{ij}=\frac{1}{2}\{b_{i;j}-b_{j;i}\},\quad s^i_j=a^{ih}s_{hj},\quad s_j=b_is^i_j,$$

$$b^i=a^{ih}b_h,\quad b^2=b^ib_i,$$

where $b_{i;j}$ is covarient derivative of b_i with respect to Levi-Civita connection of α.

Lemma (2.1) [1] The spray coefficients G^i are related to G^i_α by

$$G^i=G^i_\alpha+\alpha Qs^i_0+J(-2\alpha Qs_0+r_{00})\frac{y^i}{\alpha}+H(-2\alpha Qs_0+r_{00})(b^i-\frac{y^i}{\alpha}), \tag{2.3}$$

where $Q=\frac{\phi'}{\phi-s\phi'}$, $J=\frac{1}{2}\frac{(\phi-s\phi')\phi'}{\phi((\phi-s\phi')+(b^2-s^2)\phi'')}$, $H=\frac{1}{2}\frac{\phi''}{((\phi-s\phi')+(b^2-s^2)\phi'')}$

and subscript $'0'$ represents contraction with y^i, for instance, $s_0 = s_i y^i$.

A Finsler metric $F = F(x, y)$ on an open subset $U \subset \mathrm{R}^n$ is said to be projectively flat if all the geodesics are straight in U. In [5], it is shown that a Finsler metric $F = F(x, y)$ is projectively flat on an open subset $U \subset \mathrm{R}^n$ if and only if

$$F_{x^k y^l} y^k - F_{x^l} = 0. \tag{2.4}$$

In view of equation (2.3) and (2.4), we have the following lemma [6]

Lemma (2.2) An (α,β)-metric $F = \phi(s)$, where $s = \dfrac{\beta}{\alpha}$ is projectively flat on an open subset $U \subset \mathrm{R}^n$ if and only if

$$(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha + \alpha^3 Q s_{l0} + H\alpha(-2\alpha Q s_0 + r_{00})(b_l\alpha - s y_l) = 0. \tag{2.5}$$

Lemma (2.3) [7] If $(a_{ml}\alpha^2 - y_m y_l)G^m_\alpha = 0$, then α is projectively flat.

In [8], for a Finsler metric $F = \dfrac{(\alpha+\beta)^{m+1}}{\alpha^m}$, C. H. Xiang and X. Y. Cheng investigated:

Proposition (2.1) The following conditions are equivalent for the (α,β)-metrics (1.1)

(i) F is of isotropic S-curvature, $S = (n+1)cF$;

(ii) F is of isotropic mean Berwald curvature, $E = \dfrac{n+1}{2} c F^{-1} h$;

(iii) β is a Killing 1-form with b=constant with respect to α, that is $r_{ij} = 0, s_i = 0$;

(iv) S =0;

(v) F is weakly-Berwald,that is E=0;

where $c = c(x)$ is a scalar function on M.

III. Proof of theorem (1.1)

In view of the equation (2.3), the spray coefficients $G^i(x, y)$ of F^n with an (α,β)-metric can also be written in the following form [9],

$$G^i = G^i_\alpha + B^i, \tag{3.1}$$

where

$$B^i = \frac{\alpha F_\beta s^i_0}{F_\alpha} + C^* \left\{ \frac{\beta F_\beta y^i}{\alpha F} - \frac{\alpha F_{\alpha\alpha}}{F_\alpha}\left(\frac{y^i}{\alpha} - \frac{\alpha b^i}{\beta}\right)\right\}, \tag{3.2}$$

$$C^* = \frac{\alpha\beta(r_{00}F_\alpha - 2s_0\alpha F_\beta)}{2(\beta^2 F_\alpha + \alpha\gamma^2 F_{\alpha\alpha})},$$

$\gamma^2 = b^2\alpha^2 - \beta^2$, $F_\alpha = \dfrac{\partial F}{\partial \alpha}$, $F_\beta = \dfrac{\partial F}{\partial \beta}$ and $F_{\alpha\alpha} = \dfrac{\partial F_\alpha}{\partial \alpha}$, provided $\beta^2 F_\alpha + \alpha\gamma^2 F_{\alpha\alpha} \neq 0$. The vector $B^i(x, y)$ is called the difference vector. Differentiation of spray coefficients G^i with respect to y^j and y^k successively gives $G^i_j = \gamma^i_{0j} + B^i_j$ and $G^i_{jk} = \gamma^i_{jk} + B^i_{jk}$ where $B^i_j = \dot{\partial}_j B^i$ and $B^i_{jk} = \dot{\partial}_k B^i_j$. Thus a Finsler space with an (α,β)-metric is a Berwald space if and only if $G^i_{jk} = G^i_{jk}(x)$ equivalently $B^i_{jk} = B^i_{jk}(x)$. Moreover on account of [10] B^i_j is determined by

$$F_\alpha B^t_{ji} y^j y_t + \alpha F_\beta (B^t_{ji} b_t - b_{j;i}) y^j = 0. \tag{3.3}$$

where $y_k = a_{ik} y^i$. For the Binomial (α,β)-metrics (1.1), we have

$$F_\alpha = (\alpha+\beta)^m \alpha^{-m-1}(\alpha - m\beta), F_{\alpha\alpha} = m(m+1)(\alpha+\beta)^{m-1}\alpha^{-m-2}\beta^2,$$

$$F_\beta = (m+1)(\alpha+\beta)^m \alpha^{-m} \quad and \quad F_{\beta\beta} = m(m+1)(\alpha+\beta)^{m-1}\alpha^{-m}. \tag{3.4}$$

Substituting (3.4) in equation (3.3), we have

$$(\alpha - m\beta)B^t_{ji} y^j y_t + (m+1)\alpha^2 (B^t_{ji} b_t - b_{j;i}) y^j = 0. \tag{3.5}$$

Assume that F^n is a Berwald space, that is, $B^i_{jk} = B^i_{jk}(x)$. Separating equation (3.5) in rational and irrational terms of y^i, which yields two equations

$$-m\beta B^t_{ji} y^j y_t + (m+1)\alpha^2 (B^t_{ji} b_t - b_{j,i}) y^j = 0 \tag{3.6}$$

and

$$\alpha B^t_{ji} y^j y_t = 0. \tag{3.7}$$

Equation (3.7) yields $B^t_{ji} y^j y_t = 0$, that is,

$$B^t_{ji} a_{th} + B^t_{hi} a_{tj} = 0 \quad and \quad B^t_{ji} b_t - b_{j;i} = 0. \tag{3.8}$$

Thus we obtain $B^t_{ji} = 0$ by Christoffel process, in the first part of equation (3.8) and from second part of equation (3.8), we have $b_{i;j} = 0$.

Conversely, if $b_{i;j} = 0$, then $B^t_{ji} = 0$ are determined from equation (3.5).

IV. Proof of theorem (1.2)

A Douglas space is a generalization of Berwald space in the sense that a Finsler space F^n with an (α,β) -metric is a Douglas space if and only if $B^{ij} = B^i y^j - B^j y^i$ are positively homogeneous of degree 3 (in short we write $hp(3)$) [9]. In view of equation (3.2), the tensor B^{ij} is written in the form

$$B^{ij} = \frac{\alpha F_\beta}{F_\alpha}(s^i_0 y^j - s^j_0 y^i) + \frac{\alpha^2 F_{\alpha\alpha}}{\beta F_\alpha} C^*(b^i y^j - b^j y^i). \tag{4.1}$$

Suppose that F^n is a Douglas space. From equations (3.4) and (4.1), we have

$$\begin{aligned}&(-2\alpha^3 + 4m\alpha^2\beta - 2\alpha^2\beta + 6m\alpha\beta^2 - 4m^2\beta^3 - 2m^2b^2\alpha^3 + m^3b^2\alpha^2\beta - 2mb^2\alpha^3 +\\ &2m^2b^2\alpha^2\beta - 2m^3\beta^3)B^{ij} = (-2m\alpha^4 - 2\alpha^4 - 2\alpha^3\beta + 2m^2\alpha^3\beta + 6m^2\alpha^2\beta^2 +\\ &4m\alpha^2\beta^2 - 2m^3b^2\alpha^4 - 4m^2b^2\alpha^4 - 2mb^2\alpha^4 + 2m^3\alpha^2\beta^2)({s_0}^i y^j - {s_0}^j y^i) +\\ &(4m^2\alpha^4 s_0 + 2m\alpha^4 s_0 + 2m^3\alpha^4 s_0 - m^2\alpha^3 r_{00} - m\alpha^3 r_{00} + m^3\alpha^2 r_{00}\beta +\\ &m^2\alpha^2\beta r_{00})(b^i y^j - b^j y^i).\end{aligned} \tag{4.2}$$

Separating equation (4.2) in rational and irrational terms of y^i, we have the following two equations

$$\begin{aligned}&(4m\alpha^2\beta - 2\alpha^2\beta - 4m^2\beta^3 + 2m^3b^2\alpha^2\beta + 2m^2b^2\alpha^2\beta - 2m^3\beta^3)B^{ij} = (-2m\alpha^4 - 2\alpha^4 +\\ &6m^2\alpha^2\beta^2 + 4m\alpha^2\beta^2 - 2m^3b^2\alpha^4 - 4m^2b^2\alpha^4 - 2mb^2\alpha^4 + 2m^3\alpha^2\beta^2)({s_0}^i y^j - {s_0}^j y^i) +\\ &(2m\alpha^4 s_0 + 4m^2\alpha^4 s_0 + 2m^3\alpha^4 s_0 + m^3\alpha^2\beta r_{00} + m^2\alpha^2\beta r_{00})(b^i y^j - b^j y^i)\end{aligned} \tag{4.3}$$

and

$$\begin{aligned}&(-2\alpha^2 + 6m\beta^2 - 2m^2b^2\alpha^2 - 2mb^2\alpha^2)B^{ij} = (-2\alpha^2\beta + 2m^2\alpha^2\beta)({s_0}^i y^j - {s_0}^j y^i)\\ &+(-m\alpha^2 r_{00} - m^2\alpha^2 r_{00})(b^i y^j - b^j y^i).\end{aligned} \tag{4.4}$$

Eliminating B^{ij} from equations (4.3) and (4.4), we obtain

$$A({s_0}^i y^j - {s_0}^j y^i) - B(b^i y^j - b^j y^i) = 0, \tag{4.5}$$

where

$$A=[4m\alpha^4+4m^5\beta^4+4\alpha^4+20m^4\beta^4+32m^3\beta^4-12m^3\alpha^2\beta^2+12m^3b^4\alpha^4- 32m^4b^2\alpha^2\beta^2-8m^5b^2\alpha^2\beta^2-40m^3b^2\alpha^2\beta^2+4m^2b^4\alpha^4+16m^2b^2\alpha^4+8m^3b^2\alpha^4+ 8mb^2\alpha^4+4m^5b^4\alpha^4-20m^2\alpha^2\beta^2-12m\alpha^2\beta^2+16m^2\beta^4+12m^4b^4\alpha^4-4\alpha^2\beta^2- 16m^2b^2\alpha^2\beta^2] \quad (4.6)$$

and

$$B=[4m\alpha^4s_0+4m^3\alpha^4s_0+2m^5\beta^3r_{00}+2m\alpha^2\beta r_{00}+8m^2\alpha^4s_0-2m^3\beta^3r_{00}- 2m^3\alpha^2\beta r_{00}-12m^2\alpha^2\beta^2s_0+12m^3b^2\alpha^4s_0+4m^2b^2\alpha^4s_0-24m^3\alpha^2\beta^2s_0+ 12m^4b^2\alpha^4s_0-12m^4\alpha^2\beta^2s_0+4m^5b^2\alpha^4s_0]. \quad (4.7)$$

Transvecting equation (4.5) by b_iy_j, we get

$$A\alpha^2s_0+B(b^2\alpha^2-\beta^2)=0. \quad (4.8)$$

The terms of equation (4.8), which does not contain α^2 are $2m^3(m^2-1)\beta^5r_{00}$. Hence there exists $hp(5):V_5$ such that

$$2m^3(m^2-1)\beta^5r_{00}=\alpha^2V_5. \quad (4.9)$$

Now we consider the following two cases:
(i) $V_5=0$ and (ii) $V_5\neq 0$.

Case (i): Let $V_5=0$ then we have $r_{00}=0$, provided $m\neq 0,\pm 1$. Substituting $r_{00}=0$ into equation (4.8), we get

$$(A+B_1\gamma^2)s_0=0, \quad (4.10)$$

where

$B_1=(4m\alpha^2+4m^3\alpha^2+8m^2\alpha^2-12m^2\beta^2+12m^3b^2\alpha^2+4m^2b^2\alpha^2-24m^3\beta^2+12m^4b^2\alpha^2-12m^4\beta^2+4m^5b^2\alpha^2)$. If $(A+B_1\gamma^2)=0$, then the terms of $(A+B_1\gamma^2)$ which do not contain α^2 are $-12m^2(1+2m+m^2)\beta^2$. Thus there exists $hp(2)$: V_2 such that $-12m^2(1+2m+m^2)\beta^2=\alpha^2V_2$. Hence we have $V_2=0$, which is a contradiction. Therefore, we must have $(A+B_1\gamma^2)s_0\neq 0$. Therefore we have $s_0=0$ from equation (4.10). Substituting $s_0=0$ and $r_{00}=0$ into equation (4.5), we get

$$A({s_0}^iy^j-{s_0}^jy^i)=0. \quad (4.11)$$

If $A=0$, then from equation (4.6), we have

$$[4m\alpha^4+4m^5\beta^4+4\alpha^4+20m^4\beta^4+32m^3\beta^4-12m^3\alpha^2\beta^2+12m^3b^4\alpha^4-32m^4b^2\alpha^2\beta^2- 8m^5b^2\alpha^2\beta^2-40m^3b^2\alpha^2\beta^2+4m^2b^4\alpha^4+16m^2b^2\alpha^4+8m^3b^2\alpha^4+8mb^2\alpha^4+4m^5b^4\alpha^4- 20m^2\alpha^2\beta^2-12m\alpha^2\beta^2+16m^2\beta^4+12m^4b^4\alpha^4-4\alpha^2\beta^2-16m^2b^2\alpha^2\beta^2]=0. \quad (4.12)$$

The terms of equation (4.12), which do not contain α^2 are $4m^2(m^3+5m^2+8m+4)\beta^4$. Thus there exists $hp(2)$: V_2 such that $4m^2(m^3+5m^2+8m+4)\beta^4=\alpha^2V_2$. Therefore we have, $V_2=0$, which is a contradiction. Therefore we must have $A\neq 0$. Hence from equation (4.11), we have $({s_0}^iy^j-{s_0}^jy^i)=0$. Transvecting the above equation by y_j gives $s_0^i=0$, which imply $s_{ij}=0$. Consequently, we have $r_{ij}=s_{ij}=0$. This implies $b_{i;j}=0$.

Case (ii): If β divides α^2 then we have a contradiction of positive definiteness of Riemannian metric α, so we assume $\alpha^2\neq 0(mod\beta)$. The equation (4.9) shows that there exists a function $k=k(x)$ such that $r_{00}=k(x)\alpha^2$. Thus we have the terms of equation (4.8) which do not contain α^2, are included in the terms $2m^3(m^2-1)\beta^5r_{00}$. Hence we get $r_{00}=0$, provided $m\neq 0,\pm 1$. From equation (4.11), we have

$A\left(s_0^i y^j - s_0{}^j y^i\right) = 0.$ If $A = 0$, then it is a contradiction. Hence $A \neq 0$. Therefore we obtain $\left(s_0{}^i y^j - s_0{}^j y^i\right) = 0$. Transvecting this equation by y_j we get $s_j^i = 0$.

Hence from both cases (i) and (ii), we have $r_{ij} = s_{ij} = 0$. This implies $b_{i;j} = 0$

Conversely if $b_{i;j} = 0$, then F^n is a Berwald space, therefore F^n is a Douglas space.

Corollary: For $m \neq 0, \pm 1$ a Binomial (α,β)-metric is Douglas iff it is Berwald.

V. Proof of theorem (1.3)

Suppose the Binomial (α,β)-metrics $F = \dfrac{(\alpha+\beta)^{m+1}}{\alpha^m}$ is locally projectively flat. By Lemma (2.1), the spray coefficients G^i of F are given by equation (2.3) with

$$Q = \frac{m+1}{1-ms}, J = \frac{1}{2}\frac{(1-ms)(m+1)}{\left(1+s-ms-2ms^2+m^2b^2+mb^2-m^2s^2\right)},$$
$$H = \frac{1}{2}\frac{m(m+1)}{\left(1+s-ms-2ms^2+m^2b^2+mb^2-m^2s^2\right)}. \tag{5.1}$$

Substituting (5.1) into equation (2.3), we obtain

$$2(a_{ml}\alpha^2 - y_m y_l)G_\alpha^m(-\alpha^3 - \alpha^2\beta + 2m\alpha^2\beta + 3m\alpha\beta^2 - m^2b^2\alpha^3 - mb^2\alpha^3 - 2m^2\beta^3 + m^3b^2\alpha^2\beta + m^2b^2\alpha^2\beta - m^3\beta^3) - 2(m+1)\alpha^4 s_{l0}(\alpha^2 + \alpha\beta - m\alpha\beta - 2m\beta^2 + m^2b^2\alpha^2 + mb^2\alpha^2 - m^2\beta^2) + 2m(m+1)^2\alpha^4 s_0(b_l\alpha^2 - \beta y_l) + m(m+1)(-\alpha + m\beta)\alpha^2 r_{00}(b_l\alpha^2 - \beta y_l) = 0. \tag{5.2}$$

Separating the rational and irrational terms of y^i in equation (5.2), we have the following two equations

$$2(a_{ml}\alpha^2 - y_m y_l)G_\alpha^m(-\alpha^2\beta + 2m\alpha^2\beta - 2m^2\beta^3 + m^3b^2\alpha^2\beta + m^2b^2\alpha^2\beta - m^3\beta^3) = 2(m+1)\alpha^4 s_{l0}(\alpha^2 - 2m\beta^2 + m^2b^2\alpha^2 + mb^2\alpha^2 - m^2\beta^2) - 2m(m+1)^2\alpha^4 s_0(b_l\alpha^2 - \beta y_l) - m^2(m+1)r_{00}(b_l\alpha^2 - \beta y_l)\alpha^2\beta \tag{5.3}$$

and

$$2(a_{ml}\alpha^2 - y_m y_l)G_\alpha^m(-\alpha^3 + 3m\alpha\beta^2 - m^2b^2\alpha^3 - mb^2\alpha^3) = 2(m+1)\alpha^4 s_{l0}(\alpha\beta - m\alpha\beta) + m(m+1)r_{oo}(b_l\alpha^2 - \beta y_l)\alpha^3. \tag{5.4}$$

Contracting equations (5.3) and (5.4) with b^l, we get

$$2(b_m\alpha^2 - y_m\beta)G_\alpha{}^m(-\alpha^2\beta + 2m\alpha^2\beta - 2m^2\beta^3 + m^3b^2\alpha^2\beta + m^2b^2\alpha^2\beta - m^3\beta^3) = 2(m+1)\alpha^4 s_0(\alpha^2 - 2m\beta^2 + m^2b^2\alpha^2 + mb^2\alpha^2 - m^2\beta^2) - 2m(m+1)^2\alpha^4 s_0(b^2\alpha^2 - \beta^2) - m^2(m+1)r_{oo}(b^2\alpha^2 - \beta^2)\alpha^2\beta \tag{5.5}$$

and

$$2(b_m\alpha^2 - y_m\beta)G_\alpha{}^m(-\alpha^3 + 3m\alpha\beta^2 - m^2b^2\alpha^3 - mb^2\alpha^3) = 2(m+1)\alpha^4 s_0(\alpha\beta - m\alpha\beta) + m(m+1)r_{oo}(b^2\alpha^2 - \beta^2)\alpha^3. \tag{5.6}$$

Multiplying equation (5.5) with α and equation (5.6) with $m\beta$, we have

$$\beta(b_m\alpha^2 - y_m\beta)G_\alpha^m(-\alpha^2 + 3\alpha^2 - 5m^2\beta^2 + 2m^3b^2\alpha^2 + 2m^2b^2\alpha^2 - m^3\beta^2) = (m+1)\alpha^4(\alpha^2 - 2m\beta^2 + m^2\beta^2)s_0. \tag{5.7}$$

Above equation shows that β must divides s_0. Therefore there exists a scalar function $\tau = \tau(x)$, such that $s_0 = \tau\beta$. Thus we obtain $s_i - \tau b_i = 0$. Which gives after contraction with b^i, $s_i b^i - \tau b_i b^i = 0$. Therefore we have $\tau = 0$ and hence

$$s_0 = 0. \tag{5.8}$$

Using equations (5.7) and (5.8), we have

$$\left(b_m\alpha^2 - y_m\beta\right)G_\alpha^m = 0. \tag{5.9}$$

Then by equation (5.9) and Lemma (2.3), α is projectively flat. Also using equations (5.6), (5.8) and (5.9), we have

$$r_{00} = 0. \tag{5.10}$$

Substituting (5.9) and (5.10) in equation (5.4), we get

$$s_{l0} = 0. \tag{5.11}$$

Thus using above two equations (5.10) and (5.11) $b_{i;j} = 0$, that is β is parallel with respect to α.

Conversely, if β is parallel with respect to α and α is locally projectively flat, then by Lemma (2.2), F is locally projectively flat.

VI. Proof of theorem (1.4)

In view of equation (2.3), the spray coefficients G^i and G_α^i of F and α respectively, can be written as:

$$G^i = G_\alpha^i + \alpha Q s_0^i + \theta(2Q\alpha s_0 + r_{00})\left[\frac{y^i}{\alpha} + \frac{Q'}{Q - sQ'}b^i\right]. \tag{6.1}$$

Further the mean Cartan torsion I_i [11] and the mean Landsberg curvature J_i [12] of an (α,β)-metrics are respectively given by

$$I_i = -\frac{\Phi(\phi - s\phi')}{2\Delta\phi\alpha^2}(\alpha b_i - sy_i) \tag{6.2}$$

and

$$J_i = -\frac{1}{2\Delta\alpha^4}[\frac{2\alpha^2}{b^2 - s^2}[\frac{\Phi}{\Delta} + (n+1)(Q - sQ')](r_0 + s_0)h^i$$
$$+\frac{\alpha}{b^2 - s^2}(\Psi_1 + s\frac{\Phi}{\Delta})(r_{00} - 2\alpha Q s_0)h_i + \alpha[-\alpha Q' s_0 h^i + \alpha Q(\alpha^2 s_i - y_i s_0) +$$
$$\alpha^2\Delta s_{i0} + \alpha^2(r_{i0} - 2\alpha Q s_i) - (r_{00} - 2\alpha Q s_0)y_i]\frac{\Phi}{\Delta}], \tag{6.3}$$

where $\Phi = -(n\Delta + 1 + sQ)(Q - sQ') - (b^2 - s^2)(1 + sQ)Q''$, $\Delta = 1 + sQ + (b^2 - s^2)Q'$ and $Q = \dfrac{\phi'}{\phi - s\phi'}$. Contracting equation (6.3) with $b^i = a^{im}b_m$, we get

$$\bar{J} = J_i b^i = -\frac{1}{2\Delta\alpha^2}[\Psi_1(r_{00} - 2\alpha Q s_0) + \alpha\Psi_2(r_0 + s_0)], \tag{6.4}$$

where $\Psi_1 = \sqrt{(b^2 - s^2)}\Delta^{\frac{1}{2}}\left[\dfrac{\sqrt{(b^2 - s^2)}\Phi}{\Delta^{\frac{3}{2}}}\right], \Psi_2 = 2(n+1)(Q - sQ') + 3\dfrac{\Phi}{\Delta}$.

In view of equation (2.10) of the paper [13], we have

$$\bar{J}_{|m}y^m - J_i a^{ik}(r_{k0} + s_{k0}) - J_l\frac{\partial(G^l - \bar{G}^l)}{\partial y^i}b^i - 2\frac{\partial\bar{J}}{\partial y^l}(G^l - \bar{G}^l) + K\alpha^2\phi^2 I_m b^m$$
$$= -\frac{n+1}{3}\alpha^2\phi^2 K_{.m}b^m. \tag{6.5}$$

Now let F is an isotropic Berwald metric with almost isotropic flag curvature. In [14], it is proved that every isotropic Berwald metric has isotropic S-curvature. Conversely, suppose that F is of isotropic S

-curvature with scalar flag curvature K. In [2], it is proved that every Finsler metric of isotropic S -curvature has almost isotropic flag curvature. Now our aim to proved that F is a Isotropic Berwald metric. In [15] it is proved that F is an isotropic Berwald metric if and only if it is a Douglas metric with isotropic mean Berwald curvature. Also every Finsler metric of isotropic S -curvature has isotropic mean Berwald curvature. Therefore, to complete the proof, we must show that F is a Douglas metric.

By proposition (2.1), we have $S=0$. By theorem (1.1) in [2], F must be of isotropic flag curvature $K=K(x)$. Also By Proposition (2.1), β is a killing 1-form with respect to α, that is $r_{ij}=0$ and $s_j=0$. Then equations (6.1), (6.3) and (6.4) reduce to

$$G^i-\bar{G}^i=\alpha Q s_0^i,\quad J_i=-\frac{\Phi s_{i0}}{2\alpha\Delta},\quad \bar{J}=0 \tag{6.6}$$

from equation (6.2), we have

$$I_i b^i=-\frac{\Phi}{2\alpha\phi\Delta}(\phi-s\phi')(b^2-s^2). \tag{6.7}$$

We consider two cases:

Case (i): Let dim $M\geq 3$. In this case, by Schur Lemma F has constant flag curvature and equation (6.5) holds. Thus by equations (6.6) and (6.7), the equation (6.5) reduces to

$$\frac{\Phi s_{i0}}{2\alpha\Delta}a^{ik}s_{k0}+\frac{\Phi s_{l0}}{2\alpha\Delta}(\alpha Q s_0^l)_{.i}b^i-KF\frac{\Phi}{2\Delta}(\phi-s\phi')(b^2-s^2)=0.$$

Assuming $\Phi\neq 0$, we have

$$s_{i0}s_0^i+s_{l0}(\alpha Q s_0^l)_{.i}b^i-KF\alpha(\phi-s\phi')(b^2-s^2)=0. \tag{6.8}$$

Now

$(\alpha Q s_0^l)_{.i}b^i=sQs_0^i+Q's_0^i(b^2-s^2)$.

Then equation (6.8) can be written as follows

$$s_{i0}s_0^i\Delta-K\alpha^2\phi(\phi-s\phi')(b^2-s^2)=0. \tag{6.9}$$

Therefore for $F=\dfrac{(\alpha+\beta)^{m+1}}{\alpha^m}$, we get $\Delta=\dfrac{1-ms-m^2s^2+s-2ms^2+m^2b^2+mb^2}{(-1+ms)^2}$.

Hence equation (6.9) becomes

$$s_{i0}s_0{}^i(1-ms-m^2s^2+s-2ms^2+m^2b^2+mb^2)-K\alpha^2(1+s)^{2m+1}(b^2-s^2-3msb^2+3ms^3+3m^2s^2b^2-3m^2s^4-m^3s^3b^2+m^3s^5)=0,$$

that is

$$\begin{aligned}&s_{i0}s_0{}^i\alpha^{2+2m}(\alpha^2-m\alpha\beta-m^2\beta^2+\alpha\beta-2m\beta^2+m^2b^2\alpha^2+mb^2\alpha^2)-\\&K\left[\sum_{k=0}^{m}\binom{2m}{2k}\beta^{2m-2k}\alpha^{2k}+\sum_{k=0}^{m-1}\binom{2m}{2k+1}\beta^{2m-2k-1}\alpha^{2k+1}\right](-b^2\alpha^6+\\&\alpha^4\beta^2+3mb^2\alpha^5\beta-3m\alpha^3\beta^3-3m^2b^2\alpha^4\beta^2+3m^2\alpha^2\beta^4+m^3b^2\alpha^3\beta^3-\\&m^3\alpha\beta^5-b^2\alpha^5\beta+\alpha^3\beta^3+3mb^2\alpha^4\beta^2-3m\alpha^2\beta^4-3m^2b^2\alpha^3\beta^3+\\&3m^2\alpha\beta^5+m^3b^2\alpha^2\beta^4-m^3\beta^6)=0.\end{aligned} \tag{6.10}$$

Above equation can also be written as: $A+\alpha B=0$,

where

$$\begin{aligned}A=&\,s_{i0}s_0{}^i\alpha^{2+2m}(\alpha^2-m^2\beta^2-2m\beta^2+m^2b^2\alpha^2+mb^2\alpha^2)-K\sum_{k=0}^{m}\binom{2m}{2k}\beta^{2m-2k}\times\\&\alpha^{2k}(-b^2\alpha^6+\alpha^4\beta^2-3m^2b^2\alpha^4\beta^2+3m^2\alpha^2\beta^4+3mb^2\alpha^4\beta^2-3m\alpha^2\beta^4+m^3b^2\alpha^2\beta^4-\\&m^3\beta^6)-\sum_{k=0}^{m-1}\binom{2m}{2k+1}\beta^{2m-2k-1}\alpha^{2k+1}(3mb^2\alpha^5\beta-3m\alpha^3\beta^3+m^3b^2\alpha^3\beta^3-m^3\alpha\beta^5-\\&b^2\alpha^5\beta+\alpha^3\beta^3-3m^2b^2\alpha^3\beta^3+3m^2\alpha\beta^5)\end{aligned}$$

and

$$B = s_{i0}s_0{}^i\alpha^{2+2m}(-m\beta+\beta) - K\sum_{k=0}^{m}\binom{2m}{2k}\beta^{2m-2k}\alpha^{2k}(3mb^2\alpha^4\beta - 3m\alpha^2\beta^3 + m^3b^2\alpha^2\beta^3 - m^3\beta^5 - b^2\alpha^4\beta + \alpha^2\beta^3 - 3m^2b^2\alpha^2\beta^3 + 3m^2\beta^5) - \sum_{k=0}^{m-1}\binom{2m}{2k+1}\beta^{2m-2k-1}\times \alpha^{2k}(-b^2\alpha^6 + \alpha^4\beta^2 - 3m^2b^2\alpha^4\beta^2 + 3m^2\alpha^2\beta^4 + 3mb^2\alpha^4\beta^2 - 3m\alpha^2\beta^4 + m^3b^2\alpha^2\beta^4 - m^3\beta^6).$$

Thus we have $A=0$ and $B=0$.

When $A=0$, the term which do not contain α^2 is $Km^3\beta^{2m+6}$. This implies β^{2m+6} is not divisible by α^2. Therefore $K=0$, hence equation (6.10) reduces to $s_{i0}s_0^i = a_{ij}s_0^j s_0^i = 0$. Thus we have $s_0^i = 0$. That is β is closed. By $r_{00}=0$ and $s_0=0$. it follows that β is parallel with respect to α. Then $F = \frac{(\alpha+\beta)^{m+1}}{\alpha^m}$ is a Berwald metric. Hence F must be locally Minkowskian.

Case (ii): Let dim $M=2$. Suppose that F has isotropic Berwald curvature. In [14], it is proved that every isotropic Berwald metric has isotropic S-curvature $S=(n+1)cF$. By proposition (2.1), $c=0$. Then by equation (1.5), F reduces to a Berwald metric. Since F is a non Riemannian, then by Szabó's rigidity theorem for Berwald surface [16] F must be locally Minkowskian.

References

[1]. Z. Shen and S. S. Chern, Riemann-Finsler Geometry, (Nankai Tracts in Mathematics, World Scientific, 2004).
[2]. B.Najafi, Z.Shen and A.Tayebi, Finsler metrics of scalar flag curvature with special non-Riemannian curvature properties, Geom. Dedicata., 131, 2008 , 87-97.
[3]. S.Bacso and M.Matsumoto, On Finsler spaces of Douglas type, A generalization of the notion of Berwald space, Publ. Math. Debrecen., 51, 1997, 385-406.
[4]. B.Li, Y.Shen and Z.Shen, On a class of Douglas metrics, Studia Sci. Math. Hungarica, 46(3), 2009, 355-365.
[5]. G.Hamel, Uber die Geometrien in denen die Geraden die Kurzesten sind, Math. Ann., 57, 1903, 231-264.
[6]. Z.Shen and G.Civi Yildirim, On a Class of Projectively Flat metrics with Constant Flag Curvature, Canadian Journal of Mathematics., 60, 2008, 443-456.
[7]. Y.Yu, Projectively flat exponential Finsler metrics, J. of Zhejiang Univ. Science A., 6, 2006, 1068-1076.
[8]. C.H.Xiang and X.Cheng, On a class of weakly-Berwald (α,β) -metrics, J. Math. Res. Expos., 29(2), 2009, 227-236.
[9]. I.Y.Lee and H. S.Park, Finsler spaces with infinite series (α,β) -metrics, J. Korean Math. Soc., 41(3), 2004, 567-589.
[10]. M.Matsumoto, The Berwald connection of a Finsler space with an (α,β) -metric, Tensor, N. S., 50, 1991, 18-21.
[11]. A.Tayebi, T.Tabatabaeifar and E.Peyghan, On the second approximate Matsumoto metric, Bull. Korean Math. Soc., 51(1), 2014, 115-128.
[12]. B.Li and Z.Shen, On a class of weak Landsberg metrics, Sci. China Ser. A., 50(1), 2007, 75-85.
[13]. X.Y.Cheng, On (α,β) -metrics of scalar Flag curvature with constant S -curvature, Acta Mathematica Sinica, English series., Vol. 26(9), 2010, 1701-1708.
[14]. A.Tayebi and M.R.Rad, S -curvature of isotropic Berwald metrics, Sci. China Ser. A., 51(2), 2008, 227-236.
[15]. X.Chen and Z.Shen, On Douglas metric, Publ. Math. Debrecen. 66, 2005, 503-512.
[16]. D.Bao, S.S.Chern and Z.Shen, An introduction to Riemann-Finsler Geometry, (Springer-Verlag, 2000).