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# Numerical Solutions of Stiff Initial Value Problems Using Modified Extended Backward Differentiation Formula

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***Abstract:*** *This paper seeks numerical solutions to a stiff initial value problem using a modified extended backward differentiation formula. Some implicit schemes were developed based on the linear multi-step method. Stiff initial value problems are solved using the various stages of the derived modified extended backward differentiation formula, the result obtained at different stages are compared with Euler implicit method and Simpson's implicit method. The results obtained are considered in determining which of the methods is more accurate and efficient in providing solutions to stiff initial value problems when compared with the exact solution.*
***Keywords:*** *Stiff initial value problems, Modified extended backward differentiation formula, Euler implicit method, Simpson's implicit method and Linear multistep method.*

## I. Introduction

A problem is stiff if the numerical solution has its step size limited more severely by the stability of the numerical technique than by the accuracy of the technique. Frequently, these problems occur in systems of differential equations that involve several components that are decaying at widely differing rates. E.g. damped stiff highly oscillatory problem [5].

Although numerous attempts have been made to give a rigorous definition of this concept, it is probably fair to say that none of these definitions is entirely satisfactory. Indeed the authoritative book of Hairer and Wanner [12] deliberately avoids trying to define stiffness and relies instead on an entirely pragmatic definition given in [10]. What is clear, however, is that numerical methods for solving stiff initial value problems have to satisfy much more stringent stability requirements than is the case for methods intended for nonstiff problems. One of the first and still one of the most important, stability requirements particularly for linear multistep methods is that of A-stability which was proposed in [11]. However, the requirement of A-stability puts a severe limitation on the choice of suitable linear multistep methods.

## II. General Formation of Linear Multistep Methods

A linear k-step of differential equation

$$y' = f(x, y) \qquad y(x_0) = y_0 \tag{1}$$

Is usually given as

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} \tag{2}$$

Where $\alpha_j$ and $\beta_j$ are constant and not both $\alpha_0$ and $\beta_0$ are zero (not zero at same time) equation $(2)$ is said to be explicit if $\beta_k = 0$ and it is implicit if $\beta_k \neq 0$.

**Definition**

The linear difference operator is defined as [13]

$$L[y(x); h] = \sum_{j=0}^{k} \left[\alpha_j y(x + jh) - h\beta_j y'(x + jh)\right] \tag{3}$$

Expanding $y(x + jh)$ and its derivation $y'(x + jh)$ about $x$, we have

$$y(x + jh) = y(x) + jhy'(x) + \frac{j^2 h^2 y''}{2!}(x) + \frac{j^3 h^3 y'''}{3!}(x) + \dots \tag{4}$$

$$y'(x + jh) = y'(x) + jhy''(x) + \frac{j^2 h^2 y'''}{2!}(x) + \frac{j^3 h^3 y^{iv}}{3!}(x) + \dots \tag{5}$$

Substituting equation $(4)$ and $(5)$ in $(3)$ we have

$$L[y(x);h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + C_3 h^3 y'''(x) + \ldots + C_q h^q y^q(x) + \ldots \qquad (6)$$

Where $C_0 = \alpha_0 + \alpha_1 + \alpha_2 + \ldots + \alpha_k$

$$C_1 = (\alpha_1 + 2\alpha_2 + \ldots + 4k\alpha_k) - (\beta_0 + \beta_1 + \beta_2 + \ldots + \beta_k)$$

$$C_2 = \frac{1}{2!}(\alpha_1 + 2^2\alpha_2 + \ldots + k^2\alpha_k) - (\beta_1 + 2\beta_2 + \ldots + k\beta_k) \qquad (7)$$

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$$C_p = \frac{1}{q!}(\alpha_1 + 2^q\alpha_2 + \ldots + k^q\alpha_k) - \frac{1}{(q-1)!}(\beta_1 + 2^{q-1}\beta_2 + \ldots + k^{q-1}\beta_k),\ q = 2,3,4\ldots$$

To derive any k-step method, we set $C_1 = C_2 = C_3 = \ldots = C_p = 0$ and solve the system then check if it $C_{p+1} \neq 0$

## III. Modified Extended Backward Differentiation Formula

Modified extended backward differentiation formula (MEBDF) was originally proposed as a class of formula to which an efficient variable order, variable step boundary value approach could easily be applied. The general k-step MEBDF is defined by [4]

$$y_{n+k} + \sum_{j=0}^{k-1} \hat{\alpha}_j y_{n+j} = h\left[\hat{\beta}_{k+1} f_{n+k+1} + \hat{\beta}_k f_{n+k}\right] \qquad (8)$$

Where the coefficients are chosen so that this formula has order $k+1$. The accuracy and stability of this method is critically dependent on the predictor used to compute $y_{n+k+1}$ and in particular this predictor must be of order of least $k$ if the whole process is to be of order $k+1$. A natural $k^{th}$ -order predictor is the $k$ -step BDF and this lead to the so-called EBDF stages.

**Stage 1:** using the standard BDF to compute $y_{n+k}$

$$y_{n+k} + \sum_{j=0}^{k-1} \alpha_j y_{n+j} = h\beta_k f(x_{n+k}, y_{n+k}) \qquad (9)$$

**Stage 2:** using a standard BDF to compute $y_{n+k+1}$

$$y_{n+k+1} + \alpha_{k-1} y_{n+k} + \sum_{j=0}^{k-2} \alpha_j y_{n+j+1} = h\beta_k f(x_{n+k+1}, y_{n+k+1}) \qquad (10)$$

**Stage 3A:** derived by computing a corrected solution of order $k+1$ at $x_{n+k}$ using

$$y_{n+k} + \sum_{j=0}^{k-1} \hat{\alpha}_j y_{n+j} = h\left(\hat{\beta}_{k+1} f_{n+k+1} + \hat{\beta}_k f_{n+k}\right) \qquad (11)$$

Note that at each of these three stages a nonlinear set of equations must be solved in order that the desired approximations can be computed. The Extended Backward Differentiation formula described above can be modified to give the so-called MEBDF approach by changing stage 3A to

Stage 3B:

$$y_{n+k} + \sum_{j=0}^{k-1} \hat{\alpha}_j y_{n+j} = h\left[\hat{\beta}_{k+1} f_{n+k+1} + \beta_k f_{n+k} + \left(\hat{\beta}_k - \beta_k\right) f_{n+k}\right] \qquad (12)$$

Fortunately, this modification not only improves the computational efficiency of this approach, it also improves its stability.

**Construction of Two Steps MEBDF Using the First Stage**

$$y_{n+k} + \sum_{j=0}^{k-1} \alpha_j y_{n+j} = h\beta_k f(x_{n+k}, y_{n+k})$$

Where $\alpha_j$ and $\beta_k$ are constants. Using the implicit case, when k=2

$$y_{n+2}+\sum_{j=0}^{1}\alpha_j y_{n+j}=h\beta_2 f\left(x_{n+2},y_{n+2}\right)$$

$$y_{n+2}+\alpha_0 y_n+\alpha_1 y_{n+1}=h\beta_2 f\left(x_{n+2},y_{n+2}\right)$$

Using the definition in equation $(7)$ if $C_1=C_2=C_3=...=C_p=0$ and $C_{p+1}\neq 0$. We have

$$y_{n+2}+\frac{1}{3}y_n-\frac{4}{3}y_{n+1}=\frac{2}{3}hf\left(x_{n+2},y_{n+2}\right) \qquad (13)$$

**Construction of Two Steps MEBDF Using the Second Stage Standard BDF to Compute $y_{n+k+1}$, when k=2**

$$y_{n+k+1}+\alpha_{k-1}y_{n+k}+\sum_{j=0}^{k-2}\alpha_j y_{n+j+1}=h\beta_k f\left(x_{n+k+!},y_{n+k+1}\right)$$

$$y_{n+3}+\alpha_1 y_{n+2}+\sum_{j=0}^{0}\alpha_j y_{n+j+1}=h\beta_2 f\left(x_{n+3},y_{n+3}\right)$$

$$y_{n+3}+\alpha_1 y_{n+2}+\alpha_0 y_{n+1}=h\beta_2 f\left(x_{n+3},y_{n+3}\right)$$

Using the definition in equation $(7)$ if $C_1=C_2=C_3=...=C_p=0$ and $C_{p+1}\neq 0$. We have

$$y_{n+3}-\frac{4}{3}y_{n+2}+\frac{1}{3}y_{n+1}=\frac{2}{3}hf\left(x_{n+3},y_{n+3}\right) \qquad (14)$$

**Construction of Two Step MEBDF Using the Third Stage, when k=2**

$$y_{n+k}+\sum_{j=0}^{k-1}\hat{\alpha}_j y_{n+j}=h\left(\hat{\beta}_{k+1}f_{n+k+1}+\hat{\beta}_k f_{n+k}\right)$$

$$y_{n+2}+\sum_{j=0}^{1}\hat{\alpha}_j y_{n+j}=h\left(\hat{\beta}_3 f_{n+3}+\hat{\beta}_2 f_{n+2}\right)$$

$$y_{n+2}+\hat{\alpha}_0 y_n+\hat{\alpha}_1 y_{n+1}=h\left(\hat{\beta}_3 f_{n+3}+\hat{\beta}_2 f_{n+2}\right)$$

Using the definition in equation $(7)$ if $C_1=C_2=C_3=........=C_p=0$ and $C_{p+1}\neq 0$. Will have

$$y_{n+2}+\frac{5}{23}y_n-\frac{28}{23}y_{n+1}=-\frac{2}{23}h\left(2f_{n+3}-\frac{11}{23}f_{n+2}\right) \qquad (15)$$

## IV. Implementation of the Schemes

Consider the initial value problem [14]

$y'=x+y$ Subjected to initial condition $y(0)=1$ $\qquad (16)$

The analytical solution to IVP in equation $(15)$ when subjected to the initial condition gives $y=2e^{-x}-x-1$. Numerical solutions are preferred to the derived stage I, stage II and stage III of the two step implicit Modified Extended Backward Differentiation Formula. We obtaining numerical solutions for the IVP in $(16)$ for $x=0(0.1)1$.

**Table1:** Solutions of MEBDF Stages with $h=0.1$

| x | Exact | Stage I | Stage II | Stage III |
|---|---|---|---|---|
| 0 | 1.00000 | 1.00000 | 1.00000 | 1.00000 |
| 0.1 | 1.11034 | 1.11000 | 1.11000 | 1.11000 |
| 0.2 | 1.24281 | 1.24207 | 1.23100 | 1.11091 |
| 0.3 | 1.39972 | 1.38294 | 1.38294 | 1.23068 |
| 0.4 | 1.58365 | 1.55123 | 1.55123 | 1.37460 |
| 0.5 | 1.79744 | 1.74636 | 1.74636 | 1.54291 |
| 0.6 | 2.04424 | 1.97099 | 1.97099 | 1.73805 |
| 0.7 | 2.32751 | 2.22809 | 2.22809 | 1.96271 |
| 0.8 | 2.65108 | 2.52090 | 2.52090 | 2.21983 |

| | | | | |
|---|---|---|---|---|
| 0.9 | 3.01921 | 2.85299 | 2.85299 | 2.51267 |
| 1 | 3.43656 | 3.22829 | 3.22829 | 2.92663 |

**Table 2:** Absolute Errors of the MEBDF Stages with $h = 0.1$

| x | **Stage I** | **Stage II** | **Stage III** |
|---|---|---|---|
| 0 | 0.00000 | 0.00000 | 0.00000 |
| 0.1 | 0.00034 | 0.00034 | 0.00034 |
| 0.2 | 0.00074 | 0.01181 | 0.13189 |
| 0.3 | 0.01678 | 0.01678 | 0.16904 |
| 0.4 | 0.03242 | 0.03242 | 0.20905 |
| 0.5 | 0.05109 | 0.05109 | 0.25453 |
| 0.6 | 0.07324 | 0.07324 | 0.30618 |
| 0.7 | 0.09941 | 0.09941 | 0.36480 |
| 0.8 | 0.13018 | 0.13018 | 0.43125 |
| 0.9 | 0.16621 | 0.16621 | 0.50654 |
| 1 | 0.20827 | 0.20827 | 0.50993 |

## V. Conclusion

Investigation carried out on the numerical solutions of stiff initial value problems in ODEs using Modified Extended Backward Differentiation Formula has shown that the stage I is the best scheme for solving stiff initial value problem. It converge better than the stage II and stage III with the lowest absolute error value. Hence is the most accurate.

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