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A Generalization of QN-Maps

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Abstract: *The notion of GQN-Maps is introduced and some results regarding these maps are obtained.*
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I. Introduction

A self mapping T of a subset C of a normed linear space X is said to be nonexpansive if $||Tx - Ty|| \leq ||x - y||$ for allx , y in C [3]. It is quasi- nonexpansiveif T has at least one fixed point p of T in C and $||Tx - p|| \leq ||x - p||$ for allx in C and for each fixed point p of T in C [5,6].Many results have been proved for nonexpansive and quasi-nonexpansive mappings. One may referBrowder and Petryshyn [1], Bruck [4], Chidume [5], Das and Debata [6], Dotson [7],Petryshyn and Williamson [8], Rhoades [9], Singh and Nelson [11], Senter and Dotson [10]and many more.

The purpose of the present paper is to introduce the notion of generalized quasi-nonexpansive mappings (GQN-maps).

Throughout the paper, unless stated otherwise, X denotes a Banach space, $\Re$, the field of real numbers, $\overline{A}$, the closure of A and F(T), the fixed point set of a mapping T. A subset C of X is locally compact if each point of C has a compact neighbourhood in C [12]. The mapping r from a set C onto A, A being a subset of C, is a retraction mapping if $ra = a$for alla in A[2].

II. Definition

2.1:A selfmapping T of a subset C of X is said to be generalised quasi-nonexpansive mapping (GQN-map) provided T has at least one fixed point and corresponding to each fixed point T, there exists a constant M depending on the fixed point p (referred as M(p)) in $\Re$ such that for each x belonging to C,

$$||Tx - p|| \leq M(p)||x - p||$$

Clearly, every quasi-nonexpansive map is a GQN map. However, the converse may not be true. Example 1.2 establishes the same. It is well known that for a linear map, the fixed point setF(T) is convex and for a continuous map, the fixed point set is closed. But there are non-linear discontinuous GQN-maps whose fixed point sets are closed and convex.

Example 2.2:

(i) Define $T:[0,\frac{\pi}{2}] \to [0,\frac{\pi}{2}]$ by

$$Tx = x + (x - \tfrac{\pi}{4})(\cos x + 1)$$

Then $F(T) = \{\frac{\pi}{4}\}$

(ii) Define $T: [0,1] \to [0,1]$ by

$$Tx = (n+1)x - 1, \quad \frac{1}{n+1} < x \leq \frac{1}{n}, \quad n = 1,2,\ldots.$$

$$T(0) = 1$$

Then $F(T) = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\ldots..\}$

(iii) Define $T: \Re^+ \to \Re^+$ by

$$Tx = \frac{1-x}{n}, \quad \frac{1}{n+1} < x \leq \frac{1}{n}, \quad n = 0,1,2,3,\ldots\ldots$$

Then $F(T) = \{\frac{1}{n+1} : n = 0,1,2,3,\ldots..\}$

(iv) Consider the Banach space $\Re^n = \{(x_1, x_2, x_3 \ldots. x_n): x_i \in \Re \text{ for all } i = 1,2,3,\ldots\ldots n\}$.
Set $C = \{(x_1, x_2, x_3 \ldots. x_n): x_n = 0 \text{ for all } n > 2, x_2 \neq 0, x_2 \neq 1\}$.

Define $T : C \to C$ by

$T(x_1, x_2, 0, 0 \ldots .0) = (2x_2 - (x_2 - 1)x_1,\ x_2, 0, 0, \ldots .0)$

Then $F(T) = \{(2, x_2, 0, 0, \ldots, 0): x_2 \in \Re \sim \{0,1\}\}$.

The above examples show that F(T) may or may not be closed and convex for a GQN-map. Note that except in example (i), the GQN-maps are discontinuous also. The exact set of conditions under which the fixed point set of a GQN-map is closed and convex, areyet to obtained, but the conditions for F(T) to be a GQN-retract are obtained in the next section.

III. Main results

IV.

In this section, C always denotes a closed, bounded and convex subset of the space X.

Definition3.1:A subset A of C is said to be a GQN-retract of C if there exists a retraction mapping r from C onto A which is a GQN-map.

To find the set of conditions for any nonempty subset of a locally weakly compact set to be aGQN-retract, we prove the following two lemmas:

Lemma3.2: Suppose A is a nonempty subset of a locally weakly compact set C and let $G(A) = \{f: C \to C \text{ is a GQN-map and } f(x) = x \text{ for each } x \text{ in } A\}$. Then G(A) is compact in the topology of weak-pointwise convergence.

Proof: Fix $x_0 \in A$. For each $f \in G(A)$, there exists a real number $M_f(x_0)$ such that

$||f(x) - f(x_0)|| \le M_f(x_0)||x - x_0||$ for all $x \in C$.

Let $M(x_0) = \sup_{f \in G(A)} M_f(x_0)$.

Case (i): Let $M(x_0)$be finite. For each $x \in C$, define $A_x = \{y \in C: ||y - x_0|| \le M(x_0)||x - x_0||\}$. Then A_x contains f(x) for each x in C and f in G(A) which gives that G(A) is a subset of the Cartesian product $P = \prod_{x \in C} A_x$. Now A_x is convex and weakly compact. So if $A_x$is given the weak topology and P is given the product topology, by Tychonoff's theorem for the product of compact sets, P is compact.

Case (ii): Let $M(x_0)$be infinite. Then P = C and hence P is compact.

Now to show that G(A) is closed in P, let f be a limit point of G(A) in Pand $\langle f_\lambda \rangle$, a net in G(A) such that $f_\lambda \to f$. Then, using lower semi-continuity of the norm function and the fact that f_λ is in G(A), we get that G(A) is a closed subset of the compact set P and hence is compact as desired.

Lemma3.3: Suppose A is nonempty subset of C and C is locally weakly compact. Then there exists an r in G(A) such that for each $f \in G(A)$ we have $||rx - ry|| \le ||f(x) - f(y)||$ for all x, yin C.

Proof: Define an order < on G(A) by setting $f < g$ if $||f(x) - f(y)|| \trianglelefteq |g(x) - g(y)||$ for each x, y in C with inequality holding for at least one pair of x and y. Also $f \le g$ means either $f < g$or $f = g$. Clearly $\le$ is a partial order on G(A).For each f in G(A), we define the initial segmentIs(f) = $\{g \in G(A): g \le f\}$. Then, as shown in lemma 2.2, Is(f) is closed and compact in G(A). Now consider a chain ξin G(A). Then T = {Is(f): $f \in \xi$} is a chain of compact sets under set- inclusion as a partial order relation.By the finite intersection property for compact sets, T is bounded below, say, by Is(h). Then$f \le h\ \forall f \in G(A)$.

Now we prove the desired result in the following form:

Theorem3.4: Suppose C is locally weakly compact and A is a nonempty subset of C. Suppose further that for each z in C, there exists an $h \in G(A)$ such that $h(z) \in A$. Then A is a GQN-retract of C.

Proof: By lemma 2.3, there exists an $r \in G(A)$ such that for each x, y in C and $f \in G(A)$

$$||r(x) - r(y)|| \trianglelefteq |f(x) - f(y)|| \quad (2.1)$$

Also, it can be easily verified that for each $f \in G(A)$, the composite map $f \circ r \in G(A)$.

Since $r \in G(A)$, it is sufficient to show that for each $x \in C$, $r(x) \in A$. For this, let $x \in C$ and put $z = r(x)$. Then as $z \in C$, the hypothesis assures the existence of an $h \in G(A)$ such that $h(r(x)) \in A$. Now, let $h(r(x)) = y$ then as $h \circ r \in G(A)$, the inequality 2.1 implies

$$||r(x) - r(y)|| \leq || h \circ r(x) - h \circ r(y)|| \qquad (2.2)$$

Since $y = h(r(x)) \in A$ and $r \in G(A)$, therefore, $r(y) = y$ which further implies $h(r(y)) = h(y) = y = h(r(x))$. So we get , in view of 2.2, that $r(x) \in A$.

Since for a GQN-map T, the fixed point set F(T) is always nonempty, so we have the following :

Corollary 3.5: Let C be a locally weakly compact set and T: C → C is a GQN-map. Suppose that for each $z \in C$ there exists an h in G(F(T)) such that $h(z) \in F(T)$. Then F(T)is a GQN-retract of C.

Theorem 3.6: Under the conditions of Theorem 2.4, the class of GQN-retracts is closed under arbitrary intersection.

Proof: By theorem 2.4, the collection { Is(f): f∈ξ}, where ξ is a chain in G(A), has a minimal element f in G(A) which is a GQN-retract of C. Let $\Lambda = \{A_f \subseteq C: f \in G(A)$ and A_f is the corresponding GQN-retract of C}. Clearly $\Lambda \neq \varphi$as $A \in \Lambda$. Order Λ by $A_f \subseteq A_g$iff $\leq$ g ∀f and g in G(A). By Zorn's lemma, Λ has a minimal element, say, A_g. It can be seen that g is minimal in G(A).

Put $F = \bigcap_{f \in G(A)} A_f$. As$A \subseteq F(f)$ for every f, therefore, F is nonempty. Also minimality of g in G(A) implies that $A_g$is contained in each GQN-retract of C and hence in F. Then F= A_g. Thus F is a GQN-retract of C.

We now establish that the set of common fixed points of an increasing sequence of GQN-maps is a GQN-retract of C .

Theorem 3.7: Let C be a locally weakly compact subset of X. If $<r_n>$ is a sequence of GQN-maps in G(A) such that the corresponding GQN-retracts $F(r_n)$ form an increasing sequence with$\bigcap_n F(r_n) \neq \varphi$ then there exists a GQN-map r from C to C such that F(r) = $\bigcap_n F(r_n)$.

Proof: Consider $\mathfrak{I} = \{F(r_n): r_n$ is a GQN-retraction of C onto $F(r_n)\}$.Order $\mathfrak{I}$as $A \leq B$ if $A \subseteq B$. By Zorn's lemma, there exists a minimal element, say, $F \in \mathfrak{I}$.ThenF= $\bigcap_n F(r_n)$. Thus $\bigcap_n F(r_n)$ is a GQN-retract of C.

By hypothesis, $\bigcap_n F(r_n) \neq \varphi$. So let $\in \bigcap_n F(r_n)$. Then $p \in F(r_n)$ for each n. Choose a sequence $<\lambda_n>$ of positive numbers such that $\sum_n \lambda_n = 1$ and let $r = \sum_n \lambda_n r_n$.For each $p \in \bigcap_n F(r_n)$ and x∈ C,

$$||r(x) - r(p)|| \leq || (\textstyle\sum_n \lambda_n r_n)(x) - (\sum_n \lambda_n r_n)(p)||$$
$$\leq \textstyle\sum_n \lambda_n ||r_n x - r_n p||$$
$$\leq M(p)||x - p||$$

as$\sum_n \lambda_n = 1$ and $M(p) = \max_n\{M_{r_n}(p): M_{r_n}(p)$ is a constant corresponding to the GQN-map $r_n\}$. Thus r is a GQN-map. Further, using $\sum_n \lambda_n = 1$, it can be shown that F(r) = $\bigcap_n F(r_n)$ which proves the result.

Definition3.8: [3]:A mapping T: C →X is said to satisfy the conditional fixed point property (CFPP) if either T has no fixed point or T has a fixed point in each nonempty bounded closed set it leaves invariant.

Definition 3.9: A nonempty subset C is said to have the hereditary fixed point property (HFPP) for GQN maps if every nonempty bounded closed convex subset of C has a fixed point for GQN- mappings.

Following Bruck [3], we prove the following:

Theorem 3.10: If C is locally weakly compact and T: C →C is a GQN-map which satisfies CFPP then F(T) is a GQN retract of C.

Proof: By definition of T, F(T) is nonempty. For a fixed z in C, define K = $\{f(z) : f \in G(F(T))\}$. In view of the compactness of G(F(T)), following [3], K is weakly compact and hence bounded. Also, K ≠ϕ. For f and g in G(F(T) and $0 \leq \lambda \leq 1$, consider $\lambda f + (1-\lambda)g$. If $y_0 \in F(T)$ then $F(y_0) = y_0 = g(y_0)$ so that for all x, y in C,

$$||(\lambda f + (1-\lambda)g)(x) - y_0|| \leq |(\lambda M_f(y_0) + (1-\lambda)M_g(y_0)|||x - x_0||$$

where $M_f(y_0)$ and$M_g(y_0)$ are real numbers corresponding to the fixed point y_0 and for mappings f and g respectively. Let us put$M_{(\lambda M_f + (1-\lambda)M_g)}(y_0) = \lambda M_f(y_0) + (1-\lambda)M_g(y_0)$ then $\lambda f + (1-\lambda)g$is a GQN-map. Also every fixed point x of T is a fixed point of $\lambda f + (1-\lambda)g$ and hence K is convex. Also for f∈G(F(T)), $T \circ f \in G(F(T))$ i.e. $T(K) \subseteq K$. Therefore, by hypothesis T has a fixed point in K i.e. ∃f∈ G(F(T)) such that $f(z) \in F(T)$ for each z∈ C. Thus, by theorem 2.4, F(T) is a GQN-retract of C.

Corollary3.11: Suppose T: C →C is a GQN-map satisfying CFPP and the convex closure $\overline{\text{conv}(T(C))}$ of the range of T is locally weakly compact then F(T) is a GQN-retract of C.

The following result can be proved following the arguments of Bruck [3].

Theorem3.12: Let C be locally weakly compact and $\{F_\alpha : \in \Lambda\}$ be a family of weakly closed GQN retracts of C. Then

(a) If this family is directed by$\supset$, then $\bigcap_\alpha F_\alpha$ is a generalised quasi-nonexpansive retract of C.

(b) If each F_α is convex and the family is directed by $\subset$ then $(\overline{\bigcup_\alpha F_\alpha})$, the closure of$(\bigcup_\alpha F_\alpha)$, is a generalised quasi-nonexpansive retract of C.

Lemma3.13: Let C be weakly compact and satisfies HFPP for GQN-maps. Let F be nonempty GQN- retract of C and T: C → C is a GQN-map which leaves F invariant. Then $F(T) \cap F$ is a nonempty GQN-retract of C.

Theorem3.14: Suppose C is weakly compact and has HFPP for GQN-maps. If $\{T_j : 1 \le j \le n\}$ is a finite family of commuting GQN-maps$T_j: C \to C$ then $\bigcap_{j=1}^{n} F(T_j)$ is a nonempty GQN-retract of C.

Theorem 3.15: Let $\{T_\alpha : \alpha \in \Lambda\}$ is a family of GQN-maps of C, where, Λ is some index set. If exactly one map, say T_α, of the family is linear and continuous and commutes with each of the remaining then $F(T_\alpha) \cap (\bigcap_{\beta \neq \alpha} \overline{\text{conv.} F(T_\beta)})$ is nonempty.

Proof: Without loss of generality, we may assume that T_1 is linear and continuous such that $T_1 T_\alpha = T_\alpha T_1$ for all$\alpha \in \Lambda$. Clearly $\overline{\text{conv}\,(F(T_1))} = F(T_1)$. Also for each $\alpha \in \Lambda$, $\overline{\text{conv}\,(F(T_\alpha))}$ is a nonempty compact convex subset of C. Linearity and continuity of T_1 implies $T_1(\overline{\text{conv}\,(F(T_\alpha))} \subset \overline{\text{conv}\,(F(T_\alpha))}$. So, by Tychonoff's theorems for fixed points, T_1 has fixed points in $\overline{\text{conv}\,(F(T_\alpha))}$ and hence the result.

Remark3.16: In the proof of the above result, the condition of the self mapping being GQN-map is required to assume that $F(T_\alpha)$'s are nonempty. So if the hypothesis of the theorem contains the fact that $F(T_\alpha) \neq \phi$ for all$\alpha \in \Lambda$, the result remains true for an ordinary family of mappings with exactly one map of the family being linear and continuous.

The result of theorem 2.15 can be extended to a countable intersection of convex closures of $F(T_i)$'s but the least conditions required are yet to be traced though the result is trivially true for the family of linear and continuous maps.

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