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n -Path Graph

P. Uma[1] and R. Kala [2]
Department of Mathematics, Einstein College of Engineering Tirunelveli 627 012, Tamil Nadu, India.
Department of Mathematics, Manonmaniam Sundaranar University,Tirunelveli 627 012, Tamil Nadu, India.

Abstract: *The n-path graph $PG_n(G)$ of a graph G is a graph having the same vertex set as G and 2 vertices u and v in $PG_n(G)$ are adjacent if and only if there exist a path of length n between u and v in G. In this paper we find n-path graph of some standard graphs. Bounds are given for the degree of a vertex in $PG_n(G)$. We further characterise graphs G with $PG_2(G) = \overline{G}$, $PG_2(G) = G$ and $PG_2(G) = K_n$*
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I. Introduction

By a graph $G = (V, E)$ we mean a finite, undirected connected graph without loops and multiple edges. Terms not defined here are used in the sense of Harary[2,].

Research in graph theory is developing in diverse aspects. One among these is the study of graphs derived from graphs. In this paper we define a new graph called n-path graph for any connected graph G. It is defined as a graph having the same vertex set as G and 2 vertices u and v are adjacent in $PG_n(G)$ if and only if there exist a path of length n between u and v in G.
The open neighbourhood $N(v)$ of a vertex v in a graph G is the set of all vertices adjacent to v in G.

II. Main Results

Definition 2.1 The n-path graph $PG_n(G)$ of a graph G is a graph having the same vertex set as G and 2 vertices u and v in $PG_n(G)$ are adjacent if and only if there exist a path of length n between u and v in G.

Example 2.2 A graph G and its $PG_n(G)$ are given in figure.
Theorem 2.3
1. $PG_2(K_{1,n}) = K_n \cup K_1$.
2. $PG_2(B_{m,n}) = K_{m+1} \cup K_{n+1}$.
3. $PG_2(K_{m,n}) = K_m \cup K_n$.
4. If G is a spider $S(K_{1,n})$ then $PG_2(G) = K_n \cup K_{1,n}$.
5. If G is a wounded spider with r wounded edges then $PG_2(G) = K_{1,n-r} \cup K_n$.
6. If G is a Wheel W_n then $PG_2(G) = K_{n+1}$.
7. $PG_2(K_n) = K_n$.

Proof.
1. Let $V(K_{1,n}) = (V_1, V_2)$, where $V_1 = \{v_0\}$, $V_2 = \{v_1, v_2, \dots, v_n\}$. All the vertices of V_2 are connected to each other by a path of length 2. So all the n vertices in V_2 are adjacent to each other in $PG_2(G)$. Also v_0 is not adjacent to any other vertex in $PG_2(G)$. Hence $PG_2(K_{1,n}) = K_n \cup K_1$.
2. Let the vertices of $B(m,n)$ be v_1, v_2, $v_{1i} (1 \le i \le m)$, $v_{2j} (1 \le j \le n)$, where v_1, v_2 are 2 centers and v_{1i}, v_{2j} are pendent vertices. Let $S_1 = \{v_1, v_{21}, v_{22}, \dots, v_{2n}\}$, $S_2 = \{v_2, v_{11}, v_{12}, \dots, v_{1m}\}$ be a partition of

$V(G)$. No vertex of S_1 is connected by a path of length 2 to a vertex of S_2. Therefore $< S_1 >$ and $< S_2 >$ are 2 components in $PG_2(G)$. Also any 2 vertices in S_1 (resp. S_2) are connected by a path of length 2 in $PG_2(G)$. Hence $PG_2(B_{m,n}) = K_{m+1} \cup K_{n+1}$.

3. Let $V(K_{m,n}) = U \cup V$, where $U = \{u_1, u_2, \ldots, u_m\}, V = \{v_1, v_2, \ldots, v_n\}$. As above, we get $PG_2(K_{m,n}) = K_m \cup K_n$.

4. Let the vertices of $K_{1,n}$ be $\{v_0, v_1, v_2, \ldots, v_n\}$ where v_0 is the center. Let G be the spider obtained by subdividing $K_{1,n}$ and u_i be the new vertex obtained by subdividing $v_0 v_i (1 \le i \le n)$. Let $S_1 = \{v_0, v_1, v_2, \ldots, v_n\}$ and $S_2 = \{u_1, u_2, \ldots, u_n\}$ be a partition of the vertex set. No vertex of S_1 is connected by a path of length 2 to a vertex of S_2. So $< S_1 >$ and $< S_2 >$ are 2 components in $PG_2(G)$. Also any 2 vertices of S_2 are connected to each other by a path of length 2. So in $PG_2(G)$, $< S_2 >= K_n$. Also all the vertices $v_1, v_2, \ldots, v_n$ are connected to v_0 by a path of length 2 and they are not connected to themselves by a path of length 2. So $< S_1 >= K_{1,n}$. Hence $PG_2(G) = K_n \cup K_{1,n}$.

5. Let the vertices of $K_{1,n}$ be $\{v_0, v_1, \ldots, v_n\}$ where v_0 is the center. Let G be the wounded spider obtained by subdividing $n-r$ edges. Let u_i be the new vertex obtained by subdividing $v_0 v_i (1 \le i \le n-r)$. By an argument similar to the above, we get $PG_2(G) = K_{1,n-r} \cup K_n$.

6. Let the vertices of the wheel W_n be $\{v, v_1, v_2, \ldots, v_n\}$, where v is the center of the wheel. Every vertex is connected by a path of length 2 to all the other vertices. Hence $PG_2(W_n) = K_{n+1}$.

7. Let the vertices of K_n be $\{v_1, v_2, \ldots, v_n\}$. It is obvious that $PG_2(K_n) = K_n$.

Theorem 2.4 Let P_n be a path on n vertices. $PG_2(P_n) = P_{n_1} \cup P_{n_2}$, where $n_1 + n_2 = n, n_1 = \left\lceil \frac{n}{2} \right\rceil$ and $n_2 = \left\lfloor \frac{n}{2} \right\rfloor$.

Proof. Let P_n be $v_1 v_2 \ldots v_n$. Let $n_1 = \left\lceil \frac{n}{2} \right\rceil$, $n_2 = \left\lfloor \frac{n}{2} \right\rfloor$. Let $S_1 = \{v_1, v_3, \ldots\}$, $S_2 = \{v_2, v_4, \ldots\}$. No vertex of S_1 is adjacent to a vertex of S_2 in $PG_2(P_n)$. So $< S_1 >$ and $< S_2 >$ are two components of $PG_2(P_n)$. Also $|S_1| = n_1$ and $|S_2| = n_2$. It is easy to observe that $< S_1 >= P_{n_1}$ and $< S_2 >= P_{n_2}$. Hence $PG_2(P_n) = P_{n_1} \cup P_{n_2}$.

Now we extend this to any positive integer r.

Theorem 2.5 Let P_n be a path of n vertices. $PG_r(P_n) = P_{n_1} \cup P_{n_2} \cup \ldots \cup P_{n_r}$, where $n_1 + n_2 + \ldots + n_r = n$ and $n_i = \left\lfloor \frac{n-i}{r} \right\rfloor + 1. (1 \le i \le r)$

Proof. Let $S_i = \{v_i, v_{r+i}, v_{2r+i} \ldots, v_{k_i r+i}\}$, $k_i = \left\lfloor \frac{n-i}{r} \right\rfloor$, $1 \le i \le r$. By the definition of $PG_r(P_n)$, v_i and $v_{k_i r+i}$ are the 2 vertices which are adjacent to one vertex and all other vertices are adjacent to 2 vertices. Therefore each $< S_i >$ is a path of length k_i. Also for any i and j $(1 \le i, j \le r)$, no vertex of S_i is adjacent to a vertex of S_j. Therefore $<S_i>$'s are disconnected components of $PG_r(P_n)$. Hence $PG_r(P_n) = P_{n_1} \cup P_{n_2} \cup \ldots \cup P_{n_k}$.

Lemma 2.6 Let C_n be a cycle of length n. Then $PG_r(C_n) = PG_{n-r}(C_n)$.

Proof. If there is a path of length r in the clockwise direction then there is a path of length $n-r$ in the anti clockwise direction. So $PG_r(C_n) = PG_{n-r}(C_n)$.

Theorem 2.7 Let C_n be a cycle of length n. Then $PG_r(C_n) = qC_{\frac{n}{q}}$, where $q = gcd(r,n)$.

Proof. Let $V(C_n) = \{1,2,3,\ldots,n\}$ and $q = gcd(r,n)$. Let us partition $V(C_n)$ into r subsets as follows. Assume that the vertices of $S_i's$ are listed in the increasing order of their indices.

$S_i = \{rs + i / i = 1,2,\ldots,k, s = 0,1,2,\ldots,\frac{n-k}{r}, n \equiv k(mod\ r)\}$,

$S_j = \{rs + j / j = k+1, k+2,\ldots,r, s = 0,1,2,\ldots,\frac{n-k}{r} - 1\}$. By the choice of S_i any 2 vertices in each S_i's are at a distance mr, $m = 1,2,\ldots$. Therefore each $<S_i>$ is connected in $PG_r(C_n)$.

Case:1 $r \mid n$

Now $n \equiv 0(mod\ r)$ and so $k = 0$.

Claim $<S_j>$'s are r components in $PG_r(C_n)$.

Suppose 2 vertices $v_m \in S_i$ and $v_n \in S_j$ are connected in $PG_2(C_n)$.

$d(v_m, v_n) = | rm_1 + i - (rm_2 + j) | = | r(m_1 - m_2) + (i - j) | \neq$ multiple of r (since $i - j < r$), which is a contradiction. $<S_j>$'s are r components in $PG_r(C_n)$ and each $<S_j>$ has $\frac{n}{r}$ vertices. Therefore $PG_r(C_n) = rC_{\frac{n}{r}} = gcd(r,n)C_l$.

If $r = \frac{n}{2}$, then $PG_r(C_n) = rP_2$.

Case:2 $r \nmid n$

Claim 1: Last vertex of S_i and first vertex of S_t are adjacent where

$$t = \begin{cases} i + r - k, & if\ i + r - k \le r \\ i + r - k - r & if\ i + r - k > r \end{cases}.$$

Let u be the vertex which is at a distance r in the clockwise direction from the last vertex of s_i. Therefore $u = r(\frac{n-k}{r} + 1) + i - n = n - k + r + i - n = r - k + i$. Hence the claim.

Claim 2: If S_i and S_j lies in the same component in $PG_r(C_n)$ then $j - i$ is a multiple of q.

Suppose S_i and S_j lies in the same component in $PG_r(C_n)$. Then by claim 1, $j - i$ is a multiple of $r - k$, which is a multiple of q. Hence $j - i$ is a multiple of q.

If $q = 1$, then by claim 2, all the S_i's are connected and hence $PG_r(C_n) = C_n$.

Suppose $q > 1$.

Claim 3: No two of $S_1, S_2, \ldots, S_q$ lie in the same component.
If S_i and S_j , $(1 \le i < j \le q)$ lies in the same component, then $j-i$ is a multiple of q, which is a contradiction.
Hence by claims 2 and 3, $PG_r(C_n)$ has q components and each has equal number of S_i 's. Therefore $PG_r(C_n) = qC_{\frac{n}{q}}$.

Theorem 2.8 Let G be any graph and let $v \in V(G)$. Then $deg_P G_2(G)(v) \le \sum_{u \in N(v)} (deg_G(u) - 1)$. Further equality holds for any vertex v iff v does not lie in a C_4, K_4 or $K_4 - e$.

Proof. By the definition of $PG_2(G)$, v is adjacent to all the vertices which are connected to it by a path of length 2. In other words all the vertices of $\{N(u) - \{v\} / u \in N(v)\}$ are adjacent to v in $PG_2(G)$. Therefore $deg_{PG_2(G)} v \le \sum_{u \in N(v)} (deg_G u - 1)$.
If v does not lie in a C_4, then $\bigcap_{u \in N(v)} (N(u) - \{v\}) = \varnothing$. Therefore $deg_{PG_2(G)} v = \sum_{u \in N(v)} (deg_G u - 1)$.
Conversely, let $v \in V(G)$ with $deg_{PG_2(G)} v = \sum_{u \in N(v)} (deg_G u - 1)$. Let $deg_G(v)$ be denoted by δ_v.

Claim v does not lie in a C_4.
If not, suppose v lies in a C_4. Let $N(v) = \{u_1, u_2, \ldots, u_{\delta_v}\}$ and $N(u_i) = \{w_1, w_2, \ldots, w_{\delta_{u_i}}\} (1 \le i \le \delta_v)$.
The cycle C_4 containing v is either of the form $vu_i u_j u_k v$ or of the form $vu_r w_s u_t v$. In both the cases $deg_{PG_2(G)}(v) < \sum_{u \in N(v)} (deg_G u - 1)$, which is a contradiction. Therefore v does not lie in a C_4.

Corollary 2.9 If G is a r -regular graph, then for every $v \in V(G)$, $deg_{PG_2(G)} v \le r(r-1)$.

Theorem 2.10 Let G be any graph. Let $v \in V(G)$ and $S_i = \{u \in V(G) / d(u,v) = i\} (1 \le i \le n)$. Then $|S_n| \le deg_{PG_n(G)}(v) \le |S_1 \bigcup S_2 \bigcup \ldots \bigcup S_n|$.

Proof. The set S_n contains all the vertices which are at a distance n from v. So there is a path of length n between them. Therefore $deg_{PG_n(G)}(v) \ge |S_n|$. Every vertex u adjacent to v in $PG_n(G)$ should lie in at least one $S_j (1 \le j \le n)$. Hence $deg_{PG_n(G)}(v) \le |S_1 \bigcup S_2 \bigcup \ldots \bigcup S_n|$.

Corollary 2.11 If G is a tree, then $deg_{PG_n(G)}(v) = |S_n|$.

Proof. For a tree, there is only one path between any 2 vertices. Hence for every $u \in S_i$, there is only one path of length n between u and v.
So $deg_{PG_n(G)} v = |S_n|$.

Lemma 2.12 Let T be a tree. For $u, v \in V(T)$, $d_T(u,v)$ is even if and only if $u \,\&\, v$ are connected by a path in $PG_2(T)$.

Proof. Assume $d_T(u,v) = 2n$. Let $u = u_1, u_2 \ldots u_{2n} u_{2n+1} = v$ be the path connecting $u \,\&\, v$. By the definition of 2-path graph, $u_1 \& u_3$ are adjacent in $PG_2(T)$. Let the edge be e_1. Likewise we have $u_3u_5 = e_2$, $u_5u_7 = e_2,..., u_{2n-1}u_{2n+1} = e_{\frac{n-1}{2}}$. Therefore there is a path of length $\frac{n-1}{2}$ between $u \& v$ in $PG_2(T)$.

Conversely assume $u \& v$ are connected by a path of length l in $PG_2(T)$. Let $u = w_1, w_2 \ldots w_{l+1} = v$ be the path connecting $u \& v$ in $PG_2(T)$. By the definition of 2-path graph, as w_1, w_2 are adjacent in $PG_2(T)$, there is a path of length 2 between $w_1 \& w_2$ in T. Let the path be $w_1, w_{1'}, w_2$. Since T is a tree, this path is the unique path between $w_1 \& w_2$ in T. by similar argument we get the path between $u \& v$ in T namely $w_1, w_{1'}, w_2, w_{2'}, w_3 \ldots w_l w_{l'} w_{l+1}$ which is of length $2l$. Hence the proof.

Theorem 2.13 For any connected tree T. $PG_2(T)$ is disconnected with 2 components. But the converse is not true.

Proof.
Let T be a connected tree. Let u be a pendant vertex & v is the support. Let the vertex set $V_1 = V(T) - \{u\}$ is partition as follows $V_1 = S^1 \cup S^2$, $S^1 = S_1^1 \cup S_2^2 \cup \ldots \cup S_l^1$ and $S^2 = S_1^2 \cup S_2^2 \cup \ldots \cup S_m^2$ and $S_i^1 = \{v_{ir}^1 \in V_1 / d(u, v_{ir}^1) = 2i, 1 \le r \le l_i\}$ $S_j^2 = \{v_{jt}^2 \in V_1 / d(u, v_{jt}^1) = 2j - 1, 1 \le t \le m_j\}$.

Claim:1 $\langle S^1 \rangle$ is connected in $PG_2(T)$.

Let $v_{is}^1 \in S_i^1 \& v_{jt}^1 \in S_j^1$. Since T is connected there exists a path $v_{is}^1 = u_1, u_2, \ldots, u_m = v_{ji}^1$ in T. $u_1 \in S_i^1$, $u_2 \in S_{i+1}^2 \cup S_{i-1}^2$, since no two vertices of S_i^1 are adjacent. i.e, $u_2 \in S^2$, $u_3 \in S^1$ etc. ie) $u_l \in S^1$ if l is odd. Here $v_{jt}^1 = u_m \in S^1$. Therefore m is odd. Therefore length of path is even. By lemma 2.12, $v_{is}^1 \& v_{jt}^1$ are connected in $PG_2(T)$. ie) $\langle s^1 \rangle$ is connected in $PG_2(T)$. Similarly $\langle s^2 \rangle$ is connected in $PG_2(T)$.

Claim:2 $\langle s^1 \rangle$ and $\langle s^2 \rangle$ are disconnected in $PG_2(T)$. Let $v_{is}^1 \in S_i^1 \& v_{jt}^2 \in S_j^2$. Let $v_{is}^1 = w_1, w_2, \ldots, w_n = v_{jt}^2$ be the path between $v_{is}^1 \& v_{jt}^2$ in T. $v_{is}^1 = w_1 \in S_i^1 \Rightarrow w_2 \in S_{i+1}^1 \cup S_{i-1}^1$. i.e., $w_2 \in S^2, w_3 \in S^1$ so on. Since $w_m = v_{jt}^2 \in S^2$ implies m is even. ie) $d(v_{is}^1, v_{jt}^2)$ is odd. By lemma $v_{is}^1 \& v_{jt}^2$ are not connected in $PG_2(T)$. ie) $\langle s^1 \rangle$ and $\langle s^2 \rangle$ are disconnected in $PG_2(T)$. Hence $PG_2(T)$ is disconnected with 2 components. Converse is not true.

Example 2.14 $PG_2(C_6)$ has 2 components.

Theorem 2.15 For any graph G, $PG_2(G) = \overline{G}$ if and only if G is a star.

Proof. Suppose $PG_2(G) = \overline{G}$. If 2 vertices. u and v are adjacent in G then they are not adjacent in $\overline{G}$ and vice versa. Since $PG_2(G) = \overline{G}$, any 2 adjacent vertices u and v in G are not connected by a path of length 2. ie., G is K_3-free. Also for any two non-adjacent vertices u and v in G, there is a path of length 2 between u and v so that distance between u and v is 2.Thus $diam(G) \le 2$. Hence $G \cong K_{1,n}$. Converse is obvious.

Theorem 2.16 For any simple graph G, $PG_2(G)$ can never be a path.

Proof. If not there exist a graph G with $PG_2(G) \cong P_p$. By theorem2.7, G does not contain C_n as a subgraph. Therefore G is a tree. By theorem 2.3, G does not contain $K_{1,n}$ $(n \geq 3)$ as a subgraph. Therefore G is a path. By theorem 2.4, $PG_2(G)$ is disconnected which is a contradiction.

Theorem 2.17 If G is an Eulerian graph which does not contain C_4, K_4 or $K_4 - e$ as an induced subgraph, then $PG_2(G)$ is also Eulerian.

Proof. G is Eulerian and so $deg_G u$ is even $\forall u \in V(G)$. Since G is C_4-free, by Theorem 2.8, $deg_{PG_2(G)} u = \sum_{v \in N(u)} (deg(v) - 1)$. Since $|N(u)|$ is even and $(deg(v) - 1)$ is odd, $deg_{PG_2(G)} u$ is even. Hence $PG_2(G)$ is Eulerian. But the converse is not true.

Example 2.18 $PG_2(G)$ is Eulerian but G is an Eulerian graph which contains K_4 as an induced subgraph.

Theorem 2.19 If G', G'' are 2 graphs such that $G' \cong G''$, then $PG_2(G') \cong PG_2(G'')$. But the converse is not true.

Proof. Let ϕ be an isomorphism of G' onto G''.
Then $(u,v) \in E(G')$ iff $(\phi(u), \phi(v)) \in E(G'')$
$(u,v) \in E(PG_2(G'))$
$\Leftrightarrow$ There is a path of length 2 between u and v in G'.
$\Leftrightarrow$ There is a path of length 2 between $\phi(u)$ and $\phi(v)$ in G''.
$\Leftrightarrow (\phi(u), \phi(v)) \in E(PG_2(G''))$.
Therefore $PG_2(G') \cong PG_2(G'')$.
Converse is not true.

Consider G' and G'' given above. we observe that $G' ® G''$, but $PG_2(G') \cong PG_2(G'')$.

Theorem 2.20 Let G be a connected graph. $PG_2(G) = G$ iff $G \cong C_{2n+1}$ or K_p.

Proof. Assume $PG_2(G) = G$.
By theorem 2.12, G is not a tree.
Also G contains no pendent edge.
If G is a unicyclic graph, then by theorem 2.7 $G \cong C_{2n+1}$.
If $G ® C_{2n+1}$ we prove that $G \cong K_p$ by induction on p.
If $p = 4$, then the only graph with $PG_2(G) = G$ is K_4.
Assume that if G is a graph with n vertices with $PG_2(G) = G$ then $G \cong K_n$.
Let G be a graph with $n+1$ vertices and $PG_2(G) = G$. Let $u \in V(G)$. Let $G' = G - \{u\}$.
By induction hypothesis $G' \cong K_p$.
claim: In G, u is adjacent to all the n vertices of G'.
If not, u is not adjacent to $v \in V(G')$. Since G is connected, u is adjacent to $v' \in V(G')$ and v' is adjacent to $v \Rightarrow u$ and v are adjacent in $PG_2(G)$ which is contradiction to $PG_2(G) = G$.
Therefore u is adjacent to all the vertices of G'.
Therefore $G \cong K_(n+1)$.
Converse is obvious.

Theorem 2.21 If G is a connected graph that contains a spanning subgraph isomorphic to $G_1, G_2 or G_3$ given below, then $PG_2(G) = K_n$. But converse is not true.

Proof. Let $G \cong G_1, G_2 or G_3$. It is clear any two vertices in G_1, G_2, G_3 are connected by a path of length 2.
Therefore $deg_{PG_2(G)} u = p - 1$
Therefore $PG_2(G) = K_n$.
Converse is not true.

Here $PG_2(G) = K_n$ but $G ® G_1, G_2 or G_3$.

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