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A Note on Alpha-Skew-Generalized Logistic Distribution

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Abstract: *In this short article alpha-skew-generalized logistic distribution of type III has been introduced as an extension of the alpha-skew-logistic distribution [Hazarika and Chakraborty (2014). Alpha Skew Logistic Distribution, IOSR Journal of Mathematics 10 (4) Ver. VI: 36-46] by considering the generalized logistic distribution of type III as the base distribution. Cumulative distribution function, moment generating function and a few moments have been derived.*

AMS Subject Classification: *60E05; 62E15; 62H10*
***Key Words**: Skew logistic distribution, moment generating function, generalized Hurwitz–Lerch Zeta function*

I. Introduction

Skew distribution is natural extension of the underlying symmetric distribution derived by adding one or more additional asymmetry parameter(s). Azzalini [1] first introduced the path breaking skew-normal distribution whose probability density function (pdf) is given by

$$f_Z(z;\lambda) = 2\phi(z)\Phi(\lambda z); \qquad z \in R,\ \lambda \in R \tag{1}$$

where $\phi(.)$ and $\Phi(.)$ are the pdf and cumulative distribution function (cdf) of standard normal distribution respectively. Here, λ is the asymmetry parameter. The general formula for the construction of skew-symmetric distributions was given by Huang and Chen [2] by introducing the concept of skew function $G(.)$. According to Huang and Chen [2] a random variable *Z* is said to be skew symmetric if its pdf is given by

$$f_Z(z) = 2h(z)G(z);\ z \in R \tag{2}$$

where, $G(.)$ (skew function) is the Lebesgue measurable function such that, $0 \le G(z) \le 1$ and $G(z) + G(-z) = 1$; $z \in R$, almost everywhere and $h(.)$ is any symmetric (about 0) pdf. One can construct many skew distributions showing both unimodal and multimodal behavior by choosing an appropriate skew function $G(z)$ in the equation (2) (for more about skew distributions see [3]).

Alpha–skew–normal (ASN) distribution was introduced by Elal-Olivero [4] as a new class of skew normal distribution that includes unimodal as well as bimodal normal distributions. A random variable *Z* is said to be alpha–skew–normal distribution if its pdf is given by

$$f_{ASN}(z;\alpha) = \frac{\{(1-\alpha z)^2 + 1\}}{2+\alpha^2}\phi(z);\ -\infty < z < \infty,\ \alpha \in R \tag{3}$$

where, $\phi(z)$ is the pdf of standard normal distribution. The generalized version of the ASN distribution has been introduced by Handam [5]. Using exactly the similar approach Harandi and Alamatsaz [6] investigated a class of alpha–skew–Laplace (ASL) distribution. Recently Hazarika and Chakraborty [7] introduced and studied many properties along with estimation of parameters and data fitting application of the alpha–skew–Logistic distribution (ASLG) having pdf

$$f_Z(z;\alpha) = \frac{\{(1-\alpha z)^2 + 1\}\exp(-z)}{C\{1+\exp(-z)\}^2},\ z \in R,\ \alpha \in R \tag{5}$$

where, $C = 2 + \frac{\alpha^2\pi^2}{3}$. Symbolically if *Z* is an alpha-skew-logistic random variable with parameter α it is denoted in this article by $Z \sim \text{ASLG}(\alpha)$.

In this note, alpha-skew-generalized logistic (ASGL) distribution has been introduced by considering the generalized logistic distribution of type III as the base distribution following the methodology of Elal-Olivero [4] and some of its properties have been studied.

II. Alpha-Skew-Generalized Logistic Distribution

Definition 2.1 A random variable *Z* is said to follow generalized logistic distribution type III (Balkrishnan, [8]; Jhonson et al., [9]) if its pdf is given by

$$f_{GL}(z) = \frac{1}{B(\beta,\beta)} \frac{\exp(-\beta z)}{\{1+\exp(-z)\}^{2\beta}}; -\infty < z < \infty,\ \beta > 0 \tag{6}$$

where $B(\beta,\beta) = \frac{\{\Gamma(\beta)\}^2}{\Gamma(2\beta)}$ and it is denoted by $Z \sim \mathrm{GL}(\beta)$.

Definition 2.2 A random variable Z is said to be alpha-skew-generalized logistic $\mathrm{ASGLG}(\alpha,\beta)$ if its pdf is given by

$$f_{\mathrm{ASLG}}(z;\alpha) = \frac{(1-\alpha z)^2+1}{D} \frac{\exp(-\beta z)}{B(\beta,\beta)(1+\exp(-z))^{2\beta}};\ -\infty < z < \infty,\ \beta > 0, \alpha \in R \tag{7}$$

where, $D = 2\,[1+\alpha^2 \psi^{(1)}(\beta)]$ and

$\psi^{(n)}(a) = \frac{d^n}{d\,a^n}\Gamma(a) = \sum_{k=0}^{\infty} \frac{(-1)^{n+1} n!}{(a+k)^{n+1}}$, for any positive integer n is the poly gamma function (Gradshteyn and Ryzhik, [10]).

Particular cases:

I. $\mathrm{ASGLG}(\alpha,1) = \mathrm{ASLG}(\alpha)$ (Hazarika and Chakraborty [7])

II. $\mathrm{ASGLG}(0,\beta) = \mathrm{GL}(\beta)$.

III. $\mathrm{ASGLG}(0,1) = \mathrm{ASLG}(0) = L(0,1)$, where, $L(0,1)$ is the standard logistic distribution and its pdf is given by $f(z) = \frac{\exp(-z)}{\{1+\exp(-x)\}^2}$, $-\infty < z < \infty$.

IV. If $Z \sim \mathrm{ASGLG}(\alpha,\beta)$ then $-Z \sim \mathrm{ASGLG}(-\alpha,\beta)$.

Remark 2.1: The Cumulative distribution function (cdf) of $\mathrm{ASGLG}(\alpha,\beta)$ is given by

$$\begin{aligned} F(z) = \frac{1}{D\,B(\beta,\beta)} \Big[& \beta^{-3} \exp(-\beta z)\{1+\exp(-z)\}^{-2\beta}\{1+\exp(z)\}^{2\beta}\Big\{\beta^2(2+\alpha z(-2+\alpha z)) \\ & {}_2F_1(\beta,2\beta,1+\beta,-\exp(z)) + 2\alpha\beta(-1+\alpha z)\ {}_3F_2(\beta,\beta,2\beta;\beta+1,\beta+1;-\exp(z)) + \\ & 2\alpha^2\ {}_4F_3(\beta,\beta,\beta,2\beta;\beta+1,\beta+1,\beta+1;-\exp(z))\Big\}\Big] \end{aligned}$$

where, ${}_pF_q(a;b;x)$ is the generalized Hypergeometric function (Gradshteyn and Ryzhik, [10]) and given by

$${}_pF_q(a;b;x) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k \ldots (a_p)_k}{(b_1)_k (b_2)_k \ldots (b_p)_k} \frac{x^k}{k!} \text{ and } (g)_k = g\,(g+1)\cdots(g+k-1)$$

Proposition 2.1. (Pogány and Nadarajah [11]) If $Z \sim \mathrm{ASGLG}(\alpha,\beta)$ then

$$E(Z^{2n-1}) = 0$$

$$E(Z^{2n}) = \frac{2^{n+1}(n!)^2}{B(\beta,\beta)} \Phi^*_{2u}(-1,2n+1,\beta);\ \forall\ n > \beta.$$

where, $\Phi^*_{\mu}(z,s,a) = \sum_{j=0}^{\infty} \frac{(\mu)_j z^j}{(a+j)^s\ j!}$ is the Goyal–Laddha generalized Hurwitz–Lerch Zeta function (see Goyal and Laddha [12])

Result 2.1 If $Z \sim \mathrm{ASGLG}(\alpha,\beta)$ then

$$E(Z^n) = \frac{1}{D\,B(\beta,\beta)} \begin{cases} -2^{\frac{n+5}{2}} \alpha^2 \left(((n+1)/2)!\right)^2 \Phi^*_{2u}(-1,n+2,\beta); & \text{if } n (> 2\beta - 1) \text{ is even} \\ 2^{\frac{n+4}{2}} \left[((n/2)!)^2 \Phi^*_{2\beta}(-1,n+1,\beta) + \alpha^2 \left(((n+1)/2)!\right)^2 \Phi^*_{2\beta}(-1,n+3,\beta)\right] & \\ & \text{if } n (> \beta) \text{is odd} \end{cases}$$

Proof: $E(Z^n) = \int_{-\infty}^{\infty} Z^n \frac{(1-\alpha z)^2+1}{D} \frac{\exp(-\beta z)}{B(\beta,\beta)(1+\exp(-z))^{2\beta}} dz$

$$=\frac{1}{D}\left[\int_{-\infty}^{\infty} 2z^n f_{GL}(z)dz + \int_{-\infty}^{\infty}\alpha^2 z^{n+2} f_{GL}(z)dz - 2\int_{-\infty}^{\infty}\alpha^2 z^{n+1} f_{GL}(z)dz\right]$$

Now using Proposition 2.1 we get for the case of when n is odd

$$E(Z^n)=\frac{1}{D}\left[-2\int_{-\infty}^{\infty}\alpha^2 z^{n+1} f_{GL}(z)dz\right]$$

$$=\frac{-2^{\frac{n+5}{2}}\alpha^2\left(\frac{(n+1)}{2}!\right)^2}{D\,B(\beta,\beta)}\Phi^*_{2u}(-1,n+2,\beta)$$

and for the case when n is even is obtained as

$$E(Z^n)=\frac{1}{D}\left[\int_{-\infty}^{\infty} 2z^n f_{GL}(z)dz + \int_{-\infty}^{\infty}\alpha^2 z^{n+2} f_{GL}(z)dz\right]$$

$$=\frac{2^{\frac{n+4}{2}}}{D\ B(\beta,\beta)}\left[\left(\frac{n}{2}!\right)^2\Phi^*_{2\beta}(-1,n+1,\beta)+\alpha^2\left(\frac{n+2}{2}!\right)^2\Phi^*_{2\beta}(-1,n+3,\beta)\right]$$

In particular:

i. $\mathrm{E}(Z)=\{-4\alpha\psi^{(1)}(\beta)\}/D=\mu(\text{say})$.

ii. $\mathrm{E}(Z^2)=\frac{2}{D}\left[2\,\psi^{(1)}(\beta)+\alpha^2\{6(\psi^{(1)}(\beta))^2+\psi^{(3)}(\beta)\}\right]$.

iii. $\mathrm{Var}(Z)=\frac{2}{D^2}\left[2\alpha^2(3D-4)(\psi^{(1)}(\beta))^2+\alpha^2 D\psi^{(3)}(\beta)+2D\psi^{(1)}(\beta)\right]$.

Remark 2.2: The parameter α can be expressed in terms of mean (μ) as

$$\alpha=\frac{-1\pm\sqrt{1-\frac{\mu^2}{\psi^{(1)}(\beta)}}}{\mu}\ \ \forall\beta>1$$

Result 2.2. If $Z\sim\mathrm{ASGLG}(\alpha,\beta)$, then the moment generating function (mgf) is given by

$$M_Z(t)=\frac{1}{D\,B(\beta,\beta)}\left[\frac{\alpha^2 I_1(t)}{(\beta+t)^3}+\frac{\alpha^2 I_1(-t)}{(\beta-t)^3}-\frac{\alpha I_2(t)}{(\beta+t)^2}+\frac{\alpha I_2(-t)}{(\beta-t)^2}+\frac{I_3(-t)}{\beta-t}+\frac{I_3(t)}{\beta+t}\right].$$

Where, $I_1(t)={}_4F_3(2\beta,t+\beta,t+\beta,t+\beta;t+\beta+1,t+\beta+1,t+\beta+1;-1)$,

$I_2(t)={}_3F_2(2\beta,t+\beta,t+\beta;t+\beta+1,t+\beta+1;-1)$,

$I_3(t)={}_2F_1(2\beta,\beta-t;-t+\beta+1;-1)$

Remark 2.3. An alternative expression for the mgf of $\mathrm{ASGLG}(\alpha,\beta)$ can be obtained in terms of gamma function and its derivatives as

$$\frac{1}{\Gamma(\beta)^2[1+\alpha^2\psi^{(1)}(\beta)]}\Big[\Gamma(\beta+t)\Gamma(\beta-t)-\alpha\{\Gamma^{/}(\beta+t)\Gamma(\beta-t)-\Gamma(\beta+t)\Gamma^{/}(\beta-t)\}$$
$$+\frac{\alpha^2}{2}\{\Gamma^{//}(\beta+t)\Gamma(\beta-t)+\Gamma^{/}(\beta+t)\Gamma^{//}(\beta-t)\}\Big] \qquad (8)$$

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