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On Least Square, Minimum Norm Generalized Inverses of Bimatrices

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***Abstract**: The characterization of g-inverses, minimum norm g-inverses and least square g-inverses of bimatrices are derived as a generalization of the generalized inverses of matrices.*
***Keywords**: g-inverse, minimum norm g-inverse, least square g-inverse.*
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I. Introduction

Let $C_{n\times n}$ be the space of $n\times n$ complex matrices of order n. A matrix $A_B = A_1 \cup A_2$ is called a bimatrix if A_1 and A_2 are matrices of same (or) different orders [7,8] and a bimatrix is called unitary if $A_B A_B^* = A_B^* A_B = I_B$ [8]. For $A_1, A_2 \in C_{n\times n}$, $A_B = A_1 \cup A_2$ and let $A_B^*, r\left(A_B\right)$ and A_B^- denote the conjugate transpose, rank and generalized inverse of the bimatrix A_B where A_B^- is the solution of the equation $A_B X_B A_B = A_B$ [1,4,6]. In general, the generalized inverse A_B^- of A_B is not uniquely determined. Since A_B^- is not unique, the set of generalized inverses of A_B is some times denoted as $\left\{A_B^-\right\}$.

In this paper we analyze the characterization of g-inverse, minimum norm g-inverse and least square g-inverses of bimatrices as a generalization of the g-inverses of matrices [2,3,5] **.**

II. Generalized Inverses of Bimatrices

In this section some of the properties of generalized inverses of matrices found in [1,2,3,5] are extended to generalized inverses of bimatrices .

Theorem: 2.1

Let $H_B = A_B A_B^-$ and $F_B = A_B^- A_B$. Then the following relations hold :

(i) $H_B^2 = H_B$ and $F_B^2 = F_B$.

(ii) $r(H_B) = r(F_B) = r(A_B)$

(iii) $r\left(A_B^-\right) \geq r\left(A_B\right)$

(iv) $r\left(A_B^- A_B A_B^-\right) = r\left(A_B\right)$.

Proof of (i)

$$
\begin{aligned}
\text{Now } H_B^2 &= H_B.H_B \\
&= \left(A_B A_B^-\right)\left(A_B A_B^-\right) \\
&= \left(\left(A_1 \cup A_2\right)\left(A_1 \cup A_2\right)^-\right)\left(\left(A_1 \cup A_2\right)\left(A_1 \cup A_2\right)^-\right) \\
&= \left(\left(A_1 \cup A_2\right)\left(A_1^- \cup A_2^-\right)\right)\left(\left(A_1 \cup A_2\right)\left(A_1^- \cup A_2^-\right)\right) \\
&= \left(A_1 A_1^- \cup A_2 A_2^-\right)\left(A_1 A_1^- \cup A_2 A_2^-\right) \\
&= \left(A_1 A_1^- A_1\right) A_1^- \cup \left(A_2 A_2^- A_2\right) A_2^- \\
&= \left(A_1 A_1^- A_1 \cup A_2 A_2^- A_2\right)\left(A_1^- \cup A_2^-\right)
\end{aligned}
$$

$$=\left[\left(A_1\cup A_2\right)\left(A_1^-\cup A_2^-\right)\left(A_1\cup A_2\right)\right]\left(A_1^-\cup A_2^-\right)$$

$$=\left[\left(A_1\cup A_2\right)\left(A_1\cup A_2\right)^-\left(A_1\cup A_2\right)\right]\left(A_1\cup A_2\right)^-$$

$$=\left(A_B A_B^- A_B\right)A_B^-$$

$$=A_B A_B^-$$

$$=H_B.$$

Hence , $H_B^2=H_B$.

Now $F_B^2=F_B.F_B$

$$=\left(A_B^- A_B\right)\left(A_B^- A_B\right)$$

$$=\left(\left(A_1^-\cup A_2^-\right)\left(A_1\cup A_2\right)\right)\left(\left(A_1^-\cup A_2^-\right)\left(A_1\cup A_2\right)\right)$$

$$=\left(A_1^- A_1\cup A_2^- A_2\right)\left(A_1^- A_1\cup A_2^- A_2\right)$$

$$=\left(A_1^-\left(A_1A_1^-\right)A_1\right)\cup\left(A_2^-\left(A_2A_2^-\right)A_2\right)$$

$$=\left(A_1^-\cup A_2^-\right)\left[\left(A_1\cup A_2\right)\left(A_1^-\cup A_2^-\right)\left(A_1\cup A_2\right)\right]$$

$$=A_B^-\left(A_B A_B^- A_B\right)$$

$$=A_B^- A_B$$

$$=F_B.$$

Hence, $F_B^2=F_B.$

Proof of (ii)

Now $r\left(A_B\right)\ge r\left(A_B A_B^-\right)$

$$r\left(A_B\right)\ge r\left(H_B\right) \qquad (1)$$

Also $r\left(A_B\right)=r\left(A_B A_B^- A_B\right)$

$$=r\left(\left(A_1\cup A_2\right)\left(A_1^-\cup A_2^-\right)\left(A_1\cup A_2\right)\right)$$

$$=r\left(\left(A_1A_1^-\right)A_1\cup\left(A_2A_2^-\right)A_2\right)$$

$$=r\left(\left(A_1A_1^-\cup A_2A_2^-\right)\left(A_1\cup A_2\right)\right)$$

$$=r\left(\left(A_1\cup A_2\right)\left(A_1^-\cup A_2^-\right)\left(A_1\cup A_2\right)\right)$$

$$=r\left(\left(A_B A_B^-\right)A_B\right)$$

$$=r\left(H_B A_B\right)$$

$$r\left(A_B\right)\le r\left(H_B\right) \qquad (2)$$

From (1) and (2), we get $r\left(A_B\right)=r\left(H_B\right)$ (3)

Now $r\left(A_B\right)\ge r\left(A_B^- A_B\right)$

$$r\left(A_B\right)\ge r\left(F_B\right) \qquad (4)$$

Also $r\left(A_B\right)=r\left(A_B A_B^- A_B\right)$

$$=r\left(\left(A_1\cup A_2\right)\left(A_1^-\cup A_2^-\right)\left(A_1\cup A_2\right)\right)$$

$$= r\left(A_1\left(A_1^- A_1\right) \cup A_2\left(A_2^- A_2\right)\right)$$
$$= r\left(\left(A_1 \cup A_2\right)\left(A_1^- A_1 \cup A_2^- A_2\right)\right)$$
$$= r\left(\left(A_1 \cup A_2\right)\left(A_1^- \cup A_2^-\right)\left(A_1 \cup A_2\right)\right)$$
$$= r\left(A_B\left(A_B^- A_B\right)\right)$$
$$= r\left(A_B F_B\right)$$
$$r\left(A_B\right) \le r\left(F_B\right) \qquad (5)$$

From (4) and (5), we get $r\left(A_B\right) = r\left(F_B\right)$ (6)

From (3) and (6) we get $r\left(A_B\right) = r\left(H_B\right) = r\left(F_B\right)$. (7)

Proof of (iii)

Now $r\left(A_B\right) = r\left(A_B A_B^- A_B\right)$
$$= r\left(\left(A_1 \cup A_2\right)\left(A_1^- \cup A_2^-\right)\left(A_1 \cup A_2\right)\right)$$
$$= r\left(A_1 A_1^- A_1 \cup A_2 A_2^- A_2\right)$$
$$\le r\left(A_1 A_1^- \cup A_2 A_2^-\right)$$
$$\le r\left(A_1^- \cup A_2^-\right)$$
$$\le r\left(A_B^-\right)$$

Hence, $r\left(A_B^-\right) \ge r\left(A_B\right)$.

Proof of (iv)

Now $r\left(A_B^- A_B A_B\right) \le r\left(A_B^- A_B\right)$

Also we have $r\left(A_B^- A_B A_B^-\right) \ge r\left(A_B^- A_B A_B^- A_B\right)$
$$= r\left(\left(A_1^- \cup A_2^-\right)\left(A_1 \cup A_2\right)\left(A_1^- \cup A_2^-\right)\left(A_1 \cup A_2\right)\right)$$
$$= r\left(A_1^-\left(A_1 A_1^- A_1\right) \cup A_2^-\left(A_2 A_2^- A_2\right)\right)$$
$$= r\left(\left(A_1^- \cup A_2^-\right)\left(A_1 A_1^- A_1 \cup A_2 A_2^- A_2\right)\right)$$
$$= r\left[\left(A_1^- \cup A_2^-\right)\left(\left(A_1 \cup A_2\right)\left(A_1^- \cup A_2^-\right)\left(A_1 \cup A_2\right)\right)\right]$$
$$= r\left[A_B^-\left(A_B A_B^- A_B\right)\right]$$
$$= r\left(A_B^- A_B\right)$$
$$= r\left(H_B\right)$$
$$= r\left(A_B\right) \qquad \text{(by (7))}$$

Hence, $r\left(A_B^- A_B A_B^-\right) = r\left(A_B\right)$.

Theorem: 2.2

Let A_B be a bimatrix, then the following relations hold for a generalized inverse A_B^- of A_B.

(i) $\left\{\left(A_B^-\right)^*\right\} = \left\{\left(A_B^*\right)^-\right\}$

(ii) $A_B\left(A_B^*A_B\right)^- A_B^*A_B = A_B$

(iii) $\left(A_B\left(A_B^*A_B\right)^- A_B^*\right)^* = A_B\left(A_B^*A_B\right)^- A_B^*$

Proof of (i)

Let $A_B = A_B A_B^- A_B$

$$A_B^* = \left(A_B A_B^- A_B\right)^*$$

$$\left(A_1 \cup A_2\right)^* = \left(\left(A_1 \cup A_2\right)\left(A_1^- \cup A_2^-\right)\left(A_1 \cup A_2\right)\right)^*$$

$$A_1^* \cup A_2^* = \left(A_1A_1^-A_1 \cup A_2A_2^-A_2\right)^*$$

$$= \left(A_1A_1^-A_1\right)^* \cup \left(A_2A_2^-A_2\right)^*$$

$$= \left(A_1^*\left(A_1^-\right)^* A_1^*\right) \cup \left(A_2^*\left(A_2^-\right)^* A_2^*\right)$$

$$A_1^* \cup A_2^* = \left(A_1^* \cup A_2^*\right)\left(\left(A_1^-\right)^* \cup \left(A_2^-\right)^*\right)\left(A_1^* \cup A_2^*\right)$$

$$A_B^* = A_B^*\left(A_B^-\right)^* A_B^*$$

Thus $\left\{\left(A_B^-\right)^*\right\} \subset \left\{\left(A_B^*\right)^-\right\}$ (8)

On the other hand, $A_B^*\left(A_B^*\right)^- A_B = A_B^*$ and so $\left(\left(A_B^*\right)^-\right)^* \in \left\{A_B^-\right\}$

$$\left\{\left(A_B^*\right)^-\right\} \subset \left\{\left(A_B^-\right)^*\right\} \tag{9}$$

From (8) and (9), we get $\left\{\left(A_B^-\right)^*\right\} = \left\{\left(A_B^*\right)^-\right\}$.

Proof of (ii)

Let $G_B = A_B\left(I_B - \left(A_B^*A_B\right)^- A_B^*A_B\right)$

$$= A_B - A_B\left(A_B^*A_B\right)^- A_B^*A_B$$

$$= (A_1 \cup A_2) - (A_1 \cup A_2)\left[\left(A_1^* \cup A_2^*\right)\left(A_1 \cup A_2\right)\right]^- \left(A_1^* \cup A_2^*\right)\left(A_1 \cup A_2\right)$$

$$G_B = (A_1 \cup A_2) - (A_1 \cup A_2)\left(A_1 \cup A_2\right)^- \left(A_1^* \cup A_2^*\right)^- \left(A_1^* \cup A_2^*\right)\left(A_1 \cup A_2\right)$$

$$G_B^* = \left[(A_1 \cup A_2) - (A_1 \cup A_2)\left(A_1 \cup A_2\right)^- \left(A_1^* \cup A_2^*\right)^- \left(A_1^* \cup A_2^*\right)\left(A_1 \cup A_2\right)\right]^*$$

$$= \left[(A_1 \cup A_2) - (A_1 \cup A_2)\left(A_1^- \cup A_2^-\right)\left(\left(A_1^*\right)^- \cup \left(A_2^*\right)^-\right)\left(A_1^* \cup A_2^*\right)\left(A_1 \cup A_2\right)\right]^*$$

$$= \left[(A_1 \cup A_2) - \left(\left(A_1A_1^-\left(A_1^*\right)^- A_1^*A_1\right) \cup \left(A_2A_2^-\left(A_2^*\right)^- A_2^*A_2\right)\right)\right]^*$$

$$= \left[\left(A_1 - A_1A_1^-\left(A_1^-\right)^* A_1^*A_1\right) \cup \left(A_2 - A_2A_2^-\left(A_2^-\right)^* A_2^*A_2\right)\right]^*$$

$$= \left(A_1 - A_1A_1^-\left(A_1^-\right)^* A_1^*A_1\right)^* \cup \left(A_2 - A_2A_2^-\left(A_2^-\right)^* A_2^*A_2\right)^*$$

$$=\left(A_1^*-\left(A_1A_1^-\left(A_1^-\right)^*A_1^*A_1\right)^*\right)\cup\left(A_2^*-\left(A_2A_2^-\left(A_2^-\right)^*A_2^*A_2\right)^*\right)$$

$$=\left(A_1^*-A_1^*A_1A_1^-\left(A_1^-\right)^*A_1^*\right)\cup\left(A_2^*-A_2^*A_2A_2^-\left(A_2^-\right)^*A_2^*\right)$$

$$=\left(A_1^*\cup A_2^*\right)-\left(A_1^*\cup A_2^*\right)\left(A_1\cup A_2\right)\left(A_1^-\cup A_2^-\right)\left(\left(A_1^*\right)^-\cup\left(A_2^*\right)^-\right)\left(A_1^*\cup A_2^*\right)$$

$$G_B^*=\left(A_1\cup A_2\right)^*-\left(A_1\cup A_2\right)^*\left(A_1\cup A_2\right)\left(A_1\cup A_2\right)^-\left(\left(A_1\cup A_2\right)^*\right)^-\left(A_1\cup A_2\right)^*$$

$$G_B^*G_B=\left[\left(I_1\cup I_2\right)^*-\left(A_1\cup A_2\right)^*\left(A_1\cup A_2\right)\left(A_1\cup A_2\right)^-\left(\left(A_1\cup A_2\right)^*\right)^-\right]\left(A_1\cup A_2\right)^*$$

$$\left[\left(A_1\cup A_2\right)-\left(A_1\cup A_2\right)\left(A_1\cup A_2\right)^-\left(A_1^*\cup A_2^*\right)^-\left(A_1^*\cup A_2^*\right)\left(A_1\cup A_2\right)\right]$$

$$G_B^*G_B=\left[\left(I_1\cup I_2\right)-\left(A_1\cup A_2\right)^*\left(A_1\cup A_2\right)\left(A_1\cup A_2\right)^-\left(\left(A_1\cup A_2\right)^*\right)^-\right]$$

$$\left[\left(A_1\cup A_2\right)^*\left(A_1\cup A_2\right)-\left(A_1\cup A_2\right)^*\left(A_1\cup A_2\right)\left(A_1\cup A_2\right)^-\left(\left(A_1\cup A_2\right)^*\right)^-\left(A_1\cup A_2\right)^*\left(A_1\cup A_2\right)\right]$$

$$G_B^*G_B=\left(I_B-A_B^*A_BA_B^-\left(A_B^*\right)^-\right)\left(A_B^*A_B-A_B^*A_BA_B^-\left(A_B^*\right)^-A_B^*A_B\right)$$

$$G_B^*G_B=\left(I_B-A_B^*A_B\left(A_B^*A_B\right)^-\right)\left(A_B^*A_B-A_B^*A_B\left(A_B^*A_B\right)^-A_B^*A_B\right)$$

Let $B_B=A_B^*A_B$.

$$G_B^*G_B=\left(I_B-B_BB_B^-\right)\left(B_B-B_BB_B^-B_B\right)$$
$$=\left(I_B-B_BB_B^-\right)\left(B_B-B_B\right)$$
$$=0$$

Hence, $G_B=0$

That is, $A_B\left(I_B-\left(A_B^*A_B\right)^-A_B^*A_B\right)=0$

$$A_B-A_B\left(A_B^*A_B\right)^-A_B^*A_B=0$$

Hence, $A_B\left(A_B^*A_B\right)^-A_B^*A_B=A_B$.

Proof of (iii)

Let G_B denote a generalized inverse of $A_B^*A_B$ .Then G_B^* is also a generalized inverse of $A_B^*A_B$ and $S_B=\dfrac{G_B+G_B^*}{2}$ is a symmetric generalized inverse of $A_B^*A_B$.

Let $H_B=A_BS_BA_B^*-A_B\left(A_B^*A_B\right)^-{A_B}^*$

$$H_B=\left(A_1\cup A_2\right)\left(S_1\cup S_2\right)\left(A_1^*\cup A_2^*\right)-\left(A_1\cup A_2\right)\left[\left(A_1^*\cup A_2^*\right)\left(A_1\cup A_2\right)\right]^-\left(A_1^*\cup A_2^*\right)$$

$$H_B^*=\left[\left(A_1\cup A_2\right)\left(S_1\cup S_2\right)\left(A_1^*\cup A_2^*\right)-\left(A_1\cup A_2\right)\left(A_1\cup A_2\right)^-\left(A_1^*\cup A_2^*\right)^-\left(A_1^*\cup A_2^*\right)\right]^*$$

$$H_B^*=\left[\left(A_1\cup A_2\right)\ \left(S_1\cup S_2\right)\left(A_1^*\cup A_2^*\right)-\left(A_1\cup A_2\right)\left(A_1^-\cup A_2^-\right)\left(\left(A_1^*\right)^-\cup\left(A_2^*\right)^-\right)\left(A_1^*\cup A_2^*\right)\right]^*$$

$$=\left[\left(A_1S_1A_1^* \cup A_2S_2A_2^*\right)-\left(A_1A_1^-\left(A_1^*\right)^- A_1^* \cup A_2A_2^-\left(A_2^*\right)^- A_2^*\right)\right]^*$$

$$=\left[\left(A_1S_1A_1^* - A_1A_1^-\left(A_1^*\right)^- A_1^*\right)\cup\left(A_2S_2A_2^* - A_2A_2^-\left(A_2^*\right)^- A_2^*\right)\right]^*$$

$$=\left(A_1S_1A_1^* - A_1A_1^-\left(A_1^*\right)^- A_1^*\right)^*\cup\left(A_2S_2A_2^* - A_2A_2^-\left(A_2^*\right)^- A_2^*\right)^*$$

$$=\left(\left(A_1S_1A_1^*\right)^* - \left(A_1A_1^-\left(A_1^*\right)^- A_1^*\right)^*\right)\cup\left(\left(A_2S_2A_2^*\right)^* - \left(A_2A_2^-\left(A_2^*\right)^- A_2^*\right)^*\right)$$

$$=\left(A_1S_1^*A_1^* - A_1A_1^-\left(A_1^-\right)^* A_1^*\right)\cup\left(A_2S_2^*A_2^* - A_2A_2^-\left(A_2^-\right)^* A_2^*\right)$$

$$=\left(A_1S_1^*A_1^* \cup A_2S_2^*A_2^*\right)-\left(A_1A_1^-\left(A_1^-\right)^* A_1^* \cup A_2A_2^-\left(A_2^-\right)^* A_2^*\right)$$

$$=\left[\left(A_1 \cup A_2\right)\left(S_1^* \cup S_2^*\right)\left(A_1^* \cup A_2^*\right)\right]-\left[\left(A_1 \cup A_2\right)\left(A_1 \cup A_2\right)^-\left(A_1^-\right)^* \cup \left(A_2^-\right)^*\left(A_1^* \cup A_2^*\right)\right]$$

$$H_B^* =\left[\left(A_1 \cup A_2\right)\left(S_1^* \cup S_2^*\right)\left(A_1^* \cup A_2^*\right)\right]-\left[\left(A_1 \cup A_2\right)\left(A_1^- \cup A_2^-\right)\left(\left(A_1^*\right)^- \cup \left(A_2^*\right)^-\right)\left(A_1^* \cup A_2^*\right)\right]$$

$$H_B^*H_B =\left[\left(A_1 \cup A_2\right)\left(S_1^* \cup S_2^*\right)-\left(A_1 \cup A_2\right)\left(A_1^- \cup A_2^-\right)\left(\left(A_1^*\right)^- \cup \left(A_2^*\right)^-\right)\right]\left(A_1^* \cup A_2^*\right)$$

$$\left[\left(A_1 \cup A_2\right)\left(S_1 \cup S_2\right)\left(A_1^* \cup A_2^*\right)-\left(A_1 \cup A_2\right)\left(A_1^- \cup A_2^-\right)\left(\left(A_1^*\right)^- \cup \left(A_2^*\right)^-\right)\left(A_1^* \cup A_2^*\right)\right]$$

$$=\left[\left(A_1 \cup A_2\right)\left(S_1^* \cup S_2^*\right)-\left(A_1 \cup A_2\right)\left(A_1^- \cup A_2^-\right)\left(\left(A_1^*\right)^- \cup \left(A_2^*\right)^-\right)\right]$$

$$\left[\left(A_1^* \cup A_2^*\right)\left(A_1 \cup A_2\right)\left(S_1 \cup S_2\right)\left(A_1^* \cup A_2^*\right)-\left(A_1^* \cup A_2^*\right)\left(A_1 \cup A_2\right)\left(A_1^- \cup A_2^-\right)\left(\left(A_1^*\right)^- \cup \left(A_2^*\right)^-\right)\left(A_1^* \cup A_2^*\right)\right]$$

$$=\left[A_BS_B^* - A_BA_B^-\left(A_B^*\right)^-\right]\left[A_B^*A_BS_BA_B^* - A_B^*A_BA_B^-\left(A_B^*\right)^- A_B^*\right]$$

$$H_B^*H_B =\left[A_BS_B^* - A_B\left(A_B^*A_B\right)^-\right]\left[A_B^*A_BS_BA_B^* - A_B^*A_B\left(A_B^*A_B\right)^- A_B^*\right]$$

Let $S_B = \left(A_B^*A_B\right)^-$

$$H_B^*H_B = \left(A_BS_B^* - A_BS_B\right)\left[A_B^*A_BS_BA_B^* - A_B^*A_BS_BA_B^*\right]$$

$H_B^*H_B = 0$.

Hence , $H_B = 0$.

That is , $A_BS_BA_B^* - A_B\left(A_B^*A_B\right)^- A_B^* = 0$.

$$\left(A_B\left(A_B^*A_B\right)^- A_B^*\right)^* - \left(A_B\left(A_B^*A_B\right)^- A_B^*\right) = 0$$

$$\left(A_B\left(A_B^*A_B\right)^- A_B^*\right)^* = A_B\left(A_B^*A_B\right)^- A_B^*.$$

III. Minimum norm generalized inverses of bimatrices

In this section some of the characteristics of minimum norm g-inverses of matrices found in [2,3] are extended to minimum norm g-inverses of bimatrices.

Definition: 3.1

A generalized inverse A_B^- that satisfies both $A_B A_B^- A_B = A_B$ and $\left(A_B^- A_B\right)^* = A_B^- A_B$ is called a minimum norm g-inverse of A_B and is denoted by $\left(A_B\right)_m^-$.

Theorem: 3.2

Let A_B be a bimatrix, then the following three conditions are equivalent:

(i) $\left(\left(A_B\right)_m^- A_B\right)^* = \left(A_B\right)_m^- A_B$ and $A_B\left(A_B\right)_m^- A_B = A_B$

(ii) $\left(A_B\right)_m^- A_B A_B^* = A_B^*$

(iii) $\left(A_B\right)_m^- A_B = A_B^*\left(A_B A_B^*\right)^- A_B$.

Proof of (i) $\Rightarrow$ (ii)

From (i), $A_B = A_B\left(A_B\right)_m^- A_B$

$$A_B^* = \left(A_B\left(A_B\right)_m^- A_B\right)^*$$
$$= \left[\left(A_1 \cup A_2\right)\left(A_1 \cup A_2\right)_m^-\left(A_1 \cup A_2\right)\right]^*$$
$$= \left[\left(A_1 \cup A_2\right)\left(\left(A_1\right)_m^- \cup \left(A_2\right)_m^-\right)\left(A_1 \cup A_2\right)\right]^*$$
$$= \left[A_1\left(A_1\right)_m^- A_1 \cup A_2\left(A_2\right)_m^- A_2\right]^*$$
$$= \left(A_1\left(A_1\right)_m^- A_1\right)^* \cup \left(A_2\left(A_2\right)_m^- A_2\right)^*$$
$$= A_1^*\left(\left(A_1\right)_m^-\right)^* A_1^* \cup A_2^*\left(\left(A_2\right)_m^-\right)^* A_2^*$$
$$= \left[\left(A_1^* \cup A_2^*\right)\left(\left(\left(A_1\right)_m^-\right)^* \cup \left(\left(A_2\right)_m^-\right)^*\right)\right]\left(A_1^* \cup A_2^*\right)$$
$$= \left[A_B^*\left(\left(A_B\right)_m^-\right)^*\right]A_B^*$$
$$= \left(\left(A_B\right)_m^- A_B\right)^* A_B^*$$
$$= \left(A_B\right)_m^- A_B A_B^*$$

Hence, $\left(A_B\right)_m^- A_B A_B^* = A_B^*$.

Proof of (ii) $\Rightarrow$ (iii)

From (ii), $A_B^* = \left(A_B\right)_m^- A_B A_B^*$

Postmultiply by $\left(A_B A_B^*\right)^- A_B$ on both sides

$$A_B^*\left(A_B A_B^*\right)^- A_B = \left(A_B\right)_m^- A_B A_B^*\left(A_B A_B^*\right)^- A_B$$
$$= \left(\left(A_1\right)_m^- \cup \left(A_2\right)_m^-\right)\left(A_1 \cup A_2\right)\left(A_1^* \cup A_2^*\right)\left(\left(A_1 \cup A_2\right)\left(A_1^* \cup A_2^*\right)\right)^-\left(A_1 \cup A_2\right)$$
$$= \left(\left(A_1\right)_m^- A_1 A_1^* \cup \left(A_2\right)_m^- A_2 A_2^*\right)\left(A_1 A_1^* \cup A_2 A_2^*\right)^-\left(A_1 \cup A_2\right)$$

$$=\left((A_1)_m^- A_1A_1^* \cup (A_2)_m^- A_2A_2^*\right)\left((A_1A_1^*)^- \cup (A_2A_2^*)^-\right)(A_1 \cup A_2)$$

$$=\left((A_1)_m^- A_1A_1^* \cup (A_2)_m^- A_2A_2^*\right)\left((A_1^*)^- A_1^- \cup (A_2^*)^- A_2\right)(A_1 \cup A_2)$$

$$=\left((A_1)_m^- A_1 \cup (A_2)_m^- A_2\right)(A_1^* \cup A_2^*)\left((A_1^*)^- \cup (A_2^*)^-\right)(A_1^- \cup A_2^-)(A_1 \cup A_2)$$

$$=\left((A_1)_m^- A_1 \cup (A_2)_m^- A_2\right)\left(A_1^*(A_1^-)^* \cup A_2^*(A_2^-)^*\right)(A_1^- \cup A_2^-)(A_1 \cup A_2)$$

$$=\left((A_1)_m^- A_1 \cup (A_2)_m^- A_2\right)\left((A_1^-A_1)^* \cup (A_2^-A_2)^*\right)(A_1^-A_1 \cup A_2^-A_2)$$

$$=\left((A_1)_m^- \cup (A_2)_m^- (A_1 \cup A_2)(A_B^-A_B)^*\right)(A_1^-A_1 \cup A_2^-A_2)$$

$$=(A_B)_m^-(A_BA_B^-A_B)A_B^-A_B$$

$$=(A_B)_m^- A_BA_B^-A_B \qquad \text{(by definition 3.1)}$$

$$=(A_B)_m^- A_B \qquad \text{(by definition 3.1)}$$

Hence, $(A_B)_m^- A_B = A_B^*(A_BA_B^*)^- A_B$.

Proof of (iii) $\Rightarrow$ (i)

From (ii) of theorem (2.2), $A_B = A_B(A_B^*A_B)^- A_B^*A_B$

Replace A_B by A_B^* we get

$$A_B^* = A_B^*\left((A_B^*)^* A_B^*\right)^-(A_B^*)^* A_B^*$$

$$A_B^* = A_B^*(A_BA_B^*)^- A_BA_B^*$$

$$A_B^* = (A_1^* \cup A_2^*)\left((A_1 \cup A_2)(A_1^* \cup A_2^*)\right)^-(A_1 \cup A_2)(A_1^* \cup A_2^*)$$

$$=(A_1^* \cup A_2^*)(A_1^* \cup A_2^*)^-(A_1 \cup A_2)^-(A_1 \cup A_2)(A_1^* \cup A_2^*)$$

$$=\left[(A_1^* \cup A_2^*)\left((A_1^*)^- \cup (A_2^*)^-\right)(A_1^- \cup A_2^-)(A_1 \cup A_2)\right](A_1^* \cup A_2^*)$$

$$=\left(A_1^*(A_1^*)^- A_1^-A_1 \cup A_2^*(A_2^*)^- A_2^-A_2\right)(A_1^* \cup A_2^*)$$

$$=\left(A_1^*(A_1A_1^*)^- A_1 \cup A_2^*(A_2A_2^*)^- A_2\right)(A_1^* \cup A_2^*)$$

$$=\left[(A_1^* \cup A_2^*)\left((A_1A_1^*)^- \cup (A_2A_2^*)^-\right)(A_1 \cup A_2)\right](A_1^* \cup A_2^*)$$

$$=\left[A_B^*(A_BA_B^*)^- A_B\right]A_B^*$$

$$A_B^* = (A_B)_m^- A_BA_B^* \qquad \text{(by (iii))}$$

$$(A_B^*)^* = \left((A_B)_m^- A_BA_B^*\right)^*$$

$$A_B = \left[\left((A_1)_m^- \cup (A_2)_m^-\right)(A_1 \cup A_2)(A_1^* \cup A_2^*)\right]^*$$

$$=\left[\left((A_1)_m^- A_1A_1^*\right) \cup \left((A_2)_m^- A_2A_2^*\right)\right]^*$$

$$=\left(\left(A_1\right)_m^- A_1A_1^*\right)^* \cup \left(\left(A_2\right)_m^- A_2A_2^*\right)^*$$

$$= A_1A_1^*\left(\left(A_1\right)_m^-\right)^* \cup A_2A_2^*\left(\left(A_2\right)_m^-\right)^*$$

$$=\left(A_1 \cup A_2\right)\left(A_1^* \cup A_2^*\right)\left(\left(\left(A_1\right)_m^-\right)^* \cup \left(\left(A_2\right)_m^-\right)^*\right)$$

$$A_B = A_BA_B^*\left(\left(A_B\right)_m^-\right)^*$$

Premultiply by $A_B^*\left(A_BA_B^*\right)^-$ on both sides

$$A_B^*\left(A_BA_B^*\right)^- A_B = \left(A_B^*\left(A_BA_B^*\right)^- A_B\right)A_B^*\left(\left(A_B\right)_m^-\right)^*$$

$$\left(A_B\right)_m^- A_B = \left(\left(A_B\right)_m^- A_BA_B^*\right)\left(\left(A_B\right)_m^-\right)^* \qquad \text{(by (iii))}$$

$$= A_B^*\left(\left(A_B\right)_m^-\right)^* \qquad \text{(by(ii))}$$

$$=\left(\left(A_B\right)_m^- A_B\right)^*$$

Hence, $\left(\left(A_B\right)_m^- A_B\right)^* = \left(A_B\right)_m^- A_B$.

Also, from (iii) of theorem (2.2),

$$A_B\left(A_B^*A_B\right)^- A_B^* = \left(A_B\left(A_B^*A_B\right)^- A_B^*\right)^*$$

Replace A_B by A_B^* we get

$$A_B^*\left(\left(A_B^*\right)^* A_B^*\right)^-\left(A_B^*\right)^* = \left(A_B^*\left(\left(A_B^*\right)^* A_B^*\right)^-\left(A_B^*\right)^*\right)^*$$

$$A_B^*\left(A_BA_B^*\right)^- A_B = \left(A_B^*\left(A_BA_B^*\right)^- A_B\right)^*$$

$$\left(A_B\right)_m^- A_B = \left[\left(A_1^* \cup A_2^*\right)\left(\left(A_1 \cup A_2\right)\left(A_1^* \cup A_2^*\right)\right)^-\left(A_1 \cup A_2\right)\right]^*$$

$$=\left[\left(A_1^* \cup A_2^*\right)\left(A_1A_1^* \cup A_2A_2^*\right)^-\left(A_1 \cup A_2\right)\right]^*$$

$$=\left[\left(A_1^* \cup A_2^*\right)\left(\left(A_1A_1^*\right)^- \cup \left(A_2A_2^*\right)^-\right)\left(A_1 \cup A_2\right)\right]^*$$

$$=\left[\left(A_1^* \cup A_2^*\right)\left(\left(A_1^*\right)^- A_1^- \cup \left(A_2^*\right)^- A_2^-\right)\left(A_1 \cup A_2\right)\right]^*$$

$$=\left[\left(A_1^*\left(A_1^*\right)^- A_1^-A_1\right) \cup \left(A_2^*\left(A_2^*\right)^- A_2^-A_2\right)\right]^*$$

$$=\left(\left(A_1^*\left(A_1^*\right)^- A_1^-A_1\right)^* \cup \left(A_2^*\left(A_2^*\right)^- A_2^-A_2\right)^*\right)$$

$$=\left(A_1^*\left(A_1^-\right)^*\left(\left(A_1^*\right)^-\right)^* A_1\right) \cup \left(A_2^*\left(A_2^-\right)^*\left(\left(A_2^*\right)^-\right)^* A_2\right)$$

$$=\left(A_1^*\left(A_1^-\right)^*\left(A_1^-\right)A_1\right) \cup \left(A_2^*\left(A_2^-\right)^* A_2^-A_2\right)$$

$$=\left(A_1^* \cup A_2^*\right)\left(\left(A_1^-\right)^* \cup \left(A_2^-\right)^*\right)\left(A_1^- \cup A_2^-\right)\left(A_1 \cup A_2\right)$$

$$= A_B^*\left(A_B^-\right)^* A_B^- A_B$$

$$=\left(A_B^- A_B\right)^* A_B^- A_B$$

$$= A_B^-\left(A_B A_B^- A_B\right) \qquad \text{(by definition 3.1)}$$

$$\left(A_B\right)_m^- A_B = A_B^- A_B \qquad \text{(by definition 3.1)}$$

Premultiply by A_B on both sides,

$$A_B\left(A_B\right)_m^- A_B = A_B A_B^- A_B$$

$$A_B\left(A_B\right)_m^- A_B = A_B \qquad \text{(by definition 3.1)}$$

Hence, $A_B\left(A_B\right)_m^- A_B = A_B$.

Theorem: 3.3

Let A_B be a bimatrix, then the following relations hold for a minimum norm generalized inverse $\left(A_B\right)_m^-$ of A_B :

(i) One choice of $\left(A_B A_B^*\right)_m^-$ is $\left(\left(A_B\right)_m^-\right)^*\left(A_B\right)_m^-$.

(ii) One choice of $\left(\lambda A_B\right)_m^- = \lambda^{-1}\left(A_B\right)_m^-$ where λ is a non- zero scalar.

(iii) One choice of $\left(U_B A_B V_B\right)_m^-$ is $V_B^*\left(A_B\right)_m^- U_B^*$ where U_B and V_B are unitary bimatrices.

(iv) $\left(A_B A_B^*\right)_m^- A_B A_B^* = \left(A_B\left(A_B\right)_m^-\right)^*$.

Proof of (i)

Now $A_B A_B^* = \left(A_1 \cup A_2\right)\left(A_1^* \cup A_2^*\right)$

$$= A_1 A_1^* \cup A_2 A_2^*$$

$$\left(A_B A_B^*\right)_m^- = \left(A_1 A_1^* \cup A_2 A_2^*\right)_m^-$$

$$=\left(A_1 A_1^*\right)_m^- \cup \left(A_2 A_2^*\right)_m^-$$

$$=\left(A_1^*\right)_m^-\left(A_1\right)_m^- \cup \left(A_2^*\right)_m^-\left(A_2\right)_m^-$$

$$=\left(\left(A_1^*\right)_m^- \cup \left(A_2^*\right)_m^-\right)\left(\left(A_1\right)_m^- \cup \left(A_2\right)_m^-\right)$$

$$=\left(\left(\left(A_1\right)_m^-\right)^* \cup \left(\left(A_2\right)_m^-\right)^*\right)\left(A_1 \cup A_2\right)_m^-$$

$$=\left(\left(A_1\right)_m^- \cup \left(A_2\right)_m^-\right)^*\left(A_B\right)_m^-$$

$$=\left(\left(A_B\right)_m^-\right)^*\left(A_B\right)_m^-.$$

Hence, $\left(A_B A_B^*\right)_m^- = \left(\left(A_B\right)_m^-\right)^*\left(A_B\right)_m^-$.

Proof of (ii)

Now $(\lambda A_B)_m^- = \left[\lambda(A_1 \cup A_2)\right]_m^-$

$$= (\lambda A_1 \cup \lambda A_2)_m^-$$

$$= (\lambda A_1)_m^- \cup (\lambda A_2)_m^-$$

$$= \lambda^{-1}(A_1)_m^- \cup \lambda^{-1}(A_2)_m^-$$

$$= \lambda^{-1}\left((A_1)_m^- \cup (A_2)_m^-\right)$$

$$= \lambda^{-1}(A_B)_m^-$$

Hence , $(\lambda A_B)_m^- = \lambda^{-1}(A_B)_m^-$.

Proof of (iii)

$$(U_B A_B V_B)_m^- = \left((U_1 \cup U_2)(A_1 \cup A_2)(V_1 \cup V_2)\right)_m^-$$

$$= (U_1 A_1 V_1 \cup U_2 A_2 V_2)_m^-$$

$$= (U_1 A_1 V_1)_m^- \cup (U_2 A_2 V_2)_m^-$$

$$= \left((V_1)_m^-\right)(A_1)_m^-(U_1)_m^- \cup (V_2)_m^-(A_2)_m^-(U_2)_m^-$$

$$= \left(V_1^*\left(V_1V_1^*\right)^-(A_1)_m^- U_1^*\left(U_1U_1^*\right)^-\right) \cup \left(V_2^*\left(V_2V_2^*\right)^-(A_2)_m^- U_2^*\left(U_2U_2^*\right)^-\right)$$

since $(A_B)_m^- = A_B^*\left(\left(A_B A_B^*\right)^-\right)$

$$= \left(V_1^*(I_1)^-(A_1)_m^- U_1^*(I_1)^-\right) \cup \left(V_2^*(I_2)^-(A_2)_m^- U_2^*(I_2)^-\right)$$

(Since $U_1, U_2, V_1 \& V_2$ are unitary matrices)

$$= \left(V_1^*(A_1)_m^- U_1^*\right) \cup \left(V_2^*(A_2)_m^- U_2^*\right)$$

$$= \left(V_1^* \cup V_2^*\right)\left((A_1)_m^- \cup (A_2)_m^-\right)\left(U_1^* \cup U_2^*\right)$$

$$= V_B^*(A_B)_m^- U_B^*.$$

Hence, $(U_B A_B V_B)_m^- = V_B^*(A_B)_m^- U_B^*$.

Proof of (iv)

$$\left(A_B A_B^*\right)_m^- A_B A_B^* = \left((A_1 \cup A_2)\left(A_1^* \cup A_2^*\right)\right)_m^-(A_1 \cup A_2)\left(A_1^* \cup A_2^*\right)$$

$$= \left(A_1A_1^* \cup A_2A_2^*\right)_m^-\left(A_1A_1^* \cup A_2A_2^*\right)$$

$$= \left(\left(A_1A_1^*\right)_m^- \cup \left(A_2A_2^*\right)_m^-\right)\left(A_1A_1^* \cup A_2A_2^*\right)$$

$$= \left(\left(A_1^*\right)_m^-(A_1)_m^- A_1A_1^*\right) \cup \left(\left(A_2^*\right)_m^-(A_2)_m^- A_2A_2^*\right)$$

$$= \left(\left((A_1)_m^-\right)^*(A_1)_m^- A_1A_1^*\right) \cup \left(\left((A_2)_m^-\right)^*\left((A_2)_m^- A_2A_2^*\right)\right)$$

$$= \left(\left((A_1)_m^-\right)^* \cup \left((A_2)_m^-\right)^*\right)\left((A_1)_m^- A_1A_1^* \cup (A_2)_m^- A_2A_2^*\right)$$

$$=\left(\left((A_1)_m^-\right)^* \cup \left((A_2)_m^-\right)^*\right)\left[\left((A_1)_m^- \cup (A_2)_m^-\right)(A_1 \cup A_2)\left(A_1^* \cup A_2^*\right)\right]$$

$$=\left((A_B)_m^-\right)^*\left((A_B)_m^- A_B A_B^*\right)$$

$$=\left((A_B)_m^-\right)^* A_B^* \qquad \text{(by (ii) of theorem 3.2)}$$

$$=\left(A_B (A_B)_m^-\right)^*$$

Hence, $\left(A_B A_B^*\right)_m^- A_B A_B^* = \left(A_B (A_B)_m^-\right)^*$.

IV. Least Square Generalized Inverses of Bimatrices

In this section some of the characteristics of least square g-inverses of matrices found in [2,3,5] are extended to least square g-inverses of bimatrices .

Definition: 4.1

A generalized inverse A_B^- that satisfies both $A_B A_B^- A_B = A_B$ and $\left(A_B A_B^-\right)^* = A_B A_B^-$ is called a least square generalized inverse bimatrix of A_B and is denoted by $\left(A_B\right)_l^-$.

Theorem: 4.2

Let A_B be a bimatrix, then the following three conditions are equivalent:

(i) $A_B (A_B)_l^- A_B = A_B$ and $\left(A_B (A_B)_l^-\right)^* = A_B (A_B)_l^-$

(ii) $A_B^* A_B (A_B)_l^- = A_B^*$

(iii) $A_B (A_B)_l^- = A_B \left(A_B^* A_B\right)^- A_B^*$.

Proof of (i) $\Rightarrow$ (ii)

From (i), $A_B = A_B (A_B)_l^- A_B$

$$A_B^* = \left(A_B (A_B)_l^- A_B\right)^*$$

$$A_B^* = \left((A_1 \cup A_2)\left((A_1)_l^- \cup (A_2)_l^- (A_1 \cup A_2)\right)\right)^*$$

$$=\left(A_1 (A_1)_l^- A_1 \cup A_2 (A_2)_l^- A_2\right)^*$$

$$=\left(A_1 (A_1)_l^- A_1\right)^* \cup \left(A_2 (A_2)_l^- A_2\right)^*$$

$$= A_1^*\left((A_1)_l^-\right)^* A_1^* \cup A_2^*\left((A_2)_l^-\right)^* A_2^*$$

$$=\left(A_1^* \cup A_2^*\right)\left[\left(\left((A_1)_l^-\right)^* \cup \left((A_2)_l^-\right)^*\right)\left(A_1^* \cup A_2^*\right)\right]$$

$$= A_B^*\left(\left((A_B)_l^-\right)^* A_B^*\right)$$

$$= A_B^*\left(A_B (A_B)_l^-\right)^*$$

$$= A_B^* A_B (A_B)_l^-$$

Hence , $A_B^* A_B \left(A_B\right)_l^- = A_B^*$.

Proof of (ii) $\Rightarrow$ (iii)

From (ii), $A_B^* A_B \left(A_B\right)_l^- = A_B^*$

Premultiply by $A_B \left(A_B^* A_B\right)^-$ on both sides

$$A_B \left(A_B^* A_B\right)^- A_B^* = A_B \left(A_B^* A_B\right)^- A_B^* A_B \left(A_B\right)_l^-$$

$$= \left(A_1 \cup A_2\right)\left(\left(A_1^* \cup A_2^*\right)\left(A_1 \cup A_2\right)\right)^- \left(A_1^* \cup A_2^*\right)\left(A_1 \cup A_2\right)\left(\left(A_1\right)_l^- \cup \left(A_2\right)_l^-\right)$$

$$= \left(A_1 \cup A_2\right)\left(A_1 \cup A_2\right)^- \left(A_1^* \cup A_2^*\right)^- \left(A_1^* \cup A_2^*\right)\left(A_1 \cup A_2\right)\left(\left(A_1\right)_l^- \cup \left(A_2\right)_l^-\right)$$

$$= \left(A_1 \cup A_2\right)\left(A_1^- \cup A_2^-\right)\left(\left(A_1^*\right)^- \cup \left(A_2^*\right)^-\right)\left(A_1^* \cup A_2^*\right)\left(A_1 \cup A_2\right)\left(\left(A_1\right)_l^- \cup \left(A_2\right)_l^-\right)$$

$$= \left(A_1 A_1^- \left(A_1^*\right)^- A_1^* A_1 \left(A_1\right)_l^-\right) \cup \left(A_2 A_2^- \left(A_2^*\right)^- A_2^* A_2 \left(A_2\right)_l^-\right)$$

$$= \left(A_1 A_1^- \left(A_1^-\right)^* A_1^* A_1 \left(A_1\right)_l^-\right)\left(A_2 A_2^- \left(A_2^-\right)^* A_2^* A_2 \left(A_2\right)_l^-\right) \qquad \text{(by (i) theorem 2.2)}$$

$$= \left(A_1 \cup A_2\right)\left(A_1^- \cup A_2^-\right)\left(\left(A_1^-\right)^* \cup \left(A_2^-\right)^*\right)\left(A_1^* \cup A_2^*\right)\left(A_1 \cup A_2\right)\left(\left(A_1\right)_l^- \cup \left(A_2\right)_l^-\right)$$

$$= A_B A_B^- \left(A_B^-\right)^* A_B^* A_B \left(A_B\right)_l^-$$

$$= A_B A_B^- \left(A_B A_B^-\right)^* A_B \left(A_B\right)_l^-$$

$$= A_B A_B^- \left(A_B A_B^- A_B\right)\left(A_B\right)_l^- \qquad \text{(by definition 4.1)}$$

$$= \left(A_B A_B^- A_B\right)\left(A_B\right)_l^- \qquad \text{(by definition 4.1)}$$

$$= A_B \left(A_B\right)_l^- \qquad \text{(by definition 4.1)}$$

Hence, $A_B \left(A_B\right)_l^- = A_B \left(A_B^* A_B\right)^- A_B^*$.

Proof of (iii) $\Rightarrow$ (i)

From (iii) of theorem (2.2),

$$A_B \left(A_B^* A_B\right)^- A_B^* = \left(A_B \left(A_B^* A_B\right)^- A_B^*\right)^*$$

$$A_B \left(A_B\right)_l^- = \left(\left(A_1 \cup A_2\right)\left(\left(A_1^* \cup A_2^*\right)\left(A_1 \cup A_2\right)\right)^- \left(A_1^* \cup A_2^*\right)\right)^* \qquad \text{(by (iii))}$$

$$= \left[\left(A_1 \cup A_2\right)\left(A_1 \cup A_2\right)^- \left(A_1^* \cup A_2^*\right)^- \left(A_1^* \cup A_2^*\right)\right]^*$$

$$= \left[\left(A_1 \cup A_2\right)\left(A_1^- \cup A_2^-\right)\left(\left(A_1^*\right)^- \cup \left(A_2^*\right)^-\right)\left(A_1^* \cup A_2^*\right)\right]^*$$

$$= \left[\left(A_1 A_1^- \left(A_1^*\right)^- A_1^*\right) \cup \left(A_2 A_2^- \left(A_2^*\right)^- A_2^*\right)\right]^*$$

$$= \left(A_1 A_1^- \left(A_1^*\right)^- A_1^*\right)^* \cup \left(A_2 A_2^- \left(A_2^*\right)^- A_2^*\right)^*$$

$$=\left(A_1\left(\left(A_1^*\right)^-\right)^*\left(A_1^-\right)^*A_1^*\right)\cup\left(A_2\left(\left(A_2^*\right)^-\right)^*\left(A_2^-\right)^*A_2^*\right)$$

$$=\left(A_1\left(\left(A_1^-\right)^*\right)^*\left(A_1^-\right)^*A_1^*\right)\cup\left(A_2\left(\left(A_2^-\right)^*\right)^*\left(A_2^-\right)^*A_2^*\right)$$

$$=\left(A_1A_1^-\left(A_1^-\right)^*A_1^*\right)\cup\left(A_2A_2^-\left(A_2^-\right)^*A_2^*\right)$$

$$=\left(A_1\cup A_2\right)\left(A_1^-\cup A_2^-\right)\left(\left(A_1^-\right)^*\cup\left(A_2^-\right)^*\right)\left(A_1^*\cup A_2^*\right)$$

$$=A_BA_B^-\left(A_B^-\right)^*A_B^*$$

$$=A_BA_B^-\left(A_BA_B^-\right)^*$$

$$=\left(A_BA_B^-A_B\right)A_B^- \qquad \text{(by definition 4.1)}$$

$$A_B\left(A_B\right)_l^-=A_BA_B^- \qquad \text{(by definition 4.1)}$$

Postmultiply by A_B on both sides

$$A_B\left(A_B\right)_l^-A_B=A_BA_B^-A_B$$

$$A_B\left(A_B\right)_l^-A_B=A_B \qquad \text{(by definition 4.1)}$$

Hence , $A_B\left(A_B\right)_l^-A_B=A_B$.

Also, from (ii) of theorem (2.4),

$$A_B=A_B\left(A_B^*A_B\right)^-A_B^*A_B$$

$$A_B^*=\left(A_B\left(A_B^*A_B\right)^-A_B^*A_B\right)^*$$

$$=\left[\left(A_1\cup A_2\right)\left(\left(A_1^*\cup A_2^*\right)\left(A_1\cup A_2\right)\right)^-\left(A_1^*\cup A_2^*\right)\left(A_1\cup A_2\right)\right]^*$$

$$=\left[\left(A_1\cup A_2\right)\left(A_1\cup A_2\right)^-\left(A_1^*\cup A_2^*\right)^-\left(A_1^*\cup A_2^*\right)\left(A_1\cup A_2\right)\right]^*$$

$$=\left[\left(A_1\cup A_2\right)\left(A_1^-\cup A_2^-\right)\left(\left(A_1^*\right)^-\cup\left(A_2^*\right)^-\right)\left(A_1^*\cup A_2^*\right)\left(A_1\cup A_2\right)\right]^*$$

$$=\left[\left(A_1A_1^-\left(A_1^*\right)^-A_1^*A_1\right)^*\cup\left(A_2A_2^-\left(A_2^*\right)^-A_2^*A_2\right)^*\right]$$

$$=\left(A_1^*A_1\left(A_1^*\right)^-\right)^*\left(A_1^-\right)^*A_1^*\cup\left(A_2^*A_2\left(A_2^*\right)^-\right)^*\left(A_2^-\right)^*A_2^*$$

$$=\left(A_1^*A_1A_1^-\left(A_1^-\right)^*A_1^*\right)\cup\left(A_2^*A_2A_2^-\left(A_2^-\right)^*A_2^*\right)$$

$$=\left(A_1^*A_1\left(A_1^-\right)\left(A_1^*\right)^-A_1^*\right)\cup\left(A_2^*A_2A_2^-\left(A_2^*\right)^-A_2^*\right)$$

$$=\left(A_1^*A_1\left(A_1^*A_1\right)^-A_1^*\right)\cup\left(A_2^*A_2\left(A_2^*A_2\right)^-A_2^*\right)$$

$$=\left(A_1^*\cup A_2^*\right)\left[\left(A_1\cup A_2\right)\left(\left(A_1^*A_1\right)^-\cup\left(A_2^*A_2\right)^-\right)\left(A_1^*\cup A_2^*\right)\right]$$

$$=\left(A_1^*\cup A_2^*\right)\left[A_B\left(A_B^*A_B\right)^-A_B^*\right]$$

$$A_B^* = A_B^* A_B \left(A_B\right)_l^- \quad \text{(by (iii))}$$

$$\left(A_B^*\right)^* = \left(A_B^* A_B \left(A_B\right)_l^-\right)^*$$

$$\left(A_1^* \cup A_2^*\right)^* = \left(\left(A_1^* \cup A_2^*\right)\left(A_1 \cup A_2\right)\left(A_1 \cup A_2\right)_l^-\right)^*$$

$$A_1 \cup A_2 = \left[\left(A_1^* \cup A_2^*\right)\left(A_1 \cup A_2\right)\left(\left(A_1\right)_l^- \cup \left(A_2\right)_l^-\right)\right]^*$$

$$A_B = \left[\left(A_1^* A_1 \left(A_1\right)_l^-\right) \cup \left(A_2^* A_2 \left(A_2\right)_l^-\right)\right]^*$$

$$= \left(A_1^* A_1 \left(A_1\right)_l^-\right)^* \cup \left(A_2^* A_2 \left(A_2\right)_l^-\right)^*$$

$$= \left(\left(A_1\right)_l^-\right)^* A_1^* A_1 \cup \left(\left(A_2\right)_l^-\right)^* A_2^* A_2$$

$$= \left(\left(\left(A_1\right)_l^-\right)^* \cup \left(\left(A_2\right)_l^-\right)^*\right)\left(A_1^* \cup A_2^*\right)\left(A_1 \cup A_2\right)$$

$$A_B = \left(\left(A_B\right)_l^-\right)^* A_B^* A_B$$

Postmultiply by $\left(A_B^* A_B\right)^- A_B^*$ on both sides

$$A_B \left(A_B^* A_B\right)^- A_B^* = \left(\left(A_B\right)_l^-\right)^* A_B^* \left[A_B \left(A_B^* A_B\right)^- A_B^*\right]$$

$$A_B \left(A_B\right)_l^- = \left(\left(A_B\right)_l^-\right)^* A_B^* A_B \left(A_B\right)_l^- \quad \text{(by (iii))}$$

$$= \left(\left(A_B\right)_l^-\right)^* A_B^* \quad \text{(by (ii))}$$

$$= \left(A_B \left(A_B\right)_l^-\right)^*$$

Hence, $\left(A_B \left(A_B\right)_l^-\right)^* = A_B \left(A_B\right)_l^-$.

Theorem: 4.3

Let A_B be a bimatrix, then the following relations hold for a least square generalized inverse $\left(A_B\right)_l^-$ of A_B:

(i) $\left(\lambda A_B\right)_l^- = \lambda^{-1}\left(A_B\right)_l^-$ where λ is a non zero scalar.

(ii) One choice of $\left(U_B A_B V_B\right)_l^- = V_B^* \left(A_B\right)_l^- U_B^*$ where U_B and V_B are unitary bimatrices.

Proof of (i)

Now $\left(\lambda A_B\right)_l^- = \left[\lambda\left(A_1 \cup A_2\right)\right]_l^-$

$$= \left[\left(\lambda A_1 \cup \lambda A_2\right)\right]_l^-$$

$$= \left(\lambda A_1\right)_l^- \cup \left(\lambda A_2\right)_l^-$$

$$= \lambda^{-1}\left(A_1\right)_l^- \cup \lambda^{-1}\left(A_2\right)_l^-$$

$$= \lambda^{-1}\left(\left(A_1\right)_l^- \cup \left(A_2\right)_l^-\right)$$

$$= \lambda^{-1}\left(A_B\right)_l^-$$

Hence, $\left(\lambda A_B\right)_l^- = \lambda^{-1}\left(A_B\right)_l^-$.

Proof of (ii)

Now $\left(U_B A_B V_B\right)_l^- = \left(\left(U_1 \cup U_2\right)\left(A_1 \cup A_2\right)\left(V_1 \cup V_2\right)\right)_l^-$

$$= \left(\left(U_1 A_1 V_1\right) \cup \left(U_2 A_2 V_2\right)\right)_l^-$$

$$= \left(U_1 A_1 V_1\right)_l^- \cup \left(U_2 A_2 V_2\right)_l^-$$

$$= \left(\left(V_1\right)_l^- \left(A_1\right)_l^- \left(U_1\right)_l^-\right) \cup \left(\left(V_2\right)_l^- \left(A_2\right)_l^- \left(U_2\right)_l^-\right)$$

$$= \left(\left(V_1^* V_1\right)^- V_1^* \left(A_1\right)_l^- \left(U_1^* U_1\right)^- U_1^*\right) \cup \left(\left(V_2^* V_2\right)^- V_2^* \left(A_2\right)_l^- \left(U_2^* U_2\right)^- U_2^*\right)$$

$$\left(\text{since} \left(A_B\right)_l^- = \left(A_B^* A_B\right)^- A_B^*\right)$$

$$= \left(\left(I_1\right)^- V_1^* \left(A_1\right)_l^- \left(I_1\right)^- U_1^*\right) \cup \left(\left(I_2\right)^- V_2^* \left(A_2\right)_l^- \left(I_2\right)^- U_2^*\right)$$

(since $U_1, U_2, V_1 \, and \, V_2$ are unitary bimatrices)

$$= \left(V_1^* \left(A_1\right)_l^- U_1^*\right) \cup \left(V_2^* \left(A_2\right)_l^- U_2^*\right)$$

$$= \left(V_1^* \cup V_2^*\right)\left(\left(A_1\right)_l^- \cup \left(A_2\right)_l^-\right)\left(U_1^* \cup U_2^*\right)$$

$$= V_B^* \left(A_B\right)_l^- U_B^*$$

Hence, $\left(U_B A_B V_B\right)_l^- = V_B^* \left(A_B\right)_l^- U_B^*$.

References

[1]. Ben Isreal.A and Greville.T.N.E, Generalized Inverse: Theory and Applications, Wiley-Interscience, New York, 1994.

[2]. Campbell.S.L and Meyer.C.D, Generalized Inverses of Linear Transformations,Society for Industrial and Applied Mathematics,Philadelphia, 2009.

[3]. Haruo Yanai, Kei Takeuchi, Yashio Takane, Projection Matrices, Geneneralized Inverse Matrices, and Singular Value Decomposition, Springer, New York, 2011.

[4]. Penrose.R, A generalized Inverse for Matrices, Proc. Cambridge Phil. Soc., Vol.51, 1955, (PP 406-413).

[5]. Radhakrishna Rao.C, Calculus of Generalized Inverses of Matrices Part-I: General Theory, Sankhya, Series A, Vol.29, No. 3, September 1967, (PP 317-342).

[6]. Radhakrishna Rao.C and Sujit Kumar Mitra, Generalised Inverse of a matrix and its Applications, Wiley, New York, 1971.

[7]. Ramesh.G. and Maduranthaki.P, On Unitary Bimatrices, International Journal of Current Research, Vol.6, Issue 09, September 2014, (PP 8395-8407).

[8]. Ramesh.G. and Maduranthaki.P, On Some properties of Unitary and Normal Bimatrices, International Journal of Recent Scientific Research, Vol.5, Issue10 , October 2014, (PP 1936-1940).