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On Local Integrable Solutions of Abstract Volterra Integral Equations

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Abstract: *The aim of this paper is to study the existence and uniqueness of solution of abstract Volterra integral equations of the second kind in complete locally convex topological space. An example illustrating the obtained result is given.*
Keywords: *Abstract Volterra integral equations, locally integrable solution*

I. Introduction

Integral equations have a lot of applications in physics, engineering, mechanics, biology, and economics (cf., e.g.[1–7,11] and references therein). By applying the theory of fixed point, some interesting results on the existence of locally integrable solutions of abstract Volterra integral equations are obtained under certain conditions, see, e.g. [2,3,7] and the references therein.

In this work we consider the abstract Volterra integral equations of first and second kind in the case when the independent variable belongs to a topological space. We find sufficient conditions for existence and uniqueness of the trivial solution of the homogeneous equation and sufficient conditions for existence and uniqueness of local integrable solution of the non homogeneous equations. Moreover, we give an example, which shows that the investigated equation has a unique locally integrable solution.

II. Preliminaries

Let Ω be a topological space, $B_\Omega \subset 2^\Omega$ denotes the σ - algebra of the Borel subsets of Ω and let $\mu : B_\Omega \to [0,\infty)$ be a nontrivial σ-finite Borel measure.

We introduce the map $M : \Omega \to 2^\Omega$ which associates every point $x \in \Omega$ a closed subset $M_x \subset \Omega$ and let denote the set $\mathcal{M} = \{M_x : x \in \Omega\}$.

We will say that the condition (A) hold if for the map $M : \Omega \to 2^\Omega$ the following conditions are fulfilled:

A1. For every $M_x \in \mathcal{M}$ the following is fulfilled

$$\omega = \sup_{x\in\Omega} \mu(M_x) < +\infty.$$

A2. For any $x \in \Omega$ and $y \in M_x$ the inclusion $M_y \subseteq M_x$ holds.

A3. For every $x \in \Omega$ and for every nonempty open subset O_x of M_x there exists a neighbourhood $W = W(x, O_x)$, such that for every $y \in W$ the following holds

$$M_y \cap O_x \neq \varnothing.$$

A4. There exists a point $x_0 \in \Omega$ such that $\mu\left(M_{x_0}\right) = 0$.

Let B be a real Banach space with norm $\|\,.\,\|_B$.

Consider the linear space $L^1_{loc} = L^1_{loc}(\Omega, B)$, where

$$L^1_{loc}(\Omega, B) = \{ f : \Omega \to B,\ f \textit{ is strongly measurable and } \int_{M_x} \| f(y)\|_B \, d\mu_y < \infty \textit{ for every } x \in \Omega\}$$

We introduce topology $\mathscr{F}^1_{loc}$ using a family of semi-norms

$$\| f \|_{M_x} = \int_{M_x} \| f(y)\|_B \, d\mu_y$$

for $x \in \Omega$ and $f \in L^1_{loc}$.

Then by theorem 3.5.4[10] and theorem 1(Ch. VII, §1)[12] it follows that L^1_{loc} is a complete locally convex topological space.
Let A be an ordered index set and $\{f_\alpha\}_{\alpha\in A}\subset L^1_{loc}$, $f_0\in L^1_{loc}$.

Proposition 1.[9] A sequence $\{f_\alpha\}_{\alpha\in A}\subset L^1_{loc}$ is convergent to $f_0\in L^1_{loc}$ in the topology $\mathscr{F}^1_{loc}$ if and only if $\lim\limits_{\alpha\to\infty}\|f_\alpha - f_0\|_{M_x}=0$ for $x\in\Omega$.

Definition 2.1. We say that the function $f\in L^1_{loc}$ is $\mathcal{M}-$almost separable - valued if for every $x\in\Omega$ and $M_x\in\mathcal{M}$ there exists a set $E_x\subseteq M_x$, $\mu(E_x)=0$, such that $f(M_x\setminus E_x)$ is separable subset of B.
We will say that the operator $Q:\Omega\times\Omega\times L^1_{loc}\to L^1_{loc}$ is continuous and satisfies the conditions (B) if the following conditions are fulfilled:
B1. For every $x,y\in\Omega$ and $f_1, f_2\in L^1_{loc}$ we have

$$Q(x,y,f_1(y)+f_2(y))=Q(x,y,f_1(y))+Q(x,y,f_2(y))$$

B1*. For every $x,y\in\Omega$ and $f_1, f_2\in L^1_{loc}$ there exists a constant $L_x>0$, such that

$$\int\limits_{M_x}\|Q(x,y,f_1(y))-Q(x,y,f_2(y))\|_B\,d\mu_y\le L_x\int\limits_{M_x}\|f_1(y)-f_2(y)\|_B\,d\mu_y$$

B2. For every $x\in\Omega$ there exists a constant $A_x>0$, such that for every $f\in L^1_{loc}$ and $y\in\Omega$ the inequality $\int\limits_{M_x}\|Q(x,y,f(y))\|_B\,d\mu_y\le A_x\int\limits_{M_x}\|f(y)\|_B\,d\mu_y$ holds.
B3. For every $\varepsilon>0$, $x\in\Omega$ and $f\in L^1_{loc}$ there exists a neighbourhood $W=W(x,f,\varepsilon)$ of the point x, such that for every $y\in W$ the inequality

$$\int\limits_{M_x\cap M_y}\|Q(x,z,f(z))-Q(y,z,f(z))\|_B\,d\mu_z\le\varepsilon\int\limits_{M_x\cap M_y}\|f(z)\|_B\,d\mu_z$$

holds.
B4. For every $\varepsilon>0$, $x\in\Omega$ and $f\in L^1_{loc}$ there exists a neighbourhood $V=V(x,f,\varepsilon)$ of the point x, such that for every $y\in V$ the inequality

$$\int\limits_{M_x\Delta M_y}\|f(z)\|_B\,d\mu_z<\varepsilon$$

holds, where $M_x\Delta M_y=\{M_x\setminus M_y\}\bigcup\{M_y\setminus M_x\}$.
B5. For every fixed element $x\in\Omega$ and every fixed function $f\in L^1_{loc}$ the mapping $\varphi_f(y)=Q(x,y,f(y))$ is $\mathcal{M}-$almost separable – valued and weakly measurable with respect to M_x for $x\in\Omega$.
Remark 1: If for the operator Q condition (B1*) hold and $Q(x,y,0)\equiv 0$ for every $x,y\in\Omega$, then condition (B2) is a consequence of condition (B1*).

III. Main Results

We consider the equations

$$f(x)=\lambda(Kf)(x) \qquad (3.1)$$

and

$$f(x)=\varphi(x)+\lambda(Kf)(x) \qquad (3.2)$$

where $f,\varphi\in L^1_{loc}$, $\lambda\in\square$ and operator $K:L^1_{loc}\to L^1_{loc}$ is defined by the equality

$$(Kf)(x) = \int_{M_x} Q(x, y, f(y))\, d\mu_y \,, \qquad (3.3)$$

where $M_x \in \mathcal{M}$.

Since for every linear and continuous functional φ^* defined on B the map $\langle \varphi^*, \varphi_f \rangle : \Omega \to \mathbb{R}$ is measurable on every $M_x \in \mathcal{M}$, then the existence of the integral in (3.3) is guaranteed by the conditions (B) and the theorems 3.4.7, p.94 and 3.5.3, p.86 in [10]

Theorem 3.1. Let the conditions (A1), (B2), (B3), (B4) and (B5) hold. Then for any $f \in L^1_{loc}$ the function $Kf(x)$ is continuous in Ω.

Proof: Let $x, x_1 \in \Omega$ and $f \in L^1_{loc}$. The constants A_x and A_{x_1} are defined in condition (B2).

Let $\tilde{A}_1 = \max\{\sup_{x \in M_x} A_x, \sup_{x \in M_{x_1}} A_{x_1}\}$.

Let $\varepsilon > 0$ be an arbitrary number. Condition (B3) implies that there exists a neighbourhood $W\left(x_1, f, \frac{\varepsilon}{\|f\|_{M_{x_1}}}\right)$ of the point x_1, such that for every $x \in W\left(x_1, f, \frac{\varepsilon}{\|f\|_{M_{x_1}}}\right)$ the following inequality

$$\int_{M_x \cap M_{x_1}} \|Q(x, z, f(z)) - Q(x_1, z, f(z))\|_B \, d\mu_z \le \frac{\varepsilon}{2\|f\|_{M_{x_1}}} \qquad (3.4)$$

holds.

Condition (B4) implies that there exists a neighbourhood $V(x_1, f, \varepsilon)$ of the point x_1, such that for every $x \in V(x_1, f, \varepsilon)$ the following estimate

$$\int_{M_x \Delta M_{x_1}} \|f(z)\|_B \, d\mu_z \le \frac{\varepsilon}{2\tilde{A}_1} \qquad (3.5)$$

holds.

Let $U(x_1, f, \varepsilon) = W(x_1, f, \varepsilon) \cap V(x_1, f, \varepsilon)$ and $x \in U(x_1, f, \varepsilon)$. Then the inequalities (3.4), (3.5) and condition (B2) yield that

$$\| Kf(x) - Kf(x_1) \|_B \le \left\| \int_{M_x} Q(x, y, f(y)) - \int_{M_{x_1}} Q(x_1, y, f(y)) \right\|_B \le$$

$$\le \int_{M_x \cap M_{x_1}} \| Q(x, y, f(y)) - Q(x_1, y, f(y)) \|_B \, d\mu_y +$$

$$+ \int_{M_x \setminus M_{x_1}} \| Q(x, y, f(y)) \|_B \, d\mu_y + \int_{M_{x_1} \setminus M_x} \| Q(x_1, y, f(y)) \|_B \, d\mu_y \le$$

$$\le \frac{\varepsilon}{2\|f\|_{M_{x_1}}} \int_{M_x \cap M_{x_1}} \| f(y) \|_B \, d\mu_y + A_x \int_{M_x \setminus M_{x_1}} \| f(y) \|_B \, d\mu_y + A_{x_1} \int_{M_{x_1} \setminus M_x} \| f(y) \|_B \, d\mu_y \le$$

$$\le \frac{\varepsilon}{2\|f\|_{M_{x_1}}} \| f \|_{M_x \cap M_{x_1}} + \tilde{A}_1 \int_{M_x \Delta M_{x_1}} \| f(y) \|_B \, d\mu_y \le \varepsilon$$

holds.

Theorem 3.2. Let the following conditions are fulfilled:
1. The conditions (A1), (B1), (B2) and (B5) hold.
2. There exists a constant $\tilde{A}_2 = \sup\limits_{x \in M_z} A_x$ for each $z \in \Omega$.

Then the operator K maps L^1_{loc} into L^1_{loc} continuously.

Proof: Let $f \in L^1_{loc}$. First we will prove that $Kf \in L^1_{loc}$.
Let $z \in \Omega$. Condition (A1) and (B2) imply that

$$\| Kf \|_{M_z} = \int\limits_{M_z} \|Kf(x)\|_B d\mu_x = \int\limits_{M_z} \| \int\limits_{M_x} Q(x, y, f(y)) d\mu_y \|_B \, d\mu_x \le$$

$$\le \int\limits_{M_z} \int\limits_{M_x} \|Q(x, y, f(y))\|_B \, d\mu_y \, d\mu_x \le \int\limits_{M_z} A_x \| f \|_{M_x} \, d\mu_x \le$$

$$\le \tilde{A}_2 \int\limits_{M_z} \| f \|_{M_x} \, d\mu_x < \infty$$

holds.
Therefore we get that $Kf \in L^1_{loc}$.

Now, we will prove that K is continuous.
Let A be an ordered index set, $\lim\limits_{\alpha \to \infty} f_\alpha = f_0$, $\{f_\alpha\}_{\alpha \in A} \subset L^1_{loc}$, $f_0 \in L^1_{loc}$ and let $z \in \Omega$. Then we have

$$\| Kf_\alpha - Kf_0 \|_{M_z} = \int\limits_{M_z} \|Kf_\alpha(x) - Kf_0(x)\|_B d\mu_x =$$

$$= \int\limits_{M_z} \| \int\limits_{M_x} Q(x, y, f_\alpha(y)) - Q(x, y, f_0(y)) d\mu_y \|_B \, d\mu_x \le$$

$$\le \int\limits_{M_z} \int\limits_{M_x} \|Q(x, y, f_\alpha(y) - f_0(y))\|_B \, d\mu_y \, d\mu_x \le \int\limits_{M_z} A_x \int\limits_{M_x} \| f_\alpha(y) - f_0(y)\|_B d\mu_y \, d\mu_x =$$

$$\le \int\limits_{M_z} A_x \, \| f_\alpha - f_0 \|_{M_x} \, d\mu_x$$

holds. Therefore we get that $\| Kf_\alpha - Kf_0 \|_{M_z} \underset{n \to \infty}{\longrightarrow} 0$. □

Theorem 3.3. Let the following conditions are fulfilled:
1. The conditions (A) and (B) hold.
2. The condition 2 of Theorem 3.2 holds.
3. The space Ω is connected.

Then the equation (3.1) has only the trivial solution $f \equiv 0$ for $\lambda \in \mathbb{R}$.

Proof: Let f^* be an arbitrary solution of (3.1), $\lambda \in \mathbb{R}$ be an arbitrary real number and let consider the set $N_{f^*} = \{x \in \Omega : f^*(y) = 0 \text{ } for \text{ } y \in M_x\}$.

Condition (A4) implies that there exist an element $x_0 \in \Omega$, such that $\mu(M_{x_0}) = 0$.

Let $\mu(M_{x_0}) \ne \varnothing$ (the case $\mu(M_{x_0}) = \varnothing$ is trivial). Condition (A2) yields that for every $x \in M_{x_0}$ we have $\mu(M_{x_0}) = 0$ i.e. $x_0 \in N_{f^*}$ and therefore $N_{f^*} \ne \varnothing$.

Now we will prove that N_{f^*} is a closed set.

Let $\{x_n\} \subset N_{f^*}$ be an arbitrary convergent sequence and let x^* be its limit in Ω.

If $M_{x^*} = \varnothing$ then we have $x^* \in N_{f^*}$. Let $M_{x^*} \ne \varnothing$, $z \in M_{x^*}$ be an arbitrary fixed point and $\varepsilon > 0$ be an arbitrary number.

From Theorem 3.1 it follows that f^* is continuous function in Ω and therefore there exists a neighbourhood $W = W\left(z, f^*, \varepsilon\right)$ of the point z, such that for every $y \in W$ we have

$$\left\| f^*(y) - f^*(z) \right\|_B < \varepsilon .$$

The condition (A3) implies that for the set $O_x = W \bigcap M_x$ there exists a neighbourhood $U\left(x^*\right)$ of the point x^*, such that for every $y \in U(x^*)$ we have $M_y \bigcap O_{x^*} \neq \varnothing$

There exists a number n_0, such that for $n \geq n_0$, $x_n \in U\left(x^*\right)$ and $M_{x_n} \bigcap O_{x^*} \neq \varnothing$.

Let $y^* \in M_{x_{n'}} \bigcap O_{x^*}$ for any $n' \geq n_0$. Taking into account that $f^*(y^*) = 0$, then we have $\left\| f^*(z) \right\|_B < \varepsilon$ i.e. $f^*(z) = 0$. Therefore $x^* \in N_{f^*}$ and we conclude that N_{f^*} is closed set.

Now, we will prove that N_{f^*} is open set. Let $a \in N_{f^*}$ be an arbitrary.

Condition (B4) yields that there exists a neighbourhood $V = V\left(a, \varepsilon\right)$, such that for every $x' \in V$ and every $f \in L^1_{loc}$ the following estimate

$$\int_{M_x \Delta M_a} \| f(y) \|_B \, d\mu_y < \varepsilon$$

holds.

Let $b \in V$. Denote by $L^1_{loc}(M_b)$ the Banach space of functions , which are strongly measurable and integrable in M_b with norm

$$\|g\|_{M_b} = \int_{M_b} \|g(y)\|_B \, d\mu_y$$

We consider the set $T = \{g \in L^1_{loc}(M_b) : g(x) = 0 \ \ for \ x \in M_a \bigcap M_b\}$.

Obviously, T is a closed subset of $L^1_{loc}(M_b)$ and the restriction of f^* over M_b is contained in it. For every $x \in M_a \bigcap M_b$ and $g \in T$ conditions (A2) implies that

$$(\lambda K g)(x) = \lambda \int_{M_x} Q(x, y, g(y)) \, d\mu_y = 0$$

i.e. K maps T into T.

Let $\tilde{A}_3 = \sup_{x \in M_b} A_x$. Let $g_1, g_2 \in T$ be any elements and let $\varepsilon_0 \in (0, \varepsilon)$, such that $\dfrac{\varepsilon_0 |\lambda| \omega \tilde{A}_3}{\|g_1 - g_2\|_{M_b}} < \dfrac{1}{2}$. Then we have

$$\begin{aligned}
&\| \lambda K g_1 - \lambda K g_2 \|_{M_b} = |\lambda| \int_{M_b} \|K g_1(x) - K g_2(x)\|_B \, d\mu_x = \\
&= |\lambda| \int_{M_b} \| \int_{M_x} Q(x, y, g_1(y)) - Q(x, y, g_2(y)) d\mu_y \|_B \, d\mu_x = \\
&\leq |\lambda| \int_{M_b} \| \int_{M_x \setminus M_a} Q(x, y, g_1(y) - g_2(y)) d\mu_y \|_B \, d\mu_x + \\
&+ |\lambda| \int_{M_b} \| \int_{M_x \cap M_a} Q(x, y, g_1(y) - g_2(y)) d\mu_y \|_B \, d\mu_x \leq \\
&\leq |\lambda| \int_{M_b} \int_{M_x \setminus M_a} \| Q(x, y, g_1(y) - g_2(y)) \|_B \, d\mu_y \, d\mu_x \leq \\
&\leq |\lambda| \int_{M_b} A_x \int_{M_x \setminus M_a} \| g_1(y) - g_2(y) \|_B \, d\mu_y \, d\mu_x \leq \\
&\leq |\lambda| \int_{M_b} A_x \int_{M_x \Delta M_a} \| g_1(y) - g_2(y) \|_B \, d\mu_y \, d\mu_x \leq \\
&\leq |\lambda| \int_{M_b} A_x \varepsilon_0 \, d\mu_x \leq |\lambda| \tilde{A}_3 \, \omega \, \varepsilon_0 < \frac{1}{2} \|g_1 - g_2\|_{M_b}
\end{aligned}$$

i.e. λK is compressive over T and by Banach's theorem it has only one fixed point $g_0 \in T$. Obviously, $g_0(x)=0$ for $x \in M_b$. Therefore $b \in N_{f^*}$ i.e. N_{f^*} is open set.

Since Ω is connected, then we get that $N_{f^*}=\Omega$ i.e. $f^*(x)=0$ for every $x \in \Omega$.

This completes the proof of the Theorem. □

Theorem 3.4 Let the following conditions are fulfilled:

1. The conditions of Theorem 3.3 hold.
2. The operator K is compact.

Then the equation (3.2) has exactly one solution for every $\varphi \in L^1_{loc}$ and for each $\lambda \in \mathbb{R}$.

Proof. The conditions of Theorem 3.4 immediately follow from Theorem 3.3, separability of L^1_{loc} and preposition 4, page 217 [14].

Suppose that $f_1, f_2 \in L^1_{loc}$ are two different solutions of equation (3.2). Then for each $x \in \Omega$ it follows

$$f_1(x)-f_2(x)=\lambda\left[Kf_1(x)-Kf_2(x)\right].$$

From conditions (B1) and (B2) it follows that for each $x \in \Omega$ the inequality

$$\|f_1(x)-f_2(x)\|_B \le |\lambda| \int_{M_x} \|Q(x,y,f_1(y))-Q(x,y,f_2(y))\|_B \, d\mu_y =$$

$$=|\lambda| \int_{M_x} \|Q(x,y,f_1(y)-f_2(y))\|_B \, d\mu_y \le |\lambda| A_x \int_{M_x} \|f_1(y)-f_2(y)\|_B \, d\mu_y$$

holds.

Using Theorem 2 and Theorem 3 from [5] we get $\|f_1(x)-f_2(x)\|_B=0$ for each $x \in \Omega$, which contradicts to our supposition. □

Remark: In the Theorem 3.2, Theorem 3.3 and Theorem 3.4 instead of condition (B1) we can use condition (B1*).

We give an example, which illustrates the main result in the work i.e. Theorem 3.4.

Example: Let $\Omega=\mathbb{R}^2_+=[0,\infty)\times[0,\infty)$, $B=\mathbb{R}$, μ be Lebesgue measure. We define map $M:\Omega \to 2^\Omega$ which associates every point $x=(x_1,x_2)\in\Omega$ a set $M_x=[0,x_1]\times[0,x_2]$ and the space

$$L^1_{loc}(\Omega,B)=\{f:\Omega\to B,\ f \textit{ is strongly measurable and } \int_{M_x} |f(y)|_B \, d\mu_y < \infty \textit{ for every } x\in\Omega\}.$$

Condition (A) is fulfilled for the map $M:\Omega\to 2^\Omega$.

Let the function $K:\Omega\times\Omega\to\mathbb{R}$ is continuous. The operator $Q:\Omega\times\Omega\times L^1_{loc}\to L^1_{loc}$ is defined by the following equality

$$Q(x,y,f(y))=k(x,y)\frac{f^2(y)}{1+f^2(y)}$$

We will show that for the operator Q the conditions (B) hold.

Let $f_1, f_2 \in L^1_{loc}$. Then the condition (B1*) is verified by

$$\int_{M_x} |Q(x,y,f_1(y))-Q(x,y,f_2(y))| \, d\mu_y =$$

$$=\int_{M_x} \left| k(x,y)\left(\frac{f_1^2(y)}{1+f_1^2(y)}-\frac{f_2^2(y)}{1+f_2^2(y)}\right)\right| d\mu_y \le$$

$$\leq \int_{M_x} |k(x,y)| \left| \frac{(f_1(y)-f_2(y))(f_1(y)+f_2(y))}{(1+f_1^2(y))(1+f_2^2(y))} \right| d\mu_y \leq$$

$$\leq 2\sup_{y\in M_x} |k(x,y)| \int_{M_x} |f_1(y)-f_2(y)|\, d\mu_y$$

Condition (B2) is fulfilled because $Q(x,y,0)=0$ for every $x,y\in\Omega$.

Condition (B3) immediately follows from the continuity of function $k(x,y)$.

Let $\varepsilon>0, x\in\Omega$ and $f\in L^1_{loc}$. From definition of the set $M_x(x\in\Omega)$ it follows that there exists a neighbourhood $V=V(x,f,\varepsilon)$ of the point x, such that for $y\in V$ the following

$$\int_{M_x\Delta M_y} |f(z)|\, d\mu_z < \varepsilon$$

is fulfilled. Consequently, the condition (B5) automatically holds.

Let the operator $K: L^1_{loc}\to L^1_{loc}$ is defined by

$$(Kf)(x)=\int_{M_x} k(x,y)\frac{f^2(y)}{1+f^2(y)}\, d\mu_y.$$

We will show that the operator K is compact.

We introduce topology $\mathscr{F}^1_{loc}$ using a family of seminorms

$$\|f\|_{x_1,x_2}=\int_0^{x_1}\int_0^{x_2} |f(y_1,y_2)|\, dy_2\, dy_1, \text{ for } f\in L^1_{loc} \text{ and } x_1,x_2\geq 0 \qquad (*)$$

Let $T_1\geq 0$, $T_2\geq 0$. We denote the set $D=[0,T_1]\times[0,T_2]$, the Banach space $L^1(D)=\{f:D\to\square : f \textit{ is strongly measurable and } \int_D |f(y)|\, d\mu_y<\infty\}$ with norm $\|f\|_1=\int_D |f(y)|\, d\mu_y$ and the operator of restriction $\pi_D: L^1_{loc}\to L^1(D)$ by $\pi_D(f):=f|_D$, i.e. $\pi_D(f)$ is the restriction of the function $f\in L^1_{loc}$ to domain D.

Proposition 2.[9] A set $C\subset L^1_{loc}$ is relatively compact in the topology $\mathscr{F}^1_{loc}$ if and only if $\pi_D(C)$ is relatively compact in the Banach space $L^1(D)$ for $T_1\geq 0$, $T_2\geq 0$.

Let $C_D=\{g\in L^1(D): g(x)=Kf(x),\, x\in D,\ \|f\|_1\leq 1\}$

From Chapter VII, § 2, [13] follows that it is sufficient to show that the set C is uniformly bounded and equicontinuous.

Let $f\in L^1(D)$, $\|f\|_1\leq 1$. Then

$$|(Kf)(x)|=\left|\int_D k(x,y)\frac{f^2(y)}{1+f^2(y)}\, d\mu_y\right| \leq \max_{(x,y)\in D\times D}|k(x,y)|\int_D |f(y)|\, d\mu_y = \max_{(x,y)\in D\times D}|k(x,y)|\|f\|_1 \leq$$

$$\leq \max_{(x,y)\in D\times D}|k(x,y)| \qquad \textit{for } x\in D.$$

Consequently, the set C is uniformly bounded.

Let $x',x''\in D$. Then

$$|(Kf)(x')-(Kf)(x'')|\leq \int_D |k(x',y)-k(x'',y)|\left|\frac{f^2(y)}{1+f^2(y)}\right| d\mu_y \leq \max_{y\in D}|k(x',y)-k(x'',y)|\int_D |f(y)|\, d\mu_y =$$

$$=\max_{y\in D}|k(x',y)-k(x'',y)|\|f\|_1 \leq \max_{y\in D}|k(x',y)-k(x'',y)|$$

The function $k(x,y)$ is equicontinuous and therefore the set C is equicontinuous.

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