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On The Effect of Perturbations, Radiation and Triaxiality on the Stability of Triangular Libration Points In the Restricted Three-Body Problem

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Abstract: *This study is aimed at studying the effects of small perturbations in the coriolis (ε) and the centrifugal (ε') forces, the triaxiality (σ_1, σ_2) of the bigger primary and the radiation (q) pressure force of the smaller primary on the stability of libration point in the Restricted Three-Body Problem(RTBP) in particular to study the effect of perturbations in the coriolis and the centrifugal forces on the of the libration points in the restricted three body problem when the bigger primary is a triaxial rigid body and the smaller primary a source of radiation. The equations of motion of the restricted problem under influence of the perturbations in the coriolis and the centrifugal forces, triaxiality and radiation were established. These equations of motion are found to be affected by the aforesaid parameters. They generalize the classical equations of motion of restricted problem and those obtained by others. Five libration points were obtained three collinear points (L_1, L_2, L_3) and two triangular points (L_4, L_5). The libration points were found to be affected by the small perturbation for the range $0 \leq \mu < \mu_c$. The critical mass value μ_c is affected by all the aforesaid parameters in the coriolis and centrifugal forces, the triaxial nature of the bigger primary and the radiation pressure force of the smaller primary. This generalizes the equation of orbits of the classical restricted three body problem and those obtained by others.*

Keywords: *Classical mechanics, Motion, Perturbations, Coriolis, Triaxial, Triangular,Syndoic.*

I. Introduction

The Restricted Three Body Problem (RTBP), which kept mathematicians busy for over two hundred years, describes the motion of an infinitesimal body, which moves under influence of a gravitational attraction and is not influenced by the motion of the finite bodies (primaries). This problem began with Euler (1772). In connection with this lunar theories which brought about his major accomplishment in the introduction of synodic (rotating) coordinate system. This led to the discovery of the Jacobian integral by Jacobi (1836). In recent times many properties such as shape, surface area light, perturbing forces are taken into consideration in describing the motion of satellite (both artificial and natural) of the asteroid and their stability. Due to this, many authors studied the effect of coriolis and centrifugal forces on the problem.

Wintner (1941) discovered that the stability of the two equilateral points of the triangular libration point is due to the existence of the coriolis terms in the equation of motion expressed in the rotating coordinate system. Szebehely (1967a) showed that the restricted three body problem possess five libration points, three collinear points L_1, L_2, L_3 which are unstable and the two triangular points L_4, L_5 which are stable for $0 \leq \mu \leq \mu_c = 0.3852$ (critical mass value). He (1967b) further generalized his work and studied the effect of small perturbation in the coriolis force on the stability of libration points keeping the centrifugal force constant. He discovered that the collinear points remain unstable while for the stability of the triangular points he obtained a relation for the critical value of the mass parameter μ_c as

$$\mu_c = \mu_0 + \frac{16\varepsilon}{3\sqrt{69}}$$

where ε is the parameter for the coriolis force. Therefore establishing that the coriolis force is a stabilizing force. Subbarao and Sharma (1975) considered the same problem but with one of the primaries as an oblate spheroid and its equatorial plane coinciding with the plane of motion. While studying the stability of the triangular points they discovered that the coriolis force is not always a stabilizing force. Here, the centrifugal force is not kept constant and this paper is not in line with assertion that the coriolis force as a stabilizing force depends upon the fact that the centrifugal force is kept constant while changing the coriolis force. Based on

Szebehely's work, Bhatnagar and Hallan (1978) considered the effect of perturbations in the coriolis (ε) and the centrifugal (ε′) force and discovered that the collinear points are not influenced by the perturbations and hence remain unstable. But for the triangular points they obtained the relation:

$$\mu_c = \mu_0 + 4\frac{\left(36\varepsilon - 19\varepsilon'\right)}{27\sqrt{69}}$$

and then concluded that the range of stability increases or decreases depending on whether the point (ε, ε′) lies on either of the two points in which the (ε, ε′) plane is divided by the line 36ε - 19ε′ = 0.

Singh and Iswhar (19840 determined the effect of small perturbations in the coriolis and the centrifugal forces on the location of equilibrium points in the restricted three body problem with variable mass. They found that the triangular points form nearly equilateral triangles with the primaries while the collinear points lie on the line joining the primaries. They (1985) also considered the same problem and found that the range of stability of the triangular points increases or decreases depending on whether the perturbation point (ε, ε′) lies in either of the two part in which (ε, ε′) plane is divided by the line 36ε - 19ε′ = 0

Authors like Brouwer (1946), Osipov (1970), Nikoleave (1970), Hitzl and Breakwell (1971) considered the unusual shape of the primaries or infinitesimal mass in the generalization of RTBP.. They studied the effect of oblateness and triaxiality in the potential between the bodies.

Vidyakin (1974) determined the locations of five equilibrium solutions and studied their stability in the Lyponov sense; when the primaries are ablate spheroids with the equatorial plane coinciding with the plane of motion. Furthermore Sharma and Subbarao (1979), Sharma (1982) studied the concept of restricted problem when either one of the primaries is oblate spheroid. Sharma et al. (2001) studied the stationary solutions of the planar restricted three body problem when the primaries are triaxial rigid bodies with one of the axes as the axis of symmetry and its equatorial plane coinciding of motion..

The effect of radiations was also studied by several authors taking either one or both primaries as source of radiation. This is due to ability of the primaries to emit radiation. Radzievskii (1950) formulated the photogravitational RTBP. Simmons et al. (1985), Kumar and Choudry (1986) discovered the existence of libration points for the generalized photogravitational RTBP. Sharma et. al (2001) generalize their work which dealt with the stationary solution of the planar restricted three body problem when the primaries are triaxial rigid bodies, by further considering them as sources of radiation. They discovered that the state of stability of the triangular and collinear point remain unchanged.

Due to the results of various generalisation of the classical restricted three body problem, it was possible for us to consider and study a new form of generalisation.

This paper attempt to study effect of small perturbations in the coriolis and centrifugal forces on the stability of RTBP when the bigger primary is triaxial and the smaller one a source of radiation

Since the solar radiation pressure forces F_P changes with distance by the same law of gravitational attraction F_g and acts in opposite direction to it, It is possible that the force lead to reduction of the effective mass of the body. Thus the resultant force on the particle is

$F = F_g - F_p = F_g\left(1 - \frac{F_p}{F_g}\right) = qF_g$ where $q = 1 - \frac{F_p}{F_g}$ is the mass reduction factor and the force of the body is given by $F_p = (1- q)\, F_g$ such that $0 < (1- q) << 1$

II. Potential Of The Body

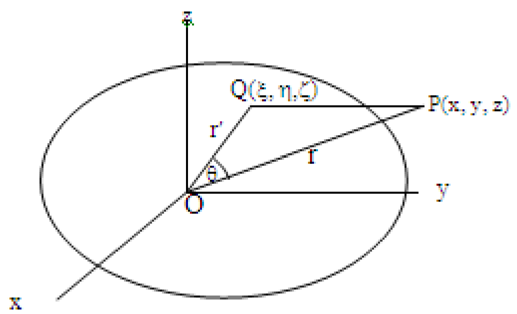

Fig.1:

We considered the potential of a body due to one solid body of arbitrary shape and mass distribution M and placed a unit mass at point P outside the body and take an elementary mass dm at the point Q within the

body.[Let O be the origin with respect to the coordinate axes P(x,y,z) and Q(ξ,η,ζ) and the centre of mass of the body].
Where OP = r and OQ = r′

Potential at P due to mass dm is $dV = -\frac{G}{PQ}dm$, (1)

where G is the gravitational constant.Therefore,
the potential at P due to the whole mass M is

$$V = -\frac{G}{r}\int_M dm - \frac{G}{r}\int_M \frac{1}{2}\left(3cos^2\theta - 1\right)\frac{r'^2}{r^2}dm$$

or

$$V = -\frac{GM}{r} - \frac{G}{2r^3}\left(2I_0 - 3I\right) \tag{2}$$

where $I_0 = \int_M r'^2 dm$ is the moment of inertia about the origin

$I = \int_M r'^2 \sin^2\theta dm$ is the moment of inertia about the line OP.

Since the body is triaxial $2I_o = I_1 + I_2 + I_3$

And so $V = -GM\left[\frac{1}{r} - \frac{1}{2Mr^3}\left(I_1 + I_2 + I_3 - 3I\right)\right]$ (3)

where I_1 , I_2 , I_3 are the principal moments of inertia about the centre of mass.
The left part of Equation (3) is the potential due to solid sphere and the right part is the potential due to the departure of the body from the spherical shape to a triaxial one.

Now,

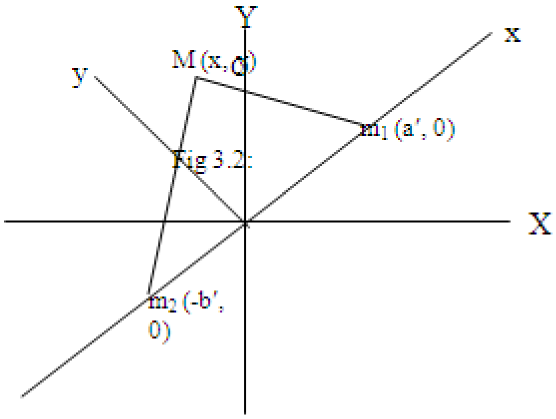

Fig. 2:

Let O be a fixed point in the space motion and r the relative distance between the primaries of masses m_1 and m_2 ($m_2<m_1$) moving under a mutual gravitational force.Let r_1, r_2 be their distances from the origin by Newton's law of Gravitation, the gravitation potential between two triaxial primaries is given by

$$V^* = -Gm_1m_2\left[\frac{1}{r} + \frac{1}{2m_1r^3}\left(I_1 + I_2 + I_3 - 3I\right) + \frac{1}{2m_2r^3}\left(I_1' + I_2' + I_3' - 3I'\right)\right]$$ (McCusky 1963)

Since m_2 is not triaxial $\left(I_1' = I_2' = I_3' = I\right)$, the potential for the system is

$$V^* = -Gm_1m_2\left[\frac{1}{r} + \frac{1}{2m_1r^3}\left(I_1 + I_2 + I_3 - 3I\right)\right] \tag{4}$$

The mean motion from Kepler's third law of motion is

$$n^2 = \frac{G\left(m_1 + m_2\right)}{r^3}.$$

$$n^2 = \frac{1}{r}\frac{(m_1+m_2}{m_1 m_2}\frac{\partial V^*}{\partial r} \tag{5}$$

Using Eqn. (4) becomes

$$n^2 = G\left(m_1 + m_2\right)\left[\frac{1}{r^3} + \frac{3}{2m_1 r^5}\left(I_1 + I_2 + I_3 - 3I\right)\right] \tag{6}$$

III. Equations of Motion

We further considered the rotating coordinate system (Oxyz), the x axis is taken along the line joining the triaxial body of the mass m_1 and the radiating smaller body of mass m_2 and that they move without rotation about their centre of mass in a circular orbit with an angular velocity n of the infinitesimal body with mass m . Then the kinetic energy is given by

$$= \frac{1}{2}m\left[\left(\dot{x}^2 + \dot{y}^2\right) + 2n\left(x\dot{y} - \dot{x}y\right) + n^2\left(x^2 + y^2\right)\right] = T_2 + T_1 + T_0 \tag{7}$$

Where

$$T_2 = \frac{1}{2}m\left(\dot{x}^2 + \dot{y}^2\right)$$

$$T_1 = mn\left(x\dot{y} - \dot{x}y\right) \tag{8}$$

$$T_0 = \frac{1}{2}mn^2\left(x^2 + y^2\right)$$

And gravitational potential energy between the infinitesimal mass m and the triaxial bigger primary and the radiating smaller primary is

$$V = -Gm\left[\frac{m_1}{r_1} + \frac{m_2 q}{r_2} + \frac{m_1}{2m_1 r_1^3}\left(I_1 + I_2 + I_3 - 3I\right)\right] \tag{9}$$

Where

$$r_1^2 = \left(x - a'\right)^2 + y^2$$

$$r_2^2 = \left(x - b'\right)^2 + y^2$$

I_1, I_2, I_3 are the principal moments of inertia of the triaxial rigid body of mass m_1 at its centre of mass with a, b, c, as lengths of its semi-axes.

$I = I_1 l^2 + I_2 m^2$ is the moment of inertia about a line joining the centre of m_1 and m_2 where l, m, are the direction cosines of the line with respect to its principal axes.

q is the mass reduction factor due to radiation.

Adapting the Hamiltonian canonical equation of motion expression

$$\dot{x} = \frac{\partial H}{\partial p_x}, \dot{y} = \frac{\partial H}{\partial p_y}, \dot{p}_x = -\frac{\partial H}{\partial x} \text{ and } \dot{p}_y = -\frac{\partial H}{\partial y} \tag{10}$$

with

$$H = T_2 - T_0 + U = \frac{1}{2m}\left[P_x^2 + P_y^2\right] + n\left(P_x y - xP_y\right) + U \quad U = V - T_0, \tag{11}$$

The equations of motion of the infinitesimal mass m are

$$\ddot{x} - 2n\dot{y} = \frac{\partial U}{\partial x}$$
$$\ddot{y} + 2n\dot{x} = \frac{\partial U}{\partial y} \qquad (12)$$

where

$$U = -Gm\left[\frac{m_1}{r_1} + \frac{m_2 q}{r_2} + \frac{m_1}{2m_1 r_1^3}\left(I_1 + I_2 + I_3 - 3I\right)\right]$$
$$r_1^2 = \left(x - a'\right)^2 + y^2$$
$$r_2^2 = \left(x - b'\right)^2 + y^2$$

we choose the sum of the masses of the primaries to be one so that if $m_2 = \mu$, $m_1 = 1 - \mu$, where μ is the ratio of the mass of the smaller primary to the total mass of the primaries $\mu = \frac{m_2}{m_1 + m_2} \le \frac{1}{2}$ and $0 < \mu \le ½$.The distance between the primaries is taken equal to one and the unit of time is so chosen as to make the gravitational constant G is unity. We have the origin as the barycentre of the masses m_1 and m_2 as defined by Szebehely (1967b). The equations of motion in the dimensionless synodic coordinate system (Oxyz) becomes

$$\ddot{x} - 2n\dot{y} = n^2 x - \frac{\partial \overline{U}}{\partial x}$$
$$\ddot{y} + 2n\dot{x} = n^2 y - \frac{\partial \overline{U}}{\partial y} \qquad (13)$$

Where $\overline{U} = \frac{1-\mu}{r_1} + \frac{\mu q}{r_2} + \frac{1-\mu}{2m_1 r_1^3}\left(I_1 + I_2 + I_3 - 3I\right)$ (14)

$$r_1^2 = \left(x - \mu\right)^2 + y^2$$
$$r_2^2 = \left(x + 1 - \mu\right)^2 + y^2 \qquad (15)$$

The mean motion n is

$$n^2 = 1 + \frac{3}{2}\left(2A_1 - A_2 - A_3\right). \qquad (16)$$

$$I_1 = \frac{m_1}{5}\left(b^2 + c^2\right), \quad I_2 = \frac{m_1}{5}\left(a^2 + c^2\right), \quad I_3 = \frac{m_1}{5}\left(b^2 + a^2\right).$$

and

$$A_1 = \frac{a^2}{5R^2}, \qquad A_2 = \frac{b^2}{5R^2}, \qquad A_3 = \frac{c^2}{5R^2},$$

where R is the dimensional distance between the primaries. Neglecting the drag effect,

Introducing small perturbations ε and ε' in the coriolis and the centrifugal forces using the parameter ϕ andψ, respectively, such that

$$\phi = 1 + \varepsilon, \left|\varepsilon\right| << 1$$
$$\psi = 1 + \varepsilon', \left|\varepsilon'\right| << 1,$$

the equations of motion in (13) and (14) become

$$\ddot{x} - 2\phi n\dot{y} = \frac{\partial\Omega}{\partial x}$$
$$\ddot{y} + 2\phi n\dot{x} = \frac{\partial\Omega}{\partial y} \qquad (17)$$

Where $\Omega = \frac{n^2\psi}{2}\left(x^2+y^2\right) + \frac{1-\mu}{r_1} + \frac{\mu q}{r_2} + \frac{1-\mu}{2m_1 r_1^3}\left(I_1+I_2+I_3-3I\right)$

or

$$\Omega = \frac{n^2\psi}{2}\left(x^2+y^2\right) + \frac{1-\mu}{r_1} + \frac{\mu q}{r_2} + \frac{(1-\mu)}{2r_1^3}(2\sigma_1-\sigma_2) - \frac{3(1-\mu)(\sigma_1-\sigma_2)}{2r_1^5}y^2 \qquad (18)$$

Where $\sigma_1 = A_1 - A_3, \sigma_2 = A_2 - A_3$, and $\sigma_1, \sigma_2 << 1$,

the mean motion, in Eqn.(16) become

$$n^2 = 1 + \frac{3}{2}(2\sigma_1 - \sigma_2) \qquad (19)$$

IV. Location of Libration Points

Equation (17) admits the Jacobi integral

$$\dot{x}^2 + \dot{y}^2 - 2\boldsymbol{\Omega} + \mathbf{C} = \mathbf{0}$$

The libration points are the singularities of the manifold

$$F(x, y, \dot{x}, \dot{y}) = \dot{x}^2 + \dot{y}^2 - 2\boldsymbol{\Omega} + \mathbf{C} = \mathbf{0}$$

These points are the solutions of the equations.

$\Omega_x = 0$, $\Omega_y = 0$

That is

$$\Omega_x = n^2\psi x - \frac{(1-\mu)}{r_1^3}(x-\mu) - \frac{\mu q}{r_2^3}(x+1-\mu) - \frac{3(1-\mu)}{2r_1^5}(x-\mu)(2\sigma_1-\sigma_2) + \frac{15(1-\mu)}{2r_1^7}(x-\mu)(\sigma_1-\sigma_2)y^2 = 0 \qquad (20)$$

$$\Omega_y = \left[n^2\psi - \frac{1-\mu}{r_1^3} - \frac{\mu q}{r_2^3} - \frac{3(1-\mu)(4\sigma_1-3\sigma_2)}{2r_1^5} + \frac{15(1-\mu)(\sigma_1-\sigma_2)y^2}{2r_1^7}\right]y = 0 \qquad (21)$$

Location of Triangular Points

the triangular points are the solutions of Eqns. (20) and (21) for $y \neq 0$ i.e.from (21)

$$n^2\psi - \frac{(1-\mu)}{r_1^3} - \frac{\mu q}{r_2^3} - \frac{3}{2r_1^5}(1-\mu)(2\sigma_1-\sigma_2) + \frac{15}{2r_1^7}(1-\mu)(\sigma_1-\sigma_2) = 0 \qquad (22)$$

If the bigger primary is not triaxial ($\sigma_1 = \sigma_2 = 0$) and the smaller one is not radiating ($q = 1$) and n=1, Eqns. (21) and (22) becomes

$$\left[\psi - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3}\right]x + \mu(1-\mu)\left[\frac{1}{r_1^3} - \frac{1}{r_2^3}\right] = 0$$

and

$$\left[\psi - \frac{1-\mu}{r_1^3} - \frac{\mu}{r_2^3}\right] = 0$$

Solving these gives $r_1 = r_2 = \frac{1}{\psi^{1/3}}$

Now, due the triaxiality of bigger primary ($\sigma_1 \neq \sigma_2 \neq 0$) and the radiation effect of the smaller one ($q \neq 1$) we can assume that the solutions to Eqns (21) and (22) to be

$r_1 = \frac{1}{\psi^{1/3}} + \alpha \quad , r_2 = \frac{1}{\psi^{1/3}} + \beta$ where σ, β are negligible (i.e. /α/, /β/, << 1) (23)

Putting the values of r_1 and r_2 from Eqn. (23) in (15), we get

$$x = \mu - \tfrac{1}{2} + \frac{1}{\psi^{1/3}}(\beta - \alpha) \qquad \text{and} \qquad y = \pm \frac{\sqrt{4-\psi^{2/3}}}{2\psi^{1/3}}\left[1 + \frac{2\psi^{1/3}}{4-\psi^{2/3}}(\alpha+\beta)\right] \tag{24}$$

Substituting Eqns. (23) and (24) in (20) and (22) neglecting second and higher order terms (since /α/, /β/, /σ_1/, /σ_2/ and /1-q/ are very small), we have

$$\alpha = -\frac{1}{\psi^{1/3}}\left[\left(1 - \psi^{2/3} - \frac{5}{8}\psi^{4/3}\right)\sigma_1 - \left(\frac{1}{2} + \frac{3}{2}\psi^{2/3} - \frac{5}{8}\psi^{4/3}\right)\sigma_2\right]$$

$$\beta = -\frac{1}{\psi^{1/3}}\left[\frac{1}{3}(1-q) + \left(1 - \frac{\psi^{2/3}}{2\mu} + \frac{\psi^{2/3}}{2}\right)\sigma_1 - \left(\frac{1}{2} - \frac{\psi^{2/3}}{2\mu} + \frac{\psi^{2/3}}{2}\right)\sigma_2\right]$$

Then we obtain the co-ordinates of the triangular libration points L_4 (x_1, + y) and L_5 (x_1, - y) as

$$x = \mu - \frac{1}{2} + \frac{1}{\psi^{1/3}}\left[-\frac{1}{3}(1-q) + \left(\frac{\psi^{2/3}}{2\mu} + \frac{\psi^{2/3}}{2} - \frac{5\psi^{4/3}}{8}\right)\sigma_1 - \left(\frac{\psi^{2/3}}{2\mu} - \psi^{2/3} - \frac{5\psi^{4/3}}{8}\right)\sigma_2\right] \tag{25}$$

$$y = \pm\frac{\sqrt{4-\psi^{2/3}}}{2\psi^{1/3}}\left[1 + \frac{2}{4-\psi^{2/3}}\left\{-\frac{1}{3}(1-q) - \left(2 - \frac{\psi^{2/3}}{2\mu} + \frac{3\psi^{2/3}}{2} - \frac{5}{8}\psi^{4/3}\right)\sigma_1 + \left(1 - \frac{\psi^{2/3}}{2\mu} + 2\psi^{2/3} - \frac{5}{8}\psi^{4/3}\right)\sigma_2\right\}\right.$$

V. Stability of the triangular libration points

Suppose we give a little displacement (a, b) to the position of the triangular points L(x,y) such that $x = a + \xi$ and $y = b + \eta$. applying this in the equations of motion(17), we obtain the variational equation of motion (first order in terms of ξ, η) as

$$\ddot{\xi} - 2n\phi\dot{\eta} = \Omega^0{}_{xx}\xi + \Omega^o{}_{xy}\eta$$

$$\ddot{\eta} + 2n\phi\dot{\xi} = \Omega^0{}_{xy}\xi + \Omega^0{}_{yy}\eta \tag{26}$$

The superscript (^{0}) indicates that the derivatives are evaluated at the libration points.
And its corresponding characteristic equation is

$$\lambda^4 - \left(\Omega^0_{xx} + \Omega^0_{yy} - 4n^2\phi^2\right)\lambda^2 + \Omega^0_{xx}\Omega^0_{yy} - \left(\Omega^0_{xy}\right)^2 = 0 \tag{27}$$

where

$$\Omega^0_{xx} = \frac{3}{4} + \frac{5}{4}\varepsilon' + (1-q) - \frac{3}{2}\mu(1-q) + \left(\frac{57}{16} - \frac{3}{2\mu} + \frac{45}{16}\mu\right)\sigma_1 + \left(-\frac{3}{16} + \frac{3}{2\mu} - \frac{93}{16}\mu\right)\sigma_2$$

$$\Omega^0_{xy} = \sqrt{3}\left[-\frac{3}{4} + \frac{3}{2}\mu + \frac{11}{6}\left(\mu - \frac{1}{2}\right)\varepsilon' - \frac{1}{3}\left(1 - \frac{\mu}{2}\right)(1-q) + \left(-\frac{47}{16} + \frac{1}{2\mu} + \frac{89}{16}\mu\right)\sigma_1 + \left(\frac{9}{16} - \frac{1}{2\mu} - \frac{37}{16}\mu\right)\sigma_2\right]$$

$$\Omega^0_{yy} = \frac{9}{4} + \frac{7}{4}\varepsilon' - (1-q) + \frac{3}{2}\mu(1-q) + \left(\frac{87}{16} + \frac{3}{2\mu} - \frac{45}{16}\mu\right)\sigma_1 + \left(\frac{-21}{16} - \frac{3}{2\mu} + \frac{45}{16}\mu\right)\sigma_2$$

We replace λ^2 in the characteristic equation (27) by Λ and thus it becomes:

$$\Lambda^2 - P\Lambda + Q = 0 \tag{28}$$

Where,

$P = 1 + 8\varepsilon - 3\varepsilon' + 3\sigma_1 + \left(\frac{-9}{2} + 3\mu\right)\sigma_2$, P>0 since 0<μ< ½ , |ε'|, |ε|, $|\sigma_1|$ and $|\sigma_2|$ << 1

$$Q = \left[\frac{27\mu}{4}(1-\mu) - \frac{33\mu}{4}(1-\mu)\varepsilon' + \frac{3\mu}{2}(1-\mu)(1-q) + \left(-\frac{45}{8} + \frac{891}{16}\mu - \frac{801}{16}\mu^2\right)\sigma_1 + \left(\frac{45}{8} - \frac{423}{16}\mu + \frac{333}{16}\mu^2\right)\sigma_2\right] \tag{29}$$

The roots of Eqn (28) are:

$$\Lambda_{1,2} = \frac{1}{2}\left[-P \pm \sqrt{P^2 - 4Q}\right]$$

Which consequently are

$$\lambda_1 = +\Lambda_1^{1/2}, \lambda_2 = -\Lambda_1^{1/2}, \lambda_3 = +\Lambda_2^{1/2}, \lambda_4 = -\Lambda_2^{1/2}$$

These roots depend on the value of the mass parameters μ, the perturbations in the coriolis and the centrifugal forces, the radiation pressure of the smaller primary and the triaxial nature of the bigger primary and are controlled by the discriminant Δ

From Eqn. (28)

$\Delta = P^2 - 4Q$

$$=1-6\varepsilon' +16\varepsilon - 27\left(1+\frac{22}{9}\varepsilon'\right)\mu(1-\mu) - 6\mu(1-\mu)(1-q)$$
$$-\frac{9}{4}\left(-\frac{38}{3}+99\mu-89\mu^2\right)\sigma_1 - \frac{9}{4}\left(14-\frac{147}{3}\mu+37\mu^2\right)\sigma_2 \qquad (30)$$

If $\mu = 0$, $\Delta > 0$ and $\mu = ½$, $\Delta < 0$ ($|\varepsilon'|$, $|\varepsilon|$ $|\sigma_1|$, and $|\sigma_2| << 1$). Since the discriminant Δ are of opposite sign at different values of μ, there is only one value of μ in the open interval (0, ½) for which Δ vanishes. The value of μ is called the critical value of the mass parameter μ denoted by μ_c

VI. Critical mass

If the bigger primary is not triaxial ($\sigma_1 = \sigma_2 = 0$) the smaller primary is not radiating (q = 1) and there are no perturbation effects in the coriolis and centrifugal forces ($\varepsilon' = \varepsilon = 0$) and the discriminant equal zero, then the root of (28) is

$$\mu = \mu_0 = \frac{1}{2}\left[1-\sqrt{\frac{23}{27}}\right] = 0.038520\text{---} \qquad \text{(Szbehely1967b)} \qquad (31)$$

Now if the bigger primary is triaxial ($\sigma_1, \sigma_2 \neq 0$), the smaller one a source of radiation ($q \neq 1$) and there are perturbations in the coriolis and centrifugal forces ($\varepsilon', \varepsilon \neq 0$), then from setting Eqn. (30) to be equal to zero

$$\mu_c == \mu_0 + \mu_p + \mu_r + \mu_t \qquad (32)$$

where

$$\mu_0 = \frac{1}{2}\left(1-\sqrt{\frac{23}{27}}\right)$$
$$\mu_p = \frac{4(36\varepsilon - 19\varepsilon')}{27\sqrt{69}}$$
$$\mu_r = -\frac{2}{27\sqrt{69}}(1-q)$$
$$\mu_t = \frac{1}{2}\left(\frac{5}{6}+\frac{59}{9\sqrt{69}}\right)\sigma_1 - \frac{1}{2}\left(\frac{19}{18}+\frac{85}{9\sqrt{69}}\right)\sigma_2$$

Equation (32) shows the effect of perturbation in the coriolis and the centrifugal forces, μ_p, the triaxial effect, μ_t from the bigger primary and the radiating effect, μ_r, from the smaller one, on the critical mass value.

We considered the mass parameter μ in three region :

i) When $0 < \mu \leq \mu_c$, we have that

$0 < \Delta \leq P^2 \qquad \Rightarrow -½P < \Lambda_1 \leq 0$

and $\quad -P^2 \leq -\Delta < 0 \quad \Rightarrow -P \leq \Lambda_2 < -\frac{1}{2}P$

written as

$$\lambda_{1,2} = \pm\sqrt{\Lambda_1} = \pm\sqrt{-(-\Lambda_1)} = \pm i(-\Lambda_1)^{1/2} = \pm i s_1 \qquad (32)$$
$$\lambda_{3,4} = \pm\sqrt{\Lambda_2} = \pm\sqrt{-(-\Lambda_2)} = \pm i(-\Lambda_2)^{1/2} = \pm i s_2$$

Where from Eqn.(29) and for very small value of μ, ($\mu \approx 0$),

$$s_1 = \left[\frac{27}{4}\mu - \frac{45}{8}\sigma_1 + \frac{45}{8}\sigma_2\right]^{1/2}, \qquad s_2 = \left[1-\frac{27}{8}\mu + 4\varepsilon - \frac{3}{2}\varepsilon' + \frac{69}{16}\sigma_1 - \frac{81}{16}\sigma_2\right]$$

The roots are purely imaginary, hence the triangular point are stable in this region.

ii) When $\mu_c < \mu \leq ½$, we have that

$\Delta < 0$ and $\Lambda_{1,2} = \frac{1}{2}\left[-P \pm \sqrt{\Delta}\right]$

This implies

$\Lambda_{1,2} = \frac{1}{2}\left[-P \pm i\delta\right]$ where $\delta = +\sqrt{\Delta} > 0$,

$$= \begin{bmatrix} 27\left(1+\frac{22}{9}\varepsilon'\right)\mu(1-\mu) - 1 + 6\varepsilon' - 16\varepsilon + 6\mu(1-\mu)(1-q) \\ + \frac{9}{4}\left(\frac{-38}{3} + 99\mu - 89\mu^2\right)\sigma_1 + \frac{9}{4}\left(14 - \frac{149}{3}\mu + 37\mu^2\right)\sigma_2 \end{bmatrix}^{1/2} \tag{33}$$

Here, the roots of the characteristic equations are:

$\lambda_{1,2} = \pm\Lambda_1^{1/2}$ and $\lambda_{3,4} = \pm\Lambda_2^{1/2}$ or

$$\lambda_1 = \left[\frac{1}{2}(-P + i\delta)\right]^{1/2} = \frac{1}{\sqrt{2}}\sqrt{-P + i\delta} = \alpha_1 + i\beta_1$$

$$\lambda_2 = -\left[\frac{1}{2}(-P + i\delta)\right]^{1/2} = -\frac{1}{\sqrt{2}}\sqrt{-P + i\delta} = \alpha_2 + i\beta_2$$

$$\lambda_3 = \left[\frac{1}{2}(-P - i\delta)\right]^{1/2} = \frac{1}{\sqrt{2}}\sqrt{-P - i\delta} = \alpha_3 + i\beta_3$$

$$\lambda_4 = -\left[\frac{1}{2}(-P - i\delta)\right]^{1/2} = -\frac{1}{\sqrt{2}}\sqrt{-P - i\delta} = \alpha_4 + i\beta_4$$

and the lengths of the roots are which are equal is given by:

$$|\lambda| = |\lambda_{1,2,3,4}| = \frac{1}{\sqrt{2}}\sqrt[4]{P^2 + \delta^2}$$

Using Eqns. (29) and (33), we have

$$|\lambda| = \frac{[27\mu(1-\mu)]^{1/4}}{\sqrt{2}}\left[1 + \frac{11}{18}\varepsilon' + \frac{1}{18}(1-q) + \frac{1}{48}\left(89 - \frac{10}{\mu}\right)\sigma_1 - \frac{1}{48}\left(37 - \frac{10}{\mu}\right)\sigma_2\right]$$

Now from the root

$$\lambda = \alpha + i\beta = \frac{1}{\sqrt{2}}\sqrt{-P + i\delta}$$

Comparing and equating the real and imaginary part, we have

$$\alpha = \frac{\delta}{2\sqrt{P + 2|\lambda|^2}} > 0 \quad \beta = \frac{1}{2}\sqrt{P + 2|\lambda|^2} > 0 \tag{34}$$

Therefore, the principal argument of the first root is

$$\arg(\lambda) = \theta = \theta_1 = \arctan\left(\beta/\alpha\right) = \arctan\left(\frac{P \pm \sqrt{P^2 + \delta^2}}{\delta}\right)$$

where P and δ are given by equations (29) and (33)

The argument of the four roots are related by

$\theta = \theta_1 = \theta_2 - \pi = 2\pi - \theta_3 = \pi - \theta_4$

The real and imaginary part of the roots α_1 and β_1 are related by

$\alpha = \alpha_1 = -\alpha_2 = \alpha_3 = -\alpha_4$ and $\beta = \beta_1 = -\beta_2 = -\beta_3 = \beta_4$

where α and β are given in Eqn.(34)

It follows that in this region the triangular points are unstable since the real parts of two of the values of the characteristic roots are positive.

iii) when $\mu = \mu_c$, we have that $\Delta = 0$.

Consequently

$\lambda_{1,} = \Lambda_1^{1/2} = +i\sqrt{\frac{1}{2}P}$ and $\lambda_{2,4} = -\Lambda_1^{1/2} = -i\sqrt{\frac{1}{2}P}$, where P>0 as given .

The double roots give secular terms in the solution of the variational equation of motion. Therefore, the triangular point are unstable.

VII. Discussion

Equations (17) and (18) describe the motion of the infinitesimal mass when the bigger primary is a triaxial rigid body and the smaller one is a source of radiation, under the influence of perturbations in the coriolis and centrifugal forces. Equation (17) depends on the parameter, ϕ, of the perturbation in the coriolis force, while equation (18), which is the force function, depends on the parameter ψ of the centrifugal force. Both equations depend on the triaxiality coefficients σ_1, σ_2 and the radiation factor, q. Equation (19) shows that the mean motion is affected by only the triaxiality of the bigger primary.

If the bigger primary is not triaxial ($\sigma_1 = \sigma_2 = 0$), the smaller primary not radiating (q= 1) and there are no perturbations in the coriolis and centrifugal forces, then the equations of motion reduce to that of classical problem (Szebehely, 1967b). In the absence of triaxiality and radiation, the equation of motion coincides with those obtained by Bhatnagar and Hallan (1979). When there are no perturbations in the coriolis and centrifugal forces, the equations of correspond to those of Sharma et al. (2001). If the bigger primary is not triaxial but an oblate spheroid ($\sigma_1 = \sigma_2 \neq 0$) the equations of motion agree with those of AbdulRaheem and Singh (2006).

Equations (32) show the effects of perturbations in the coriolis and the centrifugal forces on the critical value of the mass parameter. This value determines the stability of libration points in the restricted problem. In the absence of perturbation potentials, triaxiality of the bigger primary and radiation force of the smaller primary, the critical value of the mass parameter reduces to

$$\mu_0 = \frac{1}{2}\left(1 - \sqrt{\frac{23}{27}}\right) = 0.038520... \qquad \text{(Szebehely, 1967b)}$$

which is mass ratio for the classical restricted problem. But in the absence of only perturbation potential ($\mu_p = 0$) the critical mass value μ_c agrees with that of Sharma et al. (2001). If the bigger primary is not triaxial but oblate ($\sigma_1 = \sigma_2 \neq 0$) and ($\mu_p = 0$) then it verifies the result of Singh and Ishwar (1999). In this case $\mu_c < \mu_0$ and this implies that the range of stability decreases.

When the bigger primary is not triaxial ($\mu_t = 0$) and the smaller one is not radiating($\mu_r = 0$) the critical mass value become

$$\mu_c = \frac{4(36\varepsilon - 19\varepsilon')}{27\sqrt{69}} \qquad \text{(Bhatnagar and Hallan 1979).}$$

In this case $\mu_c > \mu_o$ and it implies that the rage of stability increases. Hence, we conclude that the perturbation potentials, the triaxiality of the bigger primary and radiation pressure force of the smaller primary have destabilizing tendency on the problem. If μ_c=0, $36\varepsilon - 19\varepsilon' = 0$ we have a straight linewhich divides the plane $(\varepsilon,\varepsilon')$ into two part; π_1 on the right for points $36\varepsilon - 19\varepsilon' > 0$ while π_2 on the left for points t $36\varepsilon - .19\varepsilon' < 0$. In the part π_1, $\mu_c < \mu_0$ (μ_t, $\mu_r < 0$), that is the range of stability decreases and for a point $(\varepsilon_1,\varepsilon')$ in the part π_2, $\mu_c < \mu_0$ this also implies that the range of stability decreases. For a point $(\varepsilon,\varepsilon')$ lying on the line $36\varepsilon - 19\varepsilon' = 0$, $\mu_c < \mu_0$, again the range of stability decreases. Therefore, the affect of perturbations in the coriolis and centrifugal forces, the triaxiality of the bigger primary and radiation force the smaller primary is that they decrease the range of stability.

As a particular case, when the point $(\varepsilon,\varepsilon')$ coincides with the origin i.e. $\varepsilon = 0$, $\varepsilon' = 0$.

$\mu_c = \mu_0 + \mu_r + \mu_t$ (Sharma et al. 2001)

.

For the point lying on the ε - axis, $\varepsilon' = 0$

$$\mu_c = \mu_0 + \mu_r + \mu_t + \frac{16\varepsilon}{3\sqrt{69}}$$

From this, we see that for $\varepsilon > 0$, $\mu_c > \mu_0$. For $\varepsilon < 0$, $\mu_c < \mu_0$ and establishes that the coriolis force is a stabilizing force provided the centrifugal force is kept constant. This verifies the result of Szebehely (1967a) though the stability ability is weakened due to the triaxiality of the bigger primary and the radiation pressure force of the smaller primary.

For point lying on the ε'- axis, $\varepsilon = 0$

$$\mu_c = \mu_0 + \mu_r + \mu_t - \frac{76\varepsilon'}{27\sqrt{69}}$$

This shows that for $\varepsilon' > 0$, $\mu_c < \mu_0$ and for $\varepsilon' < 0$ $\mu_c > \mu_0$, implying that the centrifugal force is a destabilizing force when the coriolis force is kept constant. That is, irrespective of when μ_r, μ_t are equal to zero. Here we see that the triaxiality of the bigger primary and the radiation pressure force of the smaller one increase the destabilizing tendency of the centrifugal force. This result agrees with those of Subbarao and Sharma (1975) and AbdulRaheem and Singh (2006) when $\sigma_1 = \sigma_2 \neq 0$.

VIII. Conclusion

Hence under the effect of perturbation in the coriolis and the centrifugal forces in the restricted three body problem when the bigger primary is truncated and the smaller one radiating the triangular points are stable for $0 \le \mu \le \mu_c$ and unstable for $\mu_c \le \mu \le ½$. This may be applied to examine the asteroids librate around the Lagrangian points in the Sun-Planets systems or satellite in the Earth-Moon system.

References

[1]. **AbdulRaheem, A. and Singh, J. (2006)**. Combined effects of Perturbations Radiation and Oblations on the Stability of Equilibrium Point in the Restricted Three-Body problem. Ast. J, 131: 1880-1885.

[2]. Bhatnagar, K.B and Hallan, P.P. (1979). Effect of Perturbation Potential on the Stability of Equilibrium Points in the Restricted Problem Celestial Mechanics and Dynamics Astronomy 20,95-103

[3]. **Chernikov, V.A. (1970).** The photo-gravitational Restricted Three-Body Problem, Astr. 2, 47:217-223 (Russian) Translated as Soviet Astr. A J. 14(1): 176-181

[4]. **Choudhry, R.K and Kumar,V. (1986)**. Existence of Libration Points in the Generalised Photo-gravitational Restricted Problem of Three Bodies. Bulletin of Astronomical Society, India. 14:79-90.

[5]. **Duboshin, G.N. (1968)** Celestial Mechanics Nauka, Moscow.

[6]. **Finlay, F. (1958).**Celential Mechaics Pergamon Press. California

[7]. **Fitzpatric , P.M and Crenshaw, J.W.(1966).** Application of the Methods of Celestial Mechanics to the Rigid Body Problem. Auburn Research foundation. In C. Aurbun University, Alabama

[8]. **Gupta, P.P. (1986)**. Dynamics of Rigid Bodies (Analytical Dynamic) Kishna Prakashan Mande, India :2

[9]. **Gupta, S.L., Kumar V, and Sharma, H.V. (1990)** Classical Dynamics Pragh Prakashan Meerut.

[10]. **Ishwar, B. and Elipe, A. (2001).** Secular Solution at the Triangular Equilibrium Points in the Generalize Photo-gravitational Restricted Three- Body Problem. Astrophysic and Space Science, 277: 437-446

[11]. **Khanna, M. and Bhatnagar, K.B. (1999).** Existence and Stability of Libration Points in the Restricted Three-Body Problem when the Smaller Primary is a Triaxial Rigid Body and the Bigger One is an Oblated Spheroid. Indian Journal of Pure and Applied Mathematics. 30 (7): 721-733.

[12]. **Khanna, M.L. (1976).** Dynamics of Rigid Bodies Jai Prakash Math and Co., Meerut

[13]. **Khanna, M.L. (1999),** Theory of Equations Jai Prakash Math and Co., Meerut. (xii-xviii)

[14]. **Khasan, S.N. (1996).** Labration Solution to Photo-gravitational Restricted Three- Body Problem. Cosmic Research 34(2), 146-157

[15]. **Kunitsyn, A.L. (2000).** Stability of Triangular Libration Points in the Photogravitational Three- Body Problem Journal of Applied Mathematics and Mechanics 65(5): 757-763

[16]. **Kunitsyn, A.L. (2001).** Stability of Collinear Libration Points in the Photogravitation Three–Body Problem Journal of Applied Mathematics and Mechanics 65(4): 703-706.

[17]. **Kushavah, B.S. (2008).** Liner Stability of Equilibration Points in the Generalized Photogravitational Chermnykh's Problem Astrophysics and Space Science DOI10.107

[18]. **Kushavah, B.S. (2008).** The Effect of Radiation Pressure on the Equilibrium Points in the Generalized Photogravitational Restricted Three Body Problem. Astrophysics and Space Science DOI10.107.

[19]. **Mc Cusky, S.W. (1963). "**Introduction to Celestial Mechanics", Addison Wesley.

[20]. **Moulton, F.R. (1914). "**An Introduction to Celestial Mechanics" Macmillan, New York.

[21]. **Pollard,H (1966). "**Mathematical Introduction to Celestial Mechanics". Prentice- Hall, Inc. New Jersey, 48-58

[22]. **Roy, A.E. (1978). "**Orbital Motion", Adam Hilger Lit. Bristol.

[23]. **Sharma, R.K., Taqvi, Z.A. and Bhatnagar K.B. (2001).** Existence and Stability of Libration Points in the Restricted Three-Body Problem when the Primaries are Triaxial Rigid Bodies. Celestial Mechanics and Dynamical Astronomy Netherlands, 79:119-133.

[24]. **Sharma, R.K, Taqvi, Z.A and Bhatnagar, K.B. (2001).** Existence and Stability of Libration Points in the Restricted Three-Body Problem when the Primaries are Triaxial Rigid Bodies and Source of Radiations Indian Journal of Pure and Applied Mathematics India, 32(7): 981-994.

[25]. **Subbarao, P.V. and Sharma, R.K. (1975).** A Note on the Stability of the Equilibrium Points in the Restricted Three Body Problem Astronomical Astrophysics, 43:381-383

[26]. **Szebehely, V.V. (1977a).** Stability of the Point of Equilibrium in the Restricted Problem. Astronomical Journal, 72(1): 7-9

[27]. **Szebehely, V.V. (1967b). "**Theory of Orbits the Restricted Problem of Three Bodies" Academic Press, New York.

[28]. **Vidyakin, V.V. (1974).** A Plane Circular Limited Task Pertaining to Three Spheroids. Astronomical Journal, 51(5): 1087-1094.

[29]. **Wintner, A. (1941). "**The Analytical Foundations of Celestial Mechanics" University Press, Princeton: 372