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Uniform Order Legendre Approach for Continuous Hybrid Block Methods for the Solution of First Order Ordinary Differential Equations

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Abstract: *We adopted the method of interpolation of the approximation and collocation of its differential equation and with Legendre polynomial of the first kind as basis function to yield a continuous Linear Multistep Method with constant step size. The methods are verified to be consistent and satisfies the stability condition. Our methods was tested on first order Ordinary Differential Equation (ODE) and found to give better result when compared with the analytical solution.*
Keywords: *Collocations, Legendre polynomial, Interpolation, Continuous scheme.*

I. Introduction

We consider a numerical method for solving general first order Initial Value Problems (IVPs) of Ordinary Differential Equations (ODEs) of the form:

$$y' = f(x,y),\ y(x_0) = y_0 \qquad (1)$$

Where f is a continuous function and satisfies Lipschitz condition of the existence and uniqueness of solution. A differential equation in which the unknown function is a function of two or more independent variable is called a Partial Differential Equations (PDEs). Those in which the unknown function is function of only one independent variable are called Ordinary Deferential Equations (ODEs).

Many scholars have worked extensively on the solution of (1) in literatures [1-6]. These authors proposed different method ranging from predictor corrector method to block method using different polynomials as basis functions, evaluated at some selected points.

In this paper we proposed three-step and four-step hybrid block method with two off- step points, using Legendre polynomials evaluated at grids and off-grids points to give a discrete scheme**.**

II. Derivation Of The Method

In this section, we intend to develop the Linear Multistep Method (*LMM*), by interpolating and collocating at some selected points. We consider a Legendre approximation of the form;

$$y(x) = \sum_{j=0}^{s+r-1} a_j p^j(x) \qquad (2)$$

Where r and s are interpolation and collocation point

$$y'(x) = \sum_{j=0}^{s+r-1} j a_j p^{j-1}(x) \qquad (3)$$

Substituting (3) into (1), we have

$$f(x,y) = \sum_{j=0}^{s+r-1} j a_j p^{j-1}(x) \qquad (4)$$

2.1 **Three-Step Method With Two Off-Step Points.**

Interpolating (2) at x_n and collocating (4) at x_{n+s}, $s = 0,1,\frac{3}{2},2,\frac{5}{2},3$ give the system of polynomial equation in the form .

$$AX = U \qquad (5)$$

$$\begin{bmatrix} 1 & -1 & 1 & -1 & 1 & -1 & 1 \\ 0 & \frac{2}{3} & -2 & 4 & -\frac{20}{3} & 10 & -14 \\ 0 & \frac{2}{3} & -\frac{2}{3} & -\frac{4}{9} & \frac{100}{81} & -\frac{10}{27} & -\frac{98}{81} \\ 0 & \frac{2}{3} & 0 & -1 & 0 & \frac{5}{4} & 0 \\ 0 & \frac{2}{3} & \frac{2}{3} & -\frac{4}{9} & -\frac{100}{81} & -\frac{10}{27} & \frac{98}{81} \\ 0 & \frac{2}{3} & \frac{4}{3} & \frac{11}{9} & \frac{10}{9} & -\frac{145}{108} & -\frac{343}{162} \\ 0 & \frac{2}{3} & 2 & 4 & \frac{20}{3} & 10 & 14 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_4 \\ a_5 \end{bmatrix} = \begin{bmatrix} y_n \\ hf_{n+1} \\ hf_{n+\frac{3}{2}} \\ hf_{n+2} \\ hf_{n+\frac{5}{2}} \\ hf_{n+3} \end{bmatrix}$$

Solving the above matrix by Gaussian elimination method we obtained the following results,

$$\left.\begin{aligned} a_0 &= y_n + \frac{3}{28}hf_{n+3} - \frac{18}{35}hf_{n+\frac{5}{2}} - \frac{58}{35}hf_{n+\frac{3}{2}} + \frac{71}{280}hf_n + \frac{117}{70}hf_{n+1} + \frac{459}{280}hf_{n+2} \\ a_1 &= \frac{3}{56}hf_n + \frac{3}{56}hf_{n+3} - \frac{12}{35}hf_{n+\frac{3}{2}} + \frac{243}{280}hf_{n+1} + \frac{243}{280}hf_{n+2} + \frac{459}{280}hf_{n+2} \\ a_2 &= -\frac{1}{56}hf_{n+3} + \frac{18}{35}hf_{n+\frac{5}{2}} + \frac{6}{7}hf_{n+\frac{3}{2}} - \frac{19}{280}hf_n - \frac{45}{56}hf_{n+1} - \frac{27}{56}hf_{n+2} \\ a_3 &= \frac{1}{16}hf_n + \frac{1}{16}hf_{n+3} - \frac{4}{5}hf_{n+\frac{3}{2}} + \frac{27}{80}hf_{n+1} + \frac{27}{80}hf_{n+2} \\ a_4 &= \frac{117}{6160}hf_{n+3} + \frac{54}{385}hf_{n+\frac{5}{2}} + \frac{18}{77}hf_{n+\frac{3}{2}} - \frac{261}{6160}hf_n + \frac{351}{6160}hf_{n+1} - \frac{2511}{6160}hf_{n+2} \\ a_5 &= \frac{3}{140}hf_{n+3} + \frac{12}{35}hf_{n+\frac{3}{2}} + \frac{3}{140}hf_n - \frac{27}{140}hf_{n+1} - \frac{27}{140}hf_{n+2} \\ a_6 &= \frac{9}{308}hf_{n+3} - \frac{54}{385}hf_{n+\frac{5}{2}} - \frac{18}{77}hf_{n+\frac{3}{2}} - \frac{9}{1540}hf_n + \frac{27}{308}hf_{n+1} + \frac{81}{308}hf_{n+2} \end{aligned}\right\} \quad (6)$$

Substituting (6) into (2) we obtained, the *LMM* as,

$y(z) = \alpha_0(z)y_n + \left(\sum_{j=0}^{3} \beta_0(z)f_{n+1} + \beta_{\frac{3}{2}}(z)f_{n+\frac{3}{2}} + \beta_{\frac{5}{2}}(z)f_{n+\frac{5}{2}}\right)$

where $\alpha_0(z)$ and $\beta_0(z)$ are continuous coefficients obtained as

$\alpha_0(z) = 1$

$$\left.\begin{aligned} \beta_0(z) &= z - \frac{31}{72}z^4 - \frac{29}{20}z^2 + \frac{29}{27}z^3 + \frac{4}{45}z^5 - \frac{1}{135}z^6 \\ \beta_1(z) &= \frac{119}{24}z^4 + \frac{15}{2}z^2 - \frac{19}{2}z^3 - \frac{6}{5}z^5 + \frac{1}{9}z^6 \\ \beta_{\frac{3}{2}}(z) &= -\frac{104}{9}z^4 - \frac{40}{3}z^2 + \frac{536}{27}z^3 + \frac{136}{45}z^5 - \frac{8}{27}z^6 \\ \beta_2(z) &= \frac{91}{8}z^4 + \frac{45}{4}z^2 - 18z^3 - \frac{16}{5}z^5 + \frac{1}{3}z^6 \\ \beta_{\frac{5}{2}}(z) &= -\frac{24}{5}z^2 - \frac{16}{3}z^4 + 8z^3 - \frac{8}{45}z^6 + \frac{8}{5}z^5 \\ \beta_3(z) &= \frac{71}{72}z^4 + \frac{5}{6}z^2 - \frac{77}{54}z^3 - \frac{14}{45}z^5 + \frac{1}{27}z^6 \end{aligned}\right\} \quad (7)$$

Where $z = \frac{x - x_n}{h}$

Equation (7) is known as the continuous scheme.

Evaluating (7) at x_{n+s}, $s = 1, \frac{1}{2}, 2, \frac{3}{2}, 3$ we obtained the following discrete scheme

$$
\left.\begin{aligned}
y_{n+1} &= y_n + h\left(\frac{11}{40}f_n + \frac{673}{360}f_{n+1} - \frac{104}{45}f_{n+\frac{3}{2}} + \frac{211}{120}f_{n+2} - \frac{32}{45}f_{n+\frac{5}{2}} + \frac{43}{360}f_{n+3}\right)\\
y_{n+\frac{3}{2}} &= y_n + h\left(\frac{35}{128}f_n + \frac{1323}{640}f_{n+1} - \frac{77}{40}f_{n+\frac{3}{2}} + \frac{1053}{640}f_{n+2} - \frac{27}{40}f_{n+\frac{5}{2}} + \frac{73}{640}f_{n+3}\right)\\
y_{n+2} &= y_n + h\left(\frac{37}{135}f_n + \frac{92}{45}f_{n+1} - \frac{224}{135}f_{n+\frac{3}{2}} + \frac{29}{15}f_{n+2} - \frac{32}{45}f_{n+\frac{5}{2}} + \frac{16}{145}f_{n+3}\right)\\
y_{n+\frac{5}{2}} &= y_n + h\left(\frac{35}{128}f_n + \frac{2375}{1152}f_{n+1} - \frac{125}{72}f_{n+\frac{3}{2}} + \frac{875}{384}f_{n+2} - \frac{35}{72}f_{n+\frac{5}{2}} + \frac{125}{1152}f_{n+3}\right)\\
y_{n+3} &= y_n + h\left(\frac{11}{40}f_n + \frac{81}{40}f_{n+1} - \frac{8}{5}f_{n+\frac{3}{2}} + \frac{81}{40}f_{n+2} + \frac{11}{40}f_{n+3}\right)
\end{aligned}\right\} \quad (8)
$$

Equation (8) is known as the required block method.

2.2 **Four-Step Method With Two Off-Step Points**

Following similar procedure above, we interpolate (2) at x_n and collocation (4) at x_{n+s} , s =0, 1, $\frac{5}{2}$, 2, $\frac{7}{2}$, 4 and

Evaluating at different grids and off-grid point, we obtain the method as

$$
\left.\begin{aligned}
y_{n+1} &= y_n + h\left(\frac{2039}{7056}f_n + \frac{2047}{1512}f_{n+1} - \frac{958}{315}f_{n+2} + \frac{4636}{946}f_{n+\frac{5}{2}} - \frac{3134}{2205}f_{n+\frac{7}{2}} - \frac{3359}{15120}f_{n+4}\right)\\
y_{n+2} &= y_n + h\left(\frac{83}{294}f_n + \frac{1598}{945}f_{n+1} - \frac{172}{105}f_{n+2} + \frac{3392}{945}f_{n+\frac{5}{2}} - \frac{302}{105}f_{n+3} + \frac{832}{738}f_{n+\frac{7}{2}} - \frac{341}{1890}f_{n+4}\right)\\
y_{n+3} &= y_n + h\left(\frac{1107}{3920}f_n + \frac{473}{280}f_{n+1} - \frac{51}{35}f_{n+2} + \frac{148}{35}f_{n+\frac{5}{2}} - \frac{713}{289}f_{n+3} + \frac{276}{245}f_{n+\frac{7}{2}} - \frac{101}{560}f_{n+4}\right)\\
y_{n+\frac{7}{2}} &= y_n + h\left(\frac{217}{768}f_n + \frac{5831}{3456}f_{n+1} - \frac{343}{240}f_{n+2} + \frac{4459}{1080}f_{n+\frac{5}{2}} - \frac{4459}{1920}f_{n+3} + \frac{161}{120}f_{n+\frac{7}{2}} - \frac{6517}{34560}f_{n+4}\right)\\
y_{n+\frac{5}{2}} &= y_n + h\left(\begin{aligned}&\frac{31895}{112896}f_n + \frac{40825}{24192}f_{n+1} - \frac{2875}{3016}f_{n+2} + \frac{5965}{1512}f_{n+\frac{5}{2}} - \frac{23875}{8064}f_{n+3}\\ &+ \frac{4075}{3528}f_{n+\frac{7}{2}} - \frac{8875}{48384}f_{n+4}\end{aligned}\right)
\end{aligned}\right\} \quad (9)
$$

$$
y_{n+4} = y_n + h\left(\frac{622}{2205}f_n + \frac{320}{189}f_{n+1} - \frac{472}{315}f_{n+2} + \frac{4096}{946}f_{n+\frac{5}{2}} - \frac{832}{315}f_{n+3} + \frac{4096}{2206}f_{n+\frac{7}{2}} - \frac{26}{946}f_{n+4}\right)
$$

(10)

Equation (9) and (10) together form the block method.

III. Analysis Of The Methods.

In this section we discuss the local truncation error and order, consistency and zero stability of the scheme generated.

3.1 Order And Error Constant

Let the linear operator $L[y(x);h]$ associated with the block formula be as

$L[y(x);h] = \sum_{j=0}^{k}\left[\alpha_j y(x_n + jh) - h\beta_j y'(x_n + jh)\right]$ (3.1)

Expanding in Taylor series expansion and comparing the coefficients of h gives

$$L[y(x);h] = C_0 y(x) + C_1 h y'(x) + C_1 h^2 y''(x) \ldots C_p h^p y^p(x) + C_{p+1} h^{p+1} y^{p+1}(x)$$

Definition 3.1

The Linear operator L and the associated continuous *LMM* (3.1) are said to be of order p if $C_0 = C_1 = C_2 = C_3 = C_3 \ldots = C_p = 0,\ C_{p+1} \neq 0$ is called the error constant.

Table 2.1: Features of the block method (8)

	Order	Error Constant
y_{n+1}	7	$-\frac{47}{24192}$
$y_{n+\frac{3}{2}}$	7	$-\frac{27}{14336}$
y_{n+2}	7	$-\frac{29}{15120}$
$y_{n+\frac{5}{2}}$	7	$-\frac{725}{387072}$
y_{n+3}	7	$-\frac{9}{4480}$

Table 2.2: Features of the method (10) and (11)

	Order	Error constant
y_{n+4}	7	$\frac{16}{6615}$
y_{n+3}	7	$\frac{31}{12544}$
y_{n+2}	7	$\frac{29}{11760}$
y_{n+1}	7	$\frac{965}{33860}$
$y_{n+\frac{5}{2}}$	7	$\frac{107725}{43342064}$
$y_{n+\frac{7}{2}}$	7	$\frac{24}{98304}$

3.2 Consistent And Zero Stability

Definition 3.2

The *LMM* is said to be consistent if it has order P≥1.

Definition 3.3

The block method is said to be zero stable if the roots $Z_s, s = 1,2,\dots,N$ of the characteristic $p(Z)$ defined by $p(Z) = \det(ZA^0 - A')$ Satisfied $|Z_s| \le 1$ have the multiplicity not exceeding the order of the differential equation, as $h \to 0, p(Z) = Z^{r-\mu(z-1)\mu}$

Where μ is the order of the differential equation r is the order of the matrix A^0 and A' [2].

Putting (8) in matrix form we obtain

$$\begin{bmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1 \end{bmatrix} \begin{bmatrix} y_{n+1}\\ y_{n+\frac{3}{2}}\\ y_{n+2}\\ y_{n+\frac{5}{2}}\\ y_{n+3} \end{bmatrix} = \begin{bmatrix} 0&0&0&0&1\\ 0&0&0&0&1\\ 0&0&0&0&1\\ 0&0&0&0&1\\ 0&0&0&0&1 \end{bmatrix} \begin{bmatrix} y_{n-4}\\ y_{n-3}\\ y_{n-2}\\ y_{n-1}\\ y_n \end{bmatrix} + \begin{bmatrix} \frac{673}{360} & -\frac{404}{44} & \frac{2110}{120} & -\frac{32}{45} & \frac{43}{360}\\ \frac{1323}{640} & -\frac{77}{40} & \frac{1053}{640} & -\frac{27}{40} & \frac{73}{640}\\ \frac{92}{45} & -\frac{224}{135} & \frac{29}{15} & -\frac{32}{45} & \frac{16}{135}\\ \frac{2325}{1152} & -\frac{125}{72} & \frac{875}{384} & -\frac{35}{72} & \frac{125}{1152}\\ \frac{81}{40} & -\frac{8}{5} & \frac{81}{40} & 0 & \frac{11}{40} \end{bmatrix} \begin{bmatrix} f_{n+1}\\ f_{n+\frac{3}{2}}\\ f_{n+2}\\ f_{n+\frac{5}{2}}\\ f_{n+3} \end{bmatrix}$$

$$+h \begin{bmatrix} 0&0&0&0&\frac{11}{40}\\ 0&0&0&0&\frac{35}{128}\\ 0&0&0&0&\frac{37}{135}\\ 0&0&0&0&\frac{35}{128}\\ 0&0&0&0&\frac{11}{40} \end{bmatrix} \begin{bmatrix} f_{n-4}\\ f_{n-3}\\ f_{n-2}\\ f_{n-1}\\ f_n \end{bmatrix}$$

Normalizing the matrix as $h \to 0$

$$\rho(Z) = [ZA^0 - A']$$

$$\begin{bmatrix} R & 0 & 0 & 0 & 0 \\ 0 & R & 0 & 0 & 0 \\ 0 & 0 & R & 0 & 0 \\ 0 & 0 & 0 & R & 0 \\ 0 & 0 & 0 & 0 & R \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R & 0 & 0 & 0 & -1 \\ 0 & R & 0 & 0 & -1 \\ 0 & 0 & R & 0 & -1 \\ 0 & 0 & 0 & R & -1 \\ 0 & 0 & 0 & 0 & R-1 \end{bmatrix}$$

$$Z^4(Z-1) = 0$$

$$Z = 0, Z = 1$$

The block method (8) is observed to be zero-stable.
Equation (9) and (10) in matrix form, and following the same procedure above is found to be zero-stable. [8]

3.3 Convergence
The convergence of the continuous hybrid block method is considered in the light of the basic properties discussed above in conjunction with the fundamental theorem of Dahlquist [8] for *LMM*; we state the Dahlquist theorem without proof.
Theorem 3.1
The necessary and sufficient condition for a linear multistep method to be convergent is for it to be consistent and zero stable.
Following the theorem 3.1 above shows that both methods are convergent.

IV. Numerical Examples

We now implement our derived block methods, on first order initial value problems.
In order to test the efficiency of the methods, we employed the following notations in our tables below:
3S2HBM: 3 Step two off - step Hybrid Block Method.
4S2HBM: 4 Step two off - step Hybrid Block Method.

Example 4.1
$y' = 5y$, $y(0) = 1, h = 0.01$
Exact solution y(x) = e^{5x}

Example 4.2
$y' + y = 0$, $y(0) = 1, h = 0.1$
Exact solution $y(x) = e^{-x}$

Table 4.1: Numerical results for example 4.1

X	EXACT	3S2HBM	4S2HBM
0.0	1.000000000	1.000000000	1.000000000
0.01	1.051271096	1.051271096	1.051271096
0.02	1.105170918	1.105170918	1.105170918
0.03	1.161834243	1.161834243	1.161834243
0.04	1.221402758	1.221402758	1.221402758
0.05	1.284025417	1.284025417	1.284025417
0.06	1.349858808	1.349858808	1.349858807
0.07	1.419067549	1.419067549	1.419067548
0.08	1.491824698	1.491824698	1.491824697
0.09	1.568312185	1.568312186	1.568312185
0.1	1.648721271	1.648721271	1.648721270

Table 4.2: Comparison of error for example 4.1

X	3S2HBM	4S2HBM	[7]
0.01	0.0	0.0	6.2 E -10
0.02	0.0	0.0	1.1 E -09

0.03	0.0	0.0	1.3 E -09
0.04	0.0	0.0	1.6 E -09
0.05	0.0	0.0	1.3 E -09
0.06	0.0	1.0 E -09	4.2 E -10
0.07	0.0	1.0 E -09	2.4 E -09
0.08	0.0	1.0 E -09	2.4 E -09
0.09	1.0 E -09	0.0	2.5 E -09
0.1	0.0	1.0 E -09	2.3 E -09

Table 4.3: Numerical results for example 4.2

X	EXACT	3S2HBN	4S2HBM
0.0	1.000000000	1.000000000	1.000000000
0.1	0.904837418	0.904837418	0.904837418
0.2	0.818730753	0.818730753	0.818730753
0.3	0.740818220	0.740818220	0.740818220
0.4	0.670320046	0.670320046	0.670320046
0.5	0.606530659	0.606530660	0.606530660
0.6	0.548811636	0.548811636	0.548811636
0.7	0.496575303	0.496585304	0.496583304
0.8	0.449328964	0.449328964	0.449328965
0.9	0.406569590	0.406569659	0.406569660
1.0	0.367879441	0.367879441	0.367579441

Table 4.4: Comparison of Errors for example 4.2

X	3S2HBM	4S2HBM
0.0	0.0	0.0
0.1	0.0	0.0
0.2	0.0	0.0
0.3	1.0 E -09	1.0 E -09
0.4	0.0	1.0 E -09
0.5	1.2 E -03	1.2 E -03
0.6	0.0	1.0 E -09
0.7	1.0 E -09	1.0 E -09
0.8	0.0	1.0 E-09
0.9	0.0	7.1 E -08
0.0	0.0	1.0 E -09

V. Conclusion

The desirable property of a numerical solution is to behave like the theoretical solution of the problem which can be seen in the above result. The implementation of the scheme is done with the aid of maple software. The method are tested and found to be consistent, zero stable and convergent. We implement the methods on two numerical examples and the numerical evidences shows that the methods are accurate and effective and therefore favourable.

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