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Approximate Solution of a Linear Descriptor Dynamic Control System via a non-Classical Variational Approach

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Abstract: *Descriptor dynamic control system application plays an important role in modern science and mathematics. This paper focuses on an approximate solution to some kind of descriptor dynamic control system with constant coefficient . The non-classical variational approach is developed for this purpose to obtain a very suitable approximate solution with a high degree of accuracy, and freedom. Some illustrations have been provided to show the effectiveness of this approach.*

Keywords: *consistent initial condition, Drazin inverse, linear descriptor systems, non-classical variational approach.*

I. Introduction

Differential- algebraic equations arise naturally in many application such a mechanical multibody system [19],[20], electrical circuit simulation [6],[20], chemical engineering, control theory [6],[7],[19], and other areas. Their analysis and numerical methods, therefore, plays an important role in modern mathematics. Since the differential- algebraic equations can be difficult to solve when they have index greater than one [1], the numerical solution of these types of systems has been the subject of intense activity of a lot of researchers such as [9],[10],[13],[18],[21]. The necessary requirement to find out a given problem has a variational formulation is the symmetry of its operator, if the operator is linear, or the symmetry of its gateaux derivative. This symmetry may be obtained by different approach for some problems [16],[17].

A constructive method to give a variational formulation to every linear equation or a system of linear equations by changing the associated bilinear forms was given in [15], this method has a more freedom of choice a bilinear form that makes a suitable problem has a variational formulation. The solution then may be obtained by using the inverse problem of calculus of variation .To study this problem and its freedom of choosing such a bilinear form and make it easy to be solved numerically or approximately, we have mixed this approach with some kinds of basis, for example Ritz basis of completely continuous functions in a suitable spaces, so that the solution is transform from non direct approach to direct one. The since the linear operator is then not necessary to be symmetric, this approach is named as a non-classical variational approach .This approach have been developed for a lot of applications such as integral, integro differential equations, partial differential equations, oxygen diffusion in biological tissues, moving boundary value problems with non uniform initial-boundary condition, and for solving some optimal control problems, see [12],[16],[17],[22].

We are interesting in find the solution up to some accuracy of linear descriptor dynamic control system with (without) using canonical form for open (close) loop control system and some of illustrations.

II. Description of the Problem

Consider the following general singular

$$EX^{'}(t) = AX(t) + Bu(t) \qquad \dots \ (1)$$

Here, $X(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, are constant matrices , when E is non singular the system become

$$X^{'}(t) = E^{-1}AX(t) + E^{-1}Bu(t).$$

When we mention singular systems we always mean the singular E, with $index\ E = k,\ rank(E^k) = p.$
Singular systems can be named by descriptor variable systems, generalized state space systems, semi state systems, differential-algebraic systems.

III. Some Basic Concept

Definition (3.1):[2] The System (1) is called **regular** if there exist a constant scalar $s \epsilon \mathbb{C}$ such that $det(sE - A) \neq 0.$

Remark (3.1) :

1. The regularity is very important property for descriptor linear system. It's guarantees the existence and uniqness of solutions to descriptor linear system. see [5].

2. If the descriptor linear system is irregular system , i.e. $det\ (sE - A) = 0$, this leads to no solution or non-unique solution (finite or infinite number of solutions), [7].
3. In [3] [4] if the singular system is irregular. It can transfer our system to regular one as following algorithm :

Computational Algorithm to Make the System Regular (3.1)

Step (1): For system (1) find the finite spectrum eigenvalue $\sigma_f\ (E , A)$.

Step (2): Chose $s \notin \sigma_f\ (E , A)$.

Step (3) : Define $\hat{E} = (sE - A)^{-1} E$

$$\hat{A} = (sE - A)^{-1}\ E$$

$$\hat{B} = (sE - A)^{-1}\ B .$$

Step (4): The new system $\hat{E}\, X'(t) = \hat{A}\, X(t) + \hat{B}\, u(t)$ is regular .

4. Based on previously one have $\hat{E}\hat{A} = \hat{A}\hat{E}$ and $\mathcal{N}(\hat{E}) \cap \mathcal{N}(\hat{A}) = 0$ (where $\mathcal{N}(.)$ is null space of the matrix) even when the original matrcies are not and this condition is necessary and sufficient for existence and uniqunce of solution based on [4].

Definition (3.2) :[3] For E is $n \times n$ matrix **the index** of E denoted by $ind\ (E)$, is the smallest non-negative integer k such that , $rank\ (E^k) = rank\ \ (E^{k+1})$.

Definition (3.3):[4] If $A\ \epsilon \mathbb{R}^{n\times n}$ with $ind(A) = k$, and if $A^D \epsilon \mathbb{R}^{n\times n}$ such that :

1. $AA^D = A^D A$
2. $A^D A A^D = A^D$
3. $A^D A^{k+1} = A^k$ for $k \geq ind(A)$.

Then, A^D is called **Drazin inverse** of .

Theorem (3.1) : [4] If $A \in \mathbb{R}^{n\times n}$, with $ind(A) = k$, then there exist a non-singular matrix T, such that :

$A = T \begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix} T^{-1}$, where, C is non singular and N is nilpotent of index .

Furthermore, if T, C, N be any matrices satisfying the above conditions, then

$$A^D = T \begin{bmatrix} C^{-1} & 0 \\ 0 & 0 \end{bmatrix} T^{-1}$$

Remark (3.2): Many methods have been presented in [4], [8] to find Drazin inverse for a singular matrix .

IV. core – nilpotent Decomposition

Remark (4.1) :

1- For $E \in \mathbb{R}^{nxn}$ with $ind\ (E) = k$, then there exist unique matrices C and N such that $E = C + N$ $CN = NC = 0$, N is nilpotent of index k and index of C is 0 or 1 . This decomposition is Wedderburn or core – nilpotent decomposition, for example $N = E\ (I - E\, E^D)$ and $C = E^2\, E^D$.[3]

2- Consider the non homogeneous equation

$$E\, X'(t) + A\, X(t) = f \qquad ….(2)$$

And assume $E\, A = A\, E$. Let $X = X_1 + X_2$ where

$$X_1 = E^D\, EX \text{ and } X_2 = (I - E\, E^D)\, X$$

Then the equation (2) becomes

$$(C + N)\, (X_1' - X_2') + A\, (X_1 + X_2) = f$$

Multiplying first by $C^D C$ and then by $(I - C^D C)$, we get

$$C\, X_1' + A\, X_1 = f_1$$

$$N\, X_2' + AX_2 = f_2 \qquad ….(3)$$

Where $f_1 = C^D C\, f$ and $f_2 = (I - C^D C)\, f$

The equation $C\, X_1{}' + A\, X_1 = f_1$ can be written as $X_1' + C^D A\, X_1 = C^D\, f_1$ which has a unique solution for all initial condition in $R\ (E^D E)$.

But $NX_2' + A\, X_2 = f_2$ may or may not have nontrivial solution and the solution if they exist need not be determined uniquely by initial conditions.

V. Consistent Initial Condition

Return to system (1) for a given initial condition x_0 the system (1) may or may not be consistent for a particular control u.

Having in mind the possible implicit character of equation (1) with respect $X'(t)$, it is obvious, that not all initial conditions are permissible.

The problem of consistent initial conditions is not characteristic for the systems in the normal form, but it basic one for the singular systems , we will say that an initial condition $x_0 \in \mathbb{R}^n$ is consistent if there exist a differentiable continuous solution of (1) .

The following corollary gives a characterization of consistent initial condition when $E\,A = A\,E$ and $\mathcal{N}\,(E) \cap \mathcal{N}\,(A) = \{\,0\,\}$

Corollary (5.1) : [4] Suppose that $E\,A = A\,E$ and $\mathcal{N}\,(E) \cap \mathcal{N}\,(A) = \{\,0\,\}$

Then there exist a solution to $E\,X' + A\,X = f\,, X(0) = x_0$, if and only if x_0 is of the form $x_0 = E^D E q + (I + E\,E^D)\,\sum_{n=0}^{k-1}(-1)^n\,(EA^D)^n\,A^D f^{(n)}{}_{(0)}$. For Some q .

Furthermore , the solution is unique.

Remark (5.1): [3] For a given x_0 and class of controls let the set of admissible controls $\Omega(X_0,J)$, $J = [0,\infty)$, be those u such that (1) with $x(0) = x_0$ is consistent.

If the controls are from the set of k-times continuously differentiable function on $[0,\infty)$ i.e. $C^k[0,\infty)$,

$A\Omega(X_0,J) = C^k(J) \cap \{u: (I - EE^D)x_0 = (I - EE^D)\sum_{n=0}^{k-1}(-1)^n(EA^D)\,A^D B(u)^{(n)}\}$,

And $(I - EE^D)x_0 = (I - EE^D)\sum_{n=0}^{k-1}(-1)^n(EA^D)\,A^D B(u)^{(n)}(0)$.

Computational Algorithm to Find Consist initial Space (5.1)

Step (1): Consider the descriptor system $EX^{'}(t) = AX(t) + Bu(t)$ where $E\,,A$ are nxn matrix and f is k-time continuously differentiable in $\mathbb{R}^n$.

Step (2): Find $ind\,(E) = k$.

Step (5): Using algorithm (3.1) find $\hat{E}$, $\hat{A}$ and$\hat{B}$. The new system $\hat{E}\,X^{'}(t) = \hat{A}\,X\,(t) + \hat{B}u(t)$ is regular .

Step (6): Find the Drazin inverse of $\hat{E}$ and $\hat{A}$ see remark (3.2).

Step (7): We find the class of consistent initial conditions by solve

$(I - \hat{E}\hat{E}^D)\,(x(0) - \hat{A}^D Bu) = 0$ or $(I - \hat{E}\hat{E}^D)\,x(0) = 0$.

This means

$W_k = \mathcal{N}\,(I - \hat{E}\hat{E}^D)$

i. e. $(I - \hat{E}\hat{E}^D)\,x_0 = (I - \hat{E}\hat{E}^D)\sum_{n=0}^{k-1}(-1)^n\,(\hat{E}\hat{A}^D)^n\hat{A}^D\,\hat{B}u^{(n)}(0)$

VI. Non Classical Variational for Normal System

Theorem(6.1):[16],[17] For the system $X^{'}(t) = A\,X(t) + f(t)$ consider the linear equation $LX = f$, $L = (\frac{d\,.}{dt} - A.)$ and $X = (x_1, x_2, \ldots, x_n)^T$vector in $\mathbb{R}^n$ and L denoted to be linear operator with domain $D(L)$ in linear space X and Range $R(L)$ in second linear space Y.

If L is symmetric with respect to the bilinear from $< x, y >$ then the solutions of $LX = f$ are critical points of functional

$$J[X] = 0.5 < LX, X > - < f, X > ,$$

Moreover, if the chosen bilinear satisfies:

a. If for every $x \in X, (x, \bar{y}) = 0$ then $\bar{y} = 0$.
b. If for every $y \in Y, (\bar{x}, y) = 0$ then $\bar{x} = 0$.

On $D(L)$ and $R(L)$ then the critical points of the functional J are solutions to the given equations $LX = f$.

Remark (6.1): Due to this restriction of symmetry of the linear operator there still a large number of problems have no variational formulation, in [15] and [16] shows that it is always possible to find a bilinear form that makes a given linear operator symmetric by using the following steps:

1-chose an arbitrary bilinear form (x, y) which satisfies

a. If for every $x \in X, (x, \bar{y}) = 0$ then $\bar{y} = 0$.
b. If for every $y \in Y, (\bar{x}, y) = 0$ then $\bar{x} = 0$.

2- construct a bilinear form $< x, y >$ defined for every pair of elements $x \in D(L), y \in Y$

Such that <x, y> = (x, Ly) and hence $< Lx, y > = < x, Ly >$, by theorem (6.1) $J[x]$ is defined by

$$J[x] = 0.5(Lx, Lx) - (f, Lx)$$

Examples for non-degenerate bilinear form

$1 - \quad (u\,, v) = \int_0^T u(t)v(t)dt, u\,, v\,:\; C[0,T] \to \mathbb{R}$

$2 - \quad (u\,, v) = \int_0^T \sum_{i=0}^{n} u_i(t)v_i(t)dt\,, u, v\,: C\,[0\,,T] \to \mathbb{R}^n$

$3 - (u, v) = \int_0^T u(t)v(T - t)dt,$ " convolution bilinear from " where u , v : [0, T]→ $\mathbb{R}$

VII. Generalized Ritz Method

To make this approach is a suitable for some type of application , we have mixed this approach with Ritz method , so that the solution to the inverse problem of calculus of variation is of direct approach. Ritz method is very important procedure of the so called direct method, the essence of the method is to express the unknown variables of the given initial boundary value problem as a linear combination of the elements of functions which are completely relative to the class of the feasible functions, towards this level, let$\{G_i(t)\}$ be a sequences of functions relative to the class of admissible function i.e. {linearly independent and continuous function}.

Let $x(t) = x(0) + \sum_{i=1}^{n} a_i G_i(t)$, $i = 1,2,\dots,n$,where $G_i(t)$is a suitable base function satisfied the given non homogeneous boundary and initial condition.

VIII. Solvability of Open-Loop Singular System Using non-Classical Variational Method

The difficulties for solving descriptor system with (without) control involving derivatives in equations have lead to search for variational problems equivalent to the given system in the sense that the solution of the given descriptor system is a critical point of the variational formulation.

Theorem(8.1): Consider the descriptor system $EX^{'}(t) = AX(t) + Bu(t)$ with $x(0) = x_0$, $u(t) \in A\Omega(X_0, J)$ and $ind(E) = k$. Define a linear operator L with domain $D(L)$ and range $R(L)$ such that

$$Lx = Bu(t) \qquad \text{….(4)}$$

If the conditions with $EA = AE$ and $\mathcal{N}(E) \cap \mathcal{N}(A) = \{0\}$ satisfied and L is symmetric with respect to a certain bilinear then the solution of equation (4) are critical points of functional

$$J[x] = 0.5 < Lx, x > - < Bu\,, x >$$

Moreover, if the chosen bilinear form $< x, y >$ is non-degenerate on $D(L)$ and $R(L)$ it is also true that the crtical points of the functional $J[x]$ are solution to the given equation .

Proof: for the system $X^{'}(t) = AX(t) + Bu(t)$, $ind(E) = k$, the community of E, A make the system is possible to perform a similarity so that by linear transformation vector state $w(t) = T^{-1}x(t)$, then one get

$\begin{bmatrix} C & 0 \\ 0 & N \end{bmatrix} . \begin{bmatrix} \acute{w}_1 \\ \acute{w}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} . \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t)$, $w_1 \in \mathbb{R}^p$, $w_2 \in \mathbb{R}^{n-p}$, C invertable , N nilpotent matrix , $p = rank(E)$. The equvelant system became as

$$C\acute{w}_1(t) = A_1 w_1(t) + B_1 u(t) \qquad \text{...(5)}$$

$$N\acute{w}_2(t) = A_2 w_2(t) + B_2 u(t) \qquad \text{…(6)}$$

Define the linear operator for decomposite system

$$L = \begin{bmatrix} L_c \\ L_N \end{bmatrix} = \begin{bmatrix} c\dfrac{d}{dt}. - A_1. \\ N\dfrac{d}{dt}. - A_2. \end{bmatrix}$$

Where L_c linear operator with domain $\mathbb{R}^p$ and range $\mathbb{R}^n$ and L_N linear operator with domain $\mathbb{R}^{n-p}$and range $\mathbb{R}^n$, and the linear equation be as

$$Lx = L_c w_1 + L_N w_2 = (B_1 + B_2)u(t) \qquad \text{…(7)}$$

1-The conditions $EA = AE$ and $\mathcal{N}(E) \cap \mathcal{N}(A) = \{0\}$ guarantee existence the solution .

2- Its clear that L_c and L_N are not symmetric, using remark (6.1) the new bilinear form define by $< x\,, y > = (x\,, Ly)$

Since

$$\begin{aligned} < Lx\,, y > &= (Lx\,, Ly) \\ &= (L_c w_1 + L_N w_2\,, L_c v_1 + L_N v_2) \\ &= (L_c w_1\,, L_c v_1) + (L_c w_1\,, L_N v_2) + (L_N w_2\,, L_c v_1) + (L_N w_2\,, L_N v_2) \\ &= (L_c v_1\,, L_c w_1) + (L_N v_2\,, L_C w_1) + (L_c v_1, L_N w_2\,) + (\,L_N v_2, L_N w_2) \\ &= (L_c v_1 + L_N v_2, L_c w_1 + L_N w_2) \\ &= (\,Ly, Lx\,) \\ &= < Ly, x > \end{aligned}$$

3- Lets perform the first variation of $J[X]$

$$\begin{aligned} \delta J[w_1 + w_2] &= J[w_1 + w_2 + \delta w_1 + \delta w_2] - J[w_1 + w_2] \\ &= 0.5 < L_c \delta w_1 + L_N \delta w_2, w_1 + w_2 > +0.5 < L_c w_1 + L_N w_2, \delta w_1 + \delta w_2 > \\ &\quad - < B_1 u + B_2 u, \delta w_1 + \delta w_2 > \\ &= 0.5 < L_c \delta w_1, w_1 > +0.5 < L_c \delta w_1, w_2 > +0.5 < L_N \delta w_2, w_1 > +0.5 < L_N \delta w_2, w_2 > \\ &\quad +0.5 < L_c w_1, \delta w_1 > +0.5 < L_c w_1, \delta w_2 > +0.5 < L_N w_2, \delta w_1 > +0.5 < L_N w_2, \delta w_2 > \\ &\quad - < B_1 u, \delta w_1 > - < B_1 u, \delta w_2 > - < B_2 u, \delta w_1 > - < B_2 u, \delta w_2 > \\ &= 0.5(L_c \delta w_1, L_c w_1) + 0.5(L_c \delta w_1, L_N w_2) + 0.5(L_N \delta w_2, L_c w_1) + 0.5(L_N \delta w_2, L_N w_2) \\ &\quad + 0.5(L_c w_1, L_c \delta w_1) + 0.5(L_c w_1, L_N \delta w_2) + 0.5(L_N w_2, L_c \delta w_1) + 0.5(L_N w_2, L_N \delta w_2) \end{aligned}$$

$$-(B_1u, L_c\delta w_1) - (B_1u, L_N\delta w_2) - (B_2u, L_c\delta w_1) - (B_2u, L_N\delta w_2)$$
$$= 0.5(L_cw_1, L_c\delta w_1) + 0.5(L_Nw_2, L_c\delta w_1) + 0.5(L_cw_1, L_N\delta w_2) + 0.5(L_Nw_2, L_N\delta w_2)$$
$$+ 0.5(L_cw_1, L_c\delta w_1)$$
$$+0.5(L_cw_1, L_N\delta w_2) + 0.5(L_Nw_2, L_c\delta w_1) + 0.5(L_Nw_2, L_N\delta w_2) - (B_1u, L_c\delta w_1) - (B_1u, L_N\delta w_2)$$
$$-(B_2u, L_c\delta w_1) - (B_2u, L_N\delta w_2)$$
$$= 0.5 < L_cw_1, \delta w_1 > +0.5 < L_Nw_2, \delta w_1 > +0.5 < L_cw_1, \delta w_2 > +0.5 < L_Nw_2, \delta w_2 >$$
$$+ 0.5 < L_cw_1, \delta w_1 > +0.5 < L_cw_1, \delta w_2 > +0.5 < L_Nw_2, \delta w_1 > +0.5 < L_Nw_2, \delta w_2 >$$
$$-< B_1u, \delta w_1 > -< B_1u, \delta w_2 > -< B_2u, \delta w_1 > -< B_2u, \delta w_2 >$$
$$= 0.5 < L_cw_1 + L_Nw_2, \delta w_1 + \delta w_2 > +0.5 < L_cw_1 + L_Nw_2, \delta w_1 + \delta w_2 >$$
$$-< B_1u + B_2u, \delta w_1 + \delta w_2 >$$
$$=< L_cw_1 + L_Nw_2 - B_1u - B_2u, \delta w_1 + \delta w_2 >$$

Where the symbol $\delta$is the customary symbol of variation of a function used in calculus of variation .if the $x^* = w_1^* + w_2^*$ is a solution of (7)

$$L_cw_1 + L_Nw_2 = (B_1 + B_2)u(t) = 0$$

And then $\delta J[x^*]$ =0 .

4- If the chosen bilinear form $< x , y >$ is non – degenerate on $D(L)$ and $R(L)$ let $\bar{x} = \bar{w}_1 + \bar{w}_2$ is critical point of $J[X]$ for every $\delta w_1 \in \mathbb{R}^P and\ \ \delta w_2 \in \mathbb{R}^{n-P}$

$$\delta J[\bar{x}\ ] =< L_c\bar{w}_1 - B_1u, \delta w_1 > +< L_N\bar{w}_2 - B_2u, \delta w_2 >= 0$$

And then from the non- degeneracy condition we have

$$L_c\bar{w}_1 - B_1u + L_N\bar{w}_2 - B_2u = 0$$
$$\Rightarrow L_c\bar{w}_1 + L_N\bar{w}_2 - (B_1 + B_2)u = 0$$
$$\Rightarrow L\bar{x} - Bu = 0$$

Hence if a given linear operator L is symmetric with respect to a non- degenerate bilinear form $< x , y >$ there is a variational formulation of the given linear equation (7).

Corollary(8.1): for the descriptor system $EX'(t) = AX(t) + Bu(t)$ with $x(0) = x_0$, $u(t) \in A\Omega(X_0,J)$ and $index(E) = k$. Define a linear operator L ,$L = \left(E\frac{d}{dt} - A\right)$with domain $D(L)$ and range $R(L)$.

If the conditions with $EA = AE$ and $\mathcal{N}(E) \cap \mathcal{N}(A) = \{0\}$ satisfied , then the solution of linear equation for the descriptor system are critical points of functional $J[x] = 0.5 < Lx, x > -< Bu , x >$. Moreover, if the chosen bilinear form $< x, y >$ is non-degenerate on $D(L)$ and $R(L)$ it is also true that the crtical points of the functional $J[x]$ are solution to the given equation .

Proof: since L is not symmetric remark (6.1) define $< x , y > = (x , Ly)$,one can define the linear equation

$$Lx = Bu(t) \quad \text{….(8)}$$

Then from theorem (8.1) for singular system and theorem (6.1) for normal system in a direct way one can prove the solution for (8) is critical point for $J[x]$ and for the non-degenerate $< x, y >$ the crtical points for $J[x]$ are solutions to (8).

remark(8.1): If the descriptor system (1) is irregular then one can use remark (3.1) to construct regular system and one can be continue in the proof of theorem (8.1) for the new regular system $\hat{E}\, X'(t) = \hat{A}\, X(t) + \hat{B}\, u(t)$.

Theorem(8.2): for the descriptor system $EX'(t) = AX(t) + Bu(t)$ with $x(0) = x_0$, $x_0 \in W_k$ (the class of consistent initial condition) , $u(t) \in A\Omega(X_0,J)$ and $ind(E) = k$.

If the solution $x(t)$ has been approximated by a linear combination of a suitable basis i.e. $x(t) = x(0) + \sum_{i=1}^{n} a_iG_i$ satisfies

1- $x_0 \in W_k$.

2- $G_i(x_0) = 0$.

3- G_i are continuous as required by the variational statement being.

4- $\{G_i\}_{\mathrm{i}}$ Must be linearly independent .

5- Satisfies the homogeneous from of the specified condition.

Then the solution for the system $\frac{dJ}{da_j} = 0$, $\forall j = 1,2, \dots, n$ is the approximate solution for the descriptor system.

proof:

From theorem (8.1) define the functional $J[x]$ as $J[x] = 0.5 < Lx, Lx > -< f, Lx >$ where the classical bilinear form $< Lx, Lx > = \int_0^\tau Lx(t). Lx(t)dt$, $0 \le t \le \tau$, and the linear operator

$$L = \begin{bmatrix} L_c \\ L_N \end{bmatrix} = \begin{bmatrix} c\dfrac{d}{dt}. - A_1. \\ N\dfrac{d}{dt}. - A_2. \end{bmatrix}$$

Where L_c linear operator with domain $\mathbb{R}^p$ and range $\mathbb{R}^n$ and L_N linear operator with domain $\mathbb{R}^{n-p}$ and range $\mathbb{R}^n$, and the linear equation be as

$$Lx = L_c w_1 + L_N w_2 = (B_1 + B_2)u(t)$$

Then the functional J became:

$$J[w] = 0.5\int_0^\tau (L_C.w_1, L_C.w_1) - (B_1 u(t), L_C.w_1)dt + 0.5\int_0^\tau (L_N.w_2, L_N.w_2) - (B_2 u(t), L_N.w_2)dt$$

Select suitable base satisfies Ritz method as follows

$$w_1(t) = EE^D x(t) = \sum_{i=1}^n a_i G_i(t) + w_1(0),$$
$$w_2(t) = (I - EE^D)x(t) = \sum_{i=1}^n a_i G_i(t) + w_2(0), \qquad \ldots(9)$$

Where $w_i(0) \in W_k$.

Then

$$\begin{aligned} J[w;a] = 0.5\int_0^\tau &\left\{\left[C\sum_{i=1}^n i a_i G_{i-1}(t) - w_{10}A_1 - A_1\sum_{i=1}^n a_i G_i(t)\right]^T.\left[C\sum_{i=1}^n i a_i G_{i-1}(t) - w_{10}A_1 - A_1\sum_{i=1}^n a_i G_i(t)\right]\right\}dt \\ &+ 0.5\int_0^\tau \left\{\left[N\sum_{i=1}^n i a_i G_{i-1}(t) - w_{20}A_2 - A_2\sum_{i=1}^n a_i G_i(t)\right]^T.\left[N\sum_{i=1}^n i a_i G_{i-1}(t) - w_{20}A_2 \right.\right. \\ &\left.\left. - A_2\sum_{i=1}^n a_i G_i(t)\right]\right\}dt \\ &- \int_0^\tau B_1 u\left[C\sum_{i=1}^n i a_i G_{i-1}(t) - w_{10}A_1 - A_1\sum_{i=1}^n a_i G_i(t)\right]dt \\ &- \int_0^\tau B_2 u\left[N\sum_{i=1}^n i a_i G_{i-1}(t) - w_{20}A_2 - A_2\sum_{i=1}^n a_i G_i(t)\right]dt \end{aligned}$$

In order to find the critical points of the last functional we derivative J to a_j, $\forall j = 1, \ldots, n$ and equate the result to zero, i.e. $\frac{dJ}{da_j} = 0$, $\forall j = 1,2,\ldots,n$, to get a system of algebraic equations as follows:

$$\begin{bmatrix} \int_0^\tau [CG_0(t) - A_1 w_{10} - A_1 G_1(t)]^T.[CG_0(t) - A_1 G_1(t)] & \cdots & \int_0^\tau [CnG_{n-1}(t) - A_1 w_{10} - A_1 G_n(t)]^T.[CG_0(t) - A_1 G_1(t)] \\ \vdots & \ddots & \vdots \\ \int_0^\tau [CG_0(t) - A_1 w_{10} - A_1 G_1(t)]^T.[CnG_{n-1}(t) - A_1 G_n(t)] & \cdots & \int_0^\tau [CnG_{n-1}(t) - A_1 w_{10} - A_1 G_n(t)]^T.[CnG_{n-1}(t) - A_1 G_n(t)] \end{bmatrix}$$
$$+ \begin{bmatrix} \int_0^\tau [NG_0(t) - A_2 w_{20} - A_2 G_1(t)]^T.[NG_0(t) - A_2 G_1(t)] & \cdots & \int_0^\tau [NnG_{n-1}(t) - A_2 w_{20} - AG_n(t)]^T.[NG_0(t) - A_2 G_1(t)] \\ \vdots & \ddots & \vdots \\ \int_0^\tau [NG_0(t) - A_2 w_{20} - A_2 G_1(t)]^T.[NnG_{n-1}(t) - A_2 G_n(t)] & \cdots & \int_0^\tau [NnG_{n-1}(t) - A_2 w_{20} - AG_n(t)]^T.[NnG_{n-1}(t) - A_2 G_n(t)] \end{bmatrix}.\begin{bmatrix} a_1 \\ a_2 \\ . \\ . \\ a_n \end{bmatrix}$$
$$= \begin{bmatrix} \int_0^\tau B_1 u(t)\big(CG_0(t) - A_1 G_1(t)\big) \\ \int_0^\tau B_1 u(t)\big(CG_1(t) - A_1 G_2(t)\big) \\ . \\ . \\ \int_0^\tau B_1 u(t)(CG_{n-1}(t) - A_1 G_n(t)) \end{bmatrix} + \begin{bmatrix} \int_0^\tau B_2 u(t)\big(NG_0(t) - A_2 G_1(t)\big) \\ \int_0^\tau B_2 u(t)\big(NG_1(t) - A_2 G_2(t)\big) \\ . \\ . \\ \int_0^\tau B_2 u(t)(NG_{n-1}(t) - A_2 G_n(t)) \end{bmatrix}$$

Which is written as a system

$$\sum_{i=1}^n A(i,j)a_i = b_j, \quad \forall j = 1, \ldots, n,$$

Since $w_i(0) \in W_k$ then $A(i,j)$ non singular matrix and $\sum_{i=1}^n a_i = A^{-1}(i,j)b_j$, $\forall j = 1, \ldots, n$, (for the case $w_i(0)$is arbitrary selected then $A(i,j)$ singular matrix and $\sum_{i=1}^n a_i = A^D(i,j)b_j + (I - A(i,j)A^D(i,j))y$,

$\forall j = 1, \ldots, n$, y is arbitrary in $\mathbb{R}^n$). Satisfying a_i in (9) to find the solution for decomposite system and then solution for
$EX'(t) = AX(t) + Bu(t)$.

Example (8.1): Consider the model of a chemical reactor in which a first order isomerization reaction takes place and which is externally cooled.

Denoting by c_0 the given feed reactant concentration, by T_0 the initial temperature, by c(t), T(t)
The concentration and temperature at time t, and by R(t) the reaction rate per unite volume, the model takes the form

$$\begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}\acute{C}(t)\\\acute{T}(t)\\\acute{R}(t)\end{bmatrix} = \begin{bmatrix}c_0 & 0 & 0\\0 & T_0 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}C(t)\\T(t)\\R(t)\end{bmatrix} + \begin{bmatrix}1 & 0\\0 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}sint\\cost\end{bmatrix},$$

Where $\begin{bmatrix}sint\\cost\\sint + cost\end{bmatrix}$ refers to T_c is cooling temperature .

Take $c_0 = 2$ and $T_0 = 1$ we get

$$\begin{bmatrix}1 & 0 & 0\\0 & 0 & 1\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}\acute{C}(t)\\\acute{T}(t)\\\acute{R}(t)\end{bmatrix} = \begin{bmatrix}2 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}C(t)\\T(t)\\R(t)\end{bmatrix} + \begin{bmatrix}1 & 0\\0 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}sint\\cost\end{bmatrix},\ 0 \le t \le 1.$$

Solution: let X= $\begin{bmatrix}C(t)\\T(t)\\R(t)\end{bmatrix}$, Since the eigenvalue of E are 1,0 ,0 and the eigenvector of E $(1,0,0), (0,1,0), (0,0,1)$ then $P = I$, this system is regular system and it is already in standard canonical form.

So we have $N = \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, C = 1, B_1 = [1 \quad 0], B_2 = \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}, A_1 = 2, A_2 = I_2,$
let $W = P^{-1}X$, P non singular matrix the transform system become

$$C\acute{W}_1(t) - A_1W_1(t) = B_1u(t)$$
$$N\acute{W}_2(t) - A_2W_2(t) = B_2u(t)$$

Where $W_1(t) = C(t)$ and $W_2(t) = \begin{bmatrix}T(t)\\R(t)\end{bmatrix}$ the linear operator is given by $L_E = \begin{bmatrix}L_C\\L_N\end{bmatrix}$

Where

$$L_C = (C\frac{d}{dt} - A_1)$$
$$L_N = (N\frac{d}{dt} - A_2)$$

And the J become as

$$J[w] = 0.5\int_0^1 (\acute{w}_1 - 2w_1).(\acute{w}_1 - 2w_1)dt - \int_0^1 sint\,(\acute{w}_1 - 2w_1)dt$$
$$+ 0.5\int_0^1 \left(\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}.\begin{bmatrix}\acute{w}_2\\\acute{w}_3\end{bmatrix} - \begin{bmatrix}w_2\\w_3\end{bmatrix}\right)^T.\left(\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}.\begin{bmatrix}\acute{w}_2\\\acute{w}_3\end{bmatrix} - \begin{bmatrix}w_2\\w_3\end{bmatrix}\right)dt$$
$$- \int_0^1 [cost \quad sint + cost]\left(\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}.\begin{bmatrix}\acute{w}_2\\\acute{w}_3\end{bmatrix} - \begin{bmatrix}w_2\\w_3\end{bmatrix}\right)dt.$$

$$= 0.5\int_0^1\left((\acute{w}_1 - 2w_1)^2 + \begin{bmatrix}\acute{w}_3 - w_2\\w_3\end{bmatrix}^T.\begin{bmatrix}\acute{w}_3 - w_2\\w_3\end{bmatrix}\right)dt$$
$$- \int_0^1 (sint\,(\acute{w}_1 - 2w_1) - [cost \quad sint + cost].\begin{bmatrix}\acute{w}_3 - w_2\\w_3\end{bmatrix})\,dt$$

in the following we investigate the classical solution of this system since
$-\sum_{i=0}^{1} N^i\, B_2u(0)^i = \begin{bmatrix}-2\\-1\end{bmatrix}, u = \begin{bmatrix}sint\\cost\end{bmatrix}$.

Using the algorithm in section (5) one can easily get the set of consist initial conditions as
$w_k = \{\gamma | [0 \quad I_2]\gamma = [-2 \quad -1]^T\}$

Thus, all the admissible initial value for the system takes the form
$\gamma = [\alpha \quad -2 \quad -1]^T$, $\alpha \in R$ arbitrary, particularly choosing

$$w_0 = [1 \quad -2 \quad -1]^T \in W_k$$

Now approximate the solution by:

$$w_1(t) = 1 + \sum_{i=1}^{5} a_i G_i(t), \quad G_i(t) = t^i$$

$$w_2(t) = -2 + \sum_{i=1}^{5} b_i H_i(t), \quad H_i(t) = t^i$$

$$w_3(t) = -1 + \sum_{i=1}^{5} c_i L_i(t), \quad L_i(t) = t^i$$

Substitute W_i in our functional J to get

$J[w] =$

$$0.5\int_0^1 \Big(\left(\sum_{i=1}^{5} i a_i t^{i-1} - 2 - 2\sum_{i=1}^{5} a_i t^i\right)^T . \left(\sum_{i=1}^{5} i a_i t^{i-1} - 2 - 2\sum_{i=1}^{5} a_i t^i\right) +$$

(i=15iciti–1+2–i=15biti)–i=15citiT.(i=15iciti–1+2–i=15biti)–i=15citiTdt

$$-\int_0^1 \Big(sint \left(\sum_{i=1}^{5} i a_i t^{i-1} - 2 - 2\sum_{i=1}^{5} a_i t^i\right) +$$

costsint+cost.(i=15iciti–1+2–i=15biti)(–i=15citi)dt

Now $\frac{dJ}{da_j} = \frac{dJ}{db_j} = \frac{dJ}{dc_j} = 0, \forall j = 1,2,\dots,5$, leads to system of algebraic equation $\sum_{i=1}^{n} A(i,j)Z_i = D_j, \ \forall j = 1,\dots,n, Z_i = \begin{pmatrix} a_i \\ b_i \\ c_i \end{pmatrix}$

Compute $a_i, b_i, c_i, \forall i = 1,2,\dots,5$, having $a_1 = 2.0066, a_2 = 2.4042, a_3 = 2.0906,$ $a_4 = 0.0128$, $a_5 = 0.9080$, $b_1 = 0.9995$, $b_2 = 1.0049$, $b_3 = -0.1822$, $b_4 = -0.0655$, $b_5 = 0.0040$, $c_1 = -1, c_2 = 0.4997, c_3 = 0.1685, c_4 = -0.0461, c_5 = -0.0039$.

Then find the approximate solution

$$w_1(t) = 1 + \sum_{i=1}^{5} a_i t^i$$

$$w_2(t) = -2 + \sum_{i=1}^{5} b_i t^i$$

$$w_3(t) = -1 + \sum_{i=1}^{5} c_i t^i$$

Now take $W = P^{-1}X$, then the solution by non classical variational (N.C.V.) and exist solutions are calculated along with absolute errors (Abs. Error) and present in the following tables:

Time	N.C.V. $x_1(t)$	Exact $x_1(t)$	Abs. Error	N.C.V. $x_2(t)$	Exact $x_2(t)$	Abs. Error	N.C.V. $x_3(t)$	Exact $x_3(t)$	Abs. Error
0	1	1	0	-2	-2	0	-1	-1	0
0.1	1.2268	1.2267	0.0001	-1.8902	-1.8902	0	-1.0948	-1.0948	0
0.2	1.5146	1.5147	0.0001	-1.7615	-1.7615	0	-1.1787	-1.1787	0
0.3	1.8772	1.8773	0.0001	-1.6152	-1.6152	0	-1.2509	-1.2509	0
0.4	2.3307	2.3307	0	-1.4529	-1.4528	0.0001	-1.3105	-1.3105	0
0.5	2.8948	2.8947	0.0001	-1.2760	-1.2757	0.0002	-1.3570	-1.3570	0
0.6	3.5933	3.5932	0.0001	-1.0864	-1.0860	0.0003	-1.3900	-1.3900	0
0.7	4.4554	4.4556	0.0002	-1.8859	-1.8855	0.0004	-1.4091	-1.4091	0
0.8	5.5171	5.5174	0.0003	-0.6766	-0.6766	0	-1.4141	-1.4141	0
0.9	6.8220	6.8219	0.0001	-0.4606	-0.4599	0.0007	-1.4049	-1.4049	0
1	8.4222	8.4222	0	-0.2402	-0.2401	0.0001	-1.3818	-1.3818	0

Table (8.1) show the numerical results which are compared with given analytical solution

Where $0 \le t \le 1$ and the basis are polynomial of degree 5 and exact solution

$$C(t) = \frac{-2}{5}\, sint - \frac{1}{5}\, cost + \frac{6}{5}\, e^{2t},$$

$$T(t) = sint - 2cost,$$

$$R(t) = -sint - cost$$

Example (8.2):

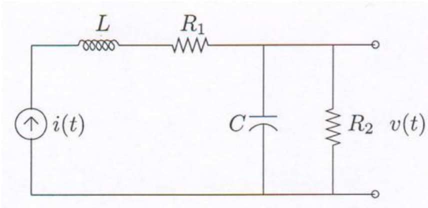

Figure (8.1): Electronic circuit

The electric circuit in figure (8.1) contain some typical components in electrical systems, the behavior of the capacitor and inductor described by the differential equations

$$C\dot{V}_C(t) = i_c(t)$$
$$L.\ddot{i}_L(t) = V_L(t)$$

Where $\dot{V}_C(t)$ and $V_L(t)$ are voltages over the capacitor and distance respectively and $i_c(t)$, $\ddot{i}_L(t)$ are corresponding currents. The resistances $R_1$and R_2 are described by

$$R_{i,j}(t) = R_j . i_{R,j}(t)\,, \quad j = 1,2$$

The current source is assumed to be ideal ,that it can provide an arbitrary current $i(t)$ independent of the voltage over it.

By choosing the state vector X as:

$$X = \begin{bmatrix} i(t) \\ V_{R,2}(t) \\ V_s(t) \end{bmatrix},$$

Where V_s is the voltage over the current source.

The matrix from of the complete circuit become

$$\begin{bmatrix} 0 & L & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \acute{X} = \begin{bmatrix} R_1 & 0 & 0 \\ 0 & \frac{1}{R_2} & 0 \\ 0 & 0 & 1 \end{bmatrix} X + \begin{bmatrix} 0 \\ -t^3 \\ -t \end{bmatrix},$$

Taking $L = 1H, R_1 = 1F, R_2 = 1\Omega$, then our system become as

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \acute{X} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} X + \begin{bmatrix} 0 \\ -t^3 \\ -t \end{bmatrix}$$

Solution: since $EA = AE,\ \mathcal{N}(E) \cap \mathcal{N}(A) = \{0\}$

Then their exist solution for our system and

$$J[x] = 0.5 < Lx, Lx > - < f, Lx >$$

Where $L = (E\frac{d}{dt} - A)$ then

$$J[x] = 0.5 \int_0^1 \begin{bmatrix} \acute{x}_2 - x_1 \\ -x_2 \\ -x_3 \end{bmatrix}^T . \begin{bmatrix} \acute{x}_2 - x_1 \\ -x_2 \\ -x_3 \end{bmatrix} dt - \int_0^1 [0 \quad -t^3 \quad -t] . \begin{bmatrix} \acute{x}_2 - x_1 \\ -x_2 \\ -x_3 \end{bmatrix} dt$$

Approximate the solution X by a linear combination of functions where

$$x_1(t) = x_{10} + \sum_{i=1}^{5} a_i G_i(t)\ , \quad G_i(t) = t^i$$

$$x_2(t) = x_{20} + \sum_{i=1}^{5} b_i H_i(t)\ , \quad H_i(t) = t^i$$

$$x_3(t) = x_{30} + \sum_{i=1}^{5} c_i L_i(t)\ , \quad L_i(t) = t^i$$

Where $(x_{10}, x_{20}, x_{30}) \in W_k = \{(x_{10}, x_{20}, x_{30}) : (x_{10}, x_{20}, x_{30}) = (0,0,0)\}$ by using an algorithm (5.1) and satisfies the condition of theorem (8.2), our functional become as follows

$$J[x] = 0.5 \int_0^1 \begin{bmatrix} (\sum_{i=1}^5 ib_i t^{i-1} - \sum_{i=1}^5 a_i t^i) \\ (-\sum_{i=1}^5 b_i t^i) \\ (-\sum_{i=1}^5 c_i t^i) \end{bmatrix}^T . \begin{bmatrix} (\sum_{i=1}^5 ib_i t^{i-1} - \sum_{i=1}^5 a_i t^i) \\ (-\sum_{i=1}^5 b_i t^i) \\ (-\sum_{i=1}^5 c_i t^i) \end{bmatrix} dt$$

$$-\int_0^1 [0 \quad -t^3 \quad -t].\begin{bmatrix} (\sum_{i=1}^5 \mathrm{i b_i t^{i-1}} - \sum_{i=1}^5 \mathrm{a_i t^i}) \\ (-\sum_{i=1}^5 \mathrm{b_i t^i}) \\ (-\sum_{i=1}^5 \mathrm{c_i t^i}) \end{bmatrix} dt$$

Now $\frac{dJ}{da_j} = \frac{dJ}{db_j} = \frac{dJ}{dc_j} = 0 \text{ , } \forall j = 1,2, \dots ,5$, leads to system of algebraic equation

$\sum_{i=1}^{n} A(i,j)Z_i = D_j , \quad \forall j = 1, \dots, n, \mathrm{Z_i} = \begin{pmatrix} \mathrm{a_i} \\ \mathrm{b_i} \\ \mathrm{c_i} \end{pmatrix}$

We compute $\mathrm{a_i}, \mathrm{b_i}, \mathrm{c_i}, \forall i = 1,2, \dots ,5$, having $\mathrm{a_1} = 0, \mathrm{a_2} = 3, \mathrm{a_3} = 0, \mathrm{a_4} = 0, \mathrm{a_5} = 0, \mathrm{b_1} = 0, \mathrm{b_2} = 0, \mathrm{b_3} = 1,$ $\mathrm{b_4} = 0, \mathrm{b_5} = 0, \mathrm{c_1} = 1, \mathrm{c_2} = 0, \mathrm{c_3} = 0, \mathrm{c_4} = 0, \mathrm{c_5} = 0.$

Then we find the approximate solution

$$\mathrm{x_1(t)} = \sum_{i=1}^{5} \mathrm{a_i t^i}$$

$$\mathrm{x_2(t)} = \sum_{i=1}^{5} \mathrm{b_i t^i}$$

$$\mathrm{x_3(t)} = \sum_{i=1}^{5} \mathrm{c_i t^i}$$

The solution by non classical variational (N.C.V.) and exist solutions are calculated along with absolute errors (Abs. Error) and present in the following tables:

Time	N.C.V. $x_1(t)$	Exact $x_1(t)$	Abs. Error	N.C.V. $x_2(t)$	Exact $x_2(t)$	Abs. Error	N.C.V. $x_3(t)$	Exact $x_3(t)$	Abs. Error
0	0	0	0	0	0	0	0	0	0
0.1	0.03	0.03	0	0.001	0.001	0	0.1	0.1	0
0.2	0.12	0.12	0	0.008	0.008	0	0.2	0.2	0
0.3	0.27	0.27	0	0.027	0.027	0	0.3	0.3	0
0.4	0.48	0.48	0	0.064	0.064	0	0.4	0.4	0
0.5	0.75	0.75	0	0.125	0.125	0	0.5	0.5	0
0.6	1.08	1.08	0	0.216	0.216	0	0.6	0.6	0
0.7	1.47	1.47	0	0.343	0.343	0	0.7	0.7	0
0.8	1.92	1.92	0	0.512	0.512	0	0.8	0.8	0
0.9	2.43	2.43	0	0.729	0.729	0	0.9	0.9	0
1	3	3	0	1	1	0	1	1	0

Table (8.2) show the numerical results which are compared with given analytical solution

Where $0 \leq t \leq 1$ and the basis are polynomial of degree 5 and exact solution

$$\mathrm{x_1(t) = 3t^2},$$
$$\mathrm{x_2(t) = t^3},$$
$$\mathrm{x_3(t) = t}.$$

Conclusion: In this paper a survey was presented of non classical variational method using bilinear forms and Ritz basis.

In this environment the non-classical method is the optimal bridge between exact solution and approximate one.

The above summarized identification algorithm have been tested with success on a lot of an examples with basis as polynomial of degree 5 as a Ritz basis and we notes that when n=5 gives very powerful technique to solve our system and gives nearly exact solutions.

As to the computational requirements several examples were presented then we compare our solution with the exact one.

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