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Fixed Point Theorm In Probabilistic Analysis

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Probabilistic operator theory is the branch of probabilistic analysis which is concerned with the study of operator-valued random variables and their properties. The development of a theory of random operators is of interest in its own right as a probabilistic generalization of (deterministic) operator theory and just as operator theory is of fundamental importance in the study of operator equations, the development of probabilistic operator theory is required for the study of various classes of random equations.

Defenition.1.1. Any $\aleph$-valued random variable x(□) which satisfies the condition □ ({□ : T (□) □ (□) = y (□)}) = 1 is said to be a random solution of the random operator equation T(□) x (□) = y (□).

Defenition.1.2: An $\aleph$-valued random variable □ (□) is said to be a fixed point of the random operator T(□) if □ (□) is a random solution of the equation T(□)□ (□) = □ (□).

The study of fixed point theorems for random operators was initiated by Špaček and Hanš[1]. The first systematic investigation of random fixed point theorems was carried out by Hanš[1]. Because of the wide applicability of Banach's contraction mapping theorem in the study of deterministic operator equations, Špaček and Hanš directed their attention to probabilistic versions of Banach's theorem and used their results to prove the existence, uniqueness, and measurability of solutions of integral equations with random kernels.

Defenitin.1.3 :A random operator T(□) on a Banach space $\aleph$ with domain D(T(□)) is said to be a random contraction operator if there exists a nonnegative real-valued random variable such that k(□) < 1, and such that as, □ □ T(□)x_1– T(□)x_2□ □ □ k(□)□ □ x_1–x_2□ □ for all x_1, x_2 □ D(T(□)). If k(□) = k (a constant) for all □ □ □ , then T(□) is called a uniform random contraction operator.

Theorem. 1.4: Let $\aleph$ be a separable Banach space, and let. T(□) be a continuous random operator on $\aleph$ to itself such that

$$\mu\left[\bigcup_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcap_{x_1\in\chi}\bigcap_{x_2\in\chi}\left\{\omega:\left\|T^n(\omega)x_1-T^n(\omega)x_2\right\|\right.\right.$$
$$\left.\left.\le\left(1-\frac{1}{m}\right)\left\|x_1-x_2\right\|\right\}\right]=1,$$

Where for every □ □ □ , x □$\aleph$, and n = 1, 2, …., we put T^1(□)x = T(□)x, and T^{n+1}(□)x = T(□)[T^n(□)x]. Then. There exists an $\aleph$-valued random variable □ (□) which is the unique fixed point of T(□); that is if □ (□) is another fixed point, then □ (□)=□ (□ .)

PROOF. Let E denote those elements of □ belonging to the set

$$\bigcup_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcap_{x_1\in\chi}\bigcap_{x_2\in\chi}\left\{\omega:\left\|T^n(\omega)x_1-T^n(\omega)x_2\right\|\le\left(1-\frac{1}{m}\right)\left\|x_1-x_2\right\|\right\}$$

for with T (□) is continuous. Clearly E □ P, and by hypothesis, □(E) = 1. Let the mapping □ (□) : □ □$\aleph$ be defined as follows : For every □ □ E, □ (□) is equal to the unique fixed point of T(□); and for every □ □- E, put □ (□) = □ (the null element of $\aleph$). Then T(□)□ (□) = □ (□)

Now,we proceed to establish the measurability of the fixed point □ (□ .) Let x_0(□) be an arbitrary $\aleph$-valued random variable, and put x_1(□) = T(□) x_0(□). x_1(□) is an $\aleph$-valued random variable and a sequence of

$\aleph$-valued random variables can be defined as follows : x_n(□) = T(□)x_{n-1}(□), n = 1, 2,… Now, since T(□) is continuous, the sequence x_n(□) converges almost surely to □ (□); hence □ (□) is an $\aleph$-valued random variable.

The uniqueness of the fixed point follows from the uniqueness of □ (□) for every □ □ E.

Theorem.1.5: Let T(□) be a continuous random operator on a separable Banach space $\aleph$ to itself, and let k(□) be a nonnegative real-valued random variable such that k(□) < 1.and □ □ T(□) x_1 – T (□) x_2 □ □ □ k(□) □ □ x_1 – x_2 □ □ for every pair of elements x_1, x_2 □ $\aleph$. Then there exists an $\aleph$-valued random variable □ (□) which is the unique fixed point of T(□).

PROOF. Let E = {□ : k(□) < 1}, F = {□ : T(□) x is continuous in x}, and

$G_{x1,x2}$ = {□ : □ □ T(□) x_1 – T(□) x_2 □ □ □ k(□) □ □ x_1 – x_2 □ □ }.

Since $\aleph$ is separable, the intersections in the expression

$$\bigcap_{x_1\in\chi}\ \bigcap_{x_2\in\chi}(G_{x1,x2}\cap E\cap F)$$

can be replaced by intersections over a countable dense set of $\aleph$. Therefore the condition of Theorem (1.4) is satisfied with n = 1.

Random contraction mapping theorems are of fundamental importance in the theory of random equations in that they can be used to establish the existence, uniqueness, and measurability of solutions of random operator equations.

Theorem.1.6. Let T(□) be a random contraction operator on a separable Banach space $\aleph$, and let k(□) be a nonnegative real-valued random variable which is bounded. Then, for every real □ □ 0 such that k(□) < □ □ □ . there exists a random operator S(□) which is the inverse of T(□) - □ I.

Proof. Since □ □ 0, T(□) - □ I is invertible whenever the random operator (I/□) T(□) - I is invertible, and vice versa. However, for every y □ $\aleph$ the random operator T_y(□) defined, for every □ □ □ and x □ $\aleph$, by T_y(□)[x] = (1/□) T(□) x - y is a random contraction operator. Therefore, by Theorem (1.6) there exists a unique random fixed point $□_y$(□) satisfying the relation $□_y$(□) = (1/□) T(□) $□_y$(□) – y a.s. However, the above statement is equivalent to the invertibility of the random operator (1/□) T(□) - I, and therefore the invertibility of the random operator T(□)- □ I.

Theorem 1.7: Let (□ □ □) be an atomic probability measure space, and let E be a compact (or closed and bounded) convex subset of a separable Banach space $\aleph$. Let T(□) be a compact random operator mapping E into itself. Then, there exists an E-valued random variable □ (□) such that T(□)□ (□) = □ (□) a.s. n, such that T($□_n$)$□_n$ = $□_n$. Put □ (□) = $□_n$ for □ □ C_n, and 0 otherwise. Then T(□)□ (□)= □ (□).

Theorem 1.8. Let E be a compact convex subset of a Banach space and T(□) be a continuous random operator mapping E into itself. Then there exists an E-valued random variable □ (□) such that T(□)□ (□) = □ (□)

PROOF. Let A(□) = {x □ E : T(□)x = x}. Then by Theorem (D) for each □ the set A(□) is nonempty. Furthermore, for any closed subset F of E

{□ : A(□) □ F is nonempty} = {□ : A(□)x = x for some x in F}

$$\bigcap_{n=1}^{\infty}\bigcup_{i=1}^{\infty}\left\{\omega:\|T(\omega)x_i-x_i\|<1/n\right\}$$

Where the x_i's form a dense sequence in F. It is therefore clear that set {□ : A(□) □ F is nonempty} is measurable for every closed subset F of E. To prove the theorem, it is sufficient to find an E-valued random variable □ (□) such that □ (□) □ A(□). It is known that we can associate with the space E a sequence of triples (C_n, P_n, $□_n$) (n a nonnegative integer) such that

(i) Each C_n is a countable set and P_n maps C_{n+1} onto C_n;

(ii) $□_n$ maps C_n into a class of nonempty closed subsets of E of diameter □ $^{2-n}$.

(iii) E = $U_{c□ C0}$ $□_0$(c);

(iv) for each n and for each c in C_n,

Without any loss of generality. We assume that C_0 and each P_n^{-1}(c) with c in C_n are naturally linearly ordered such that only finitely many elements can precede any element in this order. Now, for each n, we intend to find a suitable partition of □ . We proceed inductively as follows: For each c in C_0, we define $□_c$ by □ □ $□_c$ if and only if A (□) □ $□_0$(c) is nonempty and A(□) □ $□_0$(c□) is empty for c□ □ C_0, c□ < c. Then the $□_c$□ s arepair wise disjoint measurable sets with union □ . Suppose now that we have found a partition of □ corresponding to the elements of C_k. To do this for C_{k+1}, we define for c in C_{k+1} the set $□_c$ by □ □ $□_c$ if and only □ □ $□_{pk(c)}$ and A(□) □ $□_{k+1}$(c) is nonempty, but A(□) □ $□_{k+1}$(c□) is empty for c□ in $P_k^{-1}(P_k(c))$ and c□ < c.

For any positive n and each c □ C_n we choose an element x_n(c) □ $□_n$(c) and define $□_n$(□) = x_n(c) for □ □ $□_c$ where the $□_c$□ s are members of the partition of □ corresponding to the elements of C_n. Then each $□_n$(□ is measurable and

□ □ $□_n$(□)- $□_{n+1}$ (□)□ □ □ 2^{-n}, d ($□_n$(□), A(□)) □ 2^{-n}.

Therefore if □ (□) = lim $□_n$(□), then □ (□) □ A(□) and the theorem follows.

Theorem. 1.10: Let T(□) be a stochastically continuous random operator on a separable Banch space □ to itself. Suppose that for each □ □ □ , {x : T(□)x = x} □ Ø (the null set). Then there exists a measurable multi-valued map □ (□) : □ □ 2^X such that □ (□) = (x : T(□)x = x}.

In a number of applications of fixed point theorems in probabilistic analysis, it is assumed that a random operator T(□) satisfies the hypotheses of Schauder's theorem for each □ □ □ . Then, if T(□) is a continuous random operator it is also stochastically continuous and separable.

Theorem.1.11: Let (□ , P, □) be a probability measure space, and let E be a compact and convex subset of a separable Banach space □ . Let T(□)x_1 + T(□)x_2 □ E for all x_1, x_2 □ E and □ □ □ , (ii) there exists a nonnegative real-valued random variable k(□) such that □ □ S(□)x_1 – S(□)x_2□ □ □ k(□)□ $□_1$ – x_2□ □ for all x_1, x_2 □ E and k(□) < 1 a.s. Then there exists an X– valued random variable □ (□) such that S(□)□ (□) + T(□)□ (□) = □ (□) for all □ □ □ .

The proof of the above theorem follows easily from Theorem 1.8 observing that the operator [I – S(□)]$^{-1}$ T(□) is a well-defined continuous mapping on E into itsel

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