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Digital Roots and Their Properties

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Abstract: *In this paper we want to highlight the properties of digital root and some result based on digital root. There are many interesting properties of digital root. We can also study nature of factors of a given number based on Vedic square and then we can implement this concept for an algorithm to find out a number is prime or not.*
Keywords: *Digital Roots, factors, Integral roots, partition, Vedic square, Primality test.*

I. Introduction

In recreational number theory (A. Gray [1]]) was the first to apply "digital root" but their application had not been previously recognized as relevant. Many other authors have also discussed on the same topics on different way but here we try to discuss the same in a manner which is different from all. Generally, digital root of any natural number is the digit obtained by adding the digits of the number till a single digit is obtained (see [2]) . We can define digital root of a number as:

$dr : N \to D$ where N is the set of natural numbers and $D = \{1,2,3,4,5,6,7,8,9\}$ and $dr(a) =$ sum of digits of the number 'a' till single digit is obtained.

e.g. $dr(12567) = 3$, $dr(20) = 2$, $dr(12) = 3$ etc.

1.1 Preposition

If N be the set of natural numbers and $D = \{1,2,3,4,5,6,7,8,9\}$, then the function $dr : N \to D$ is well defined.

Proof: let $a,b \in N$ and $a = b$, then obviously digits of a and b are same. Therefore, sum of the digits of 'a'=sum of the digits of 'b'

$\Rightarrow$ sum of digits of 'a' till a single digit is obtained =sum of digits of 'b' till a single digit is obtained. $\Rightarrow dr(a) = dr(b)$.

Therefore, $dr : N \to D$ is well defined.

II. Basic terminologies:

2.1. **Equivalent number:** Two numbers 'a' and 'b' are said to be equivalent numbers if and only if both numbers have same digital root.

i.e. $a \sim b \Leftrightarrow dr(a) = dr(b)$

2.2 Inverse of digital root: Inverse of digital root is defined as

$dr^{-1}(a) = a + 9 * j; j \in W, \ a \in D$

Remark: Inverse of digital root is not unique.

$dr^{-1}(dr(a)) = dr(a) + 9 * j, \ \ j \in W$, (where W is the set of non-negative integers)

Example: $dr^{-1}(3) = 3 + 9 * j = \{3,12,21,30,----\}$.

It is not unique but certainly they have a general form and all of these are equivalent number.

III. Important Results

3.1. Digital root partition the set of non-negative integers.

Proof: Let the relation is defined as $aRb \Leftrightarrow a \sim b$

R is reflexive: $dr(a) = dr(b) \Leftrightarrow a \sim b \Rightarrow aRb$

R is symmetric: $aRb \Rightarrow dr(a) = dr(b) \Rightarrow dr(b) = dr(a) \Rightarrow b \sim a \Rightarrow bRa$

R is transitive: aRb and $bRc \Rightarrow dr(a) = dr(b)$ and $dr(b) = dr(c)$

$\Rightarrow dr(a) = dr(b) = dr(c) \Rightarrow dr(a) = dr(c) \Rightarrow a \sim c \Rightarrow aRc$

Therefore it is a equivalence relation and it partition the set of non-negative integers.
Remark: It is not anti-symmetric.

3.2 Representation of non-negative integers in terms of digital roots:
A number having digital root equals to $a \in D$ can be represented as $a+9*j:\ j=0,1,2,3,----$
A number having digital root equals to 2 can be represented as $2+9*j:\ j=0,1,2,3,----$
e.g. $11=2+9*1$, $20=2+9*2$, and $29=2+9*3$
A number having digital root equals to 3 can be represented as $\pm(3+9*j):\ j=0,1,2,3,----$
3.3. Difference of two equivalent number is always a multiple of 9. If $a\sim b$ then $a\equiv b \bmod (9)$.
'if a is equivalent to b then a is congruent to b modulo 9'
Proof: Let $a\sim b$ then we have $dr(a)=dr(b) \Rightarrow$ a and b belong to the equivalence class.
$\Rightarrow a=x+9*j$, where $j\in W$ and $\Rightarrow b=x+9*k$, where $k\in W$ and $x\in\{1,2,3,4,5,6,7,8,9\}$.
$\Rightarrow a-b=(x+9*j)-(x+9*k)=9*(j-k) \Rightarrow a\equiv b \bmod (9)$.
Remark 1: Any number of the form of $\pm(3+9*j)$, $\pm(6+9*j)$, $\pm(9+9*j)$ can never be a prime number.
Remark 2: Twin prime can have any one of the following form: $(2+9j,5+9j),(5+9j,7+9j),(8+9j,1+9j)$

IV. Important Discussion

Table - I

×	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	10	12	14	16	18
3	3	6	9	12	15	18	21	24	27
4	4	8	12	16	20	24	28	32	36
5	5	10	15	20	25	30	35	40	45
6	6	12	18	24	30	36	42	48	54
7	7	14	21	28	35	42	49	56	63
8	8	16	24	32	40	48	56	64	72
9	9	18	27	36	45	54	63	72	81

Simple multiplication table from 1 to 9

Table - II

dr	1	2	3	4	5	6	7	8	9
1	1	2	3	4	5	6	7	8	9
2	2	4	6	8	1	3	5	7	9
3	3	6	9	3	6	9	3	6	9
4	4	8	3	7	2	6	1	5	9
5	5	1	6	2	7	3	8	4	9
6	6	3	9	6	3	9	6	3	9
7	7	5	3	1	8	6	4	2	9
8	8	7	6	5	4	3	2	1	9
9	9	9	9	9	9	9	9	9	9

Corresponding digital root table (formerly known as "Vedic Square")

From this table we can observe that $a*b=c$
$\Rightarrow dr(a*b)=dr(dr(a)*dr(b))=dr(c)$

We observe that $1*1=1$, $5*4=4*5=2$, $7*4=4*7=1$, $8*8=1$
$\Rightarrow dr^{-1}(1)*dr^{-1}(1)=dr^{-1}(1)$
Also, $\Rightarrow dr^{-1}(5)\cdot dr^{-1}(2)=dr^{-1}(2)\cdot dr^{-1}(5)=dr^{-1}(1)$
and $dr^{-1}(7)\cdot dr^{-1}(4)=dr^{-1}(4)\cdot dr^{-1}(7)=dr^{-1}(1)$
and $dr^{-1}(8)\cdot dr^{-1}(8)=dr^{-1}(1)$

We can prove these results very easily. For example: let's prove $dr^{-1}(2)\cdot dr^{-1}(5)=dr^{-1}(1)$
$dr^{-1}(2)=2+9*j\ for\ j\in W$
$dr^{-1}(5)=5+9*k\ for\ k\in W$
$dr^{-1}(1)=1+9*l\ for\ l\in W$

Now

$$dr^{-1}(2)dr^{-1}(5) = (2+9\times j)\times(5+9\times k)$$
$$= 10+9\times\left(j\times(5+9\times k)+k\times(2+9\times j)\right)$$
$$= 1+9\times m,$$

where $m = 9\times\left(j\times(5+9\times k)+k\times(2+9\times j)\right)+1$, and $m\in W$

$$= dr^{-1}(1)$$

Similarly other results may be verified.

This means that, every number having digital root 1 when multiplied by a number having digital root 1 shall result in a number having digital root 1. And similarly, all the number with digital root 5 when multiplied with a number having a digital root is equal to 2 shall result in a number having digital root is equal to 1. And all number having digital root is equal to 7 when multiplied with a number having digital root equal to 4 shall result in a number having digital root equal to 1.

This also can be interpreted as :
All the number having digital root equal to 1 can be only factorized in these following cases:

- (A number having digital root equal to 1)*(A number having digital root equal to 1)
  i.e. $(1+9*j)\times(1+9*k)$ *for some* $j,k\in W$
- (A number having digital root equal to 5)* (A number having digital root equal to 2)
  i.e. $(5+9*j)\times(2+9*k)$ *for some* $j,k\in W$
- (A number having digital root equal to 7)* (A number having digital root equal to 4)
  i.e. $(7+9*j)\times(4+9*k)$ *for some* $j,k\in W$
- (A number having digital root equal to 8)* (A number having digital root equal to 8)
  i.e. $(8+9*j)\times(8+9*k)$ *for some* $j,k\in W$

Similarly from **Table II ,**
All the number having digital root equal to 2 can be only factorized in these following cases:

- $(1+9*j)\times(2+9*k)$ *for some* $j,k\in W$
- $(5+9*j)\times(4+9*k)$ *for some* $j,k\in W$
- $(7+9*j)\times(8+9*k)$ *for some* $j,k\in W$

All the number having digital root equal to 4 can be only factorized in these following cases:

- $(1+9*j)\times(4+9*k)$ *for some* $j,k\in W$
- $(2+9*j)\times(2+9*k)$ *for some* $j,k\in W$
- $(5+9*j)\times(8+9*k)$ *for some* $j,k\in W$
- $(7+9*j)\times(7+9*k)$ *for some* $j,k\in W$

All the number having digital root equal to 5 can be only factorized in any of following case(s):

- $(1+9*j)\times(5+9*k)$ *for some* $j,k\in W$
- $(7+9*j)\times(2+9*k)$ *for some* $j,k\in W$
- $(8+9*j)\times(4+9*k)$ *for some* $j,k\in W$

All the number having digital root equal to 7 can be only factorized in any of following case(s):

- $(1+9*j)\times(7+9*k)$ *for some* $j,k\in W$
- $(8+9*j)\times(2+9*k)$ *for some* $j,k\in W$
- $(4+9*j)\times(4+9*k)$ *for some* $j,k\in W$
- $(5+9*j)\times(5+9*k)$ *for some* $j,k\in W$

All the number having digital root equal to 8 can be only factorized in any of following case(s):

- $(1+9*j)\times(8+9*k)$ *for some* $j,k\in W$
- $(4+9*j)\times(2+9*k)$ *for some* $j,k\in w$
- $(7+9*j)\times(5+9*k)$ *for some* $j,k\in W$

Now, we want to discuss an important theorem based on the polynomial whose roots are integers.

Theorem 4.1. Let $p(x)$ be any n degree polynomial whose roots are only integers, if a be its root then $dr(p(dr(a))) = 9$. although inverse of theorem may not true.

Proof: Let p(x) is n degree polynomial and has only integral root.
Then let $\alpha_n \alpha_2, \alpha_3, \ldots \alpha_n$ be n roots then $p(x) = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3) \ldots (x-\alpha_n)$.
Now dr(α_1) $-$ α_1 is must be a multiple of 9. Since $dr(\alpha_1)$ *and* α_1 are equivalent numbers.
Therefore

$$dr\left(p\left(dr\left(\alpha_1\right)\right)\right) = dr[\left(dr\left(\alpha_1\right)-\alpha_1\right)\left(dr\left(\alpha_1\right)-\alpha_2\right)\times\left(dr\left(\alpha_1\right)-\alpha_3\right)\ldots\left(dr\left(\alpha_1\right)-\alpha_n\right)] = 9$$

Since every multiple of 9 has digital root is equal to 9. (See table **Table II**)

Significance of theorem 4.1.
Let us suppose a polynomial $p(x)$ of n degree is given whose roots are integers. Check the digital roots of $p(1), p(2), p(3), \ldots, p(9)$. If $dr\left(p(d)\right) \neq 9$ *for some* $d \in D$ then any number lying in the equivalence class of d, can never satisfy $p(x)$ and if). If $dr\left(p(d)\right) = 9$ *for some* $d \in D$ then numbers equivalent to d can be its root.

For example:
We have $p(x) = x^3 - 92x^2 + 2401x - 15810$
It is very hard to find roots of this polynomial, but there is computational method, so start evaluating $p(1), p(2) \ldots p(9)$

$$i.e.\quad p(1) = 13500 \Rightarrow dr(p(1)) = 9$$
$$p(2) = -11368 \Rightarrow dr(p(2)) = 1 \neq 9$$
$$p(3) = -11809 \Rightarrow dr(p(3)) = 1 \neq 9$$
$$p(4) = -7614 \Rightarrow dr(p(4)) = 9$$
$$p(5) = -5730 \Rightarrow dr(p(5)) = 6 \neq 9$$
$$p(6) = -4500 \Rightarrow dr(p(6)) = 9$$
$$p(7) = -3168 \Rightarrow dr(p(7)) = 9$$
$$p(8) = -1978 \Rightarrow dr(p(8)) = 7 \neq 9$$
$$p(9) = -924 \Rightarrow dr(p(9)) = 6 \neq 9$$

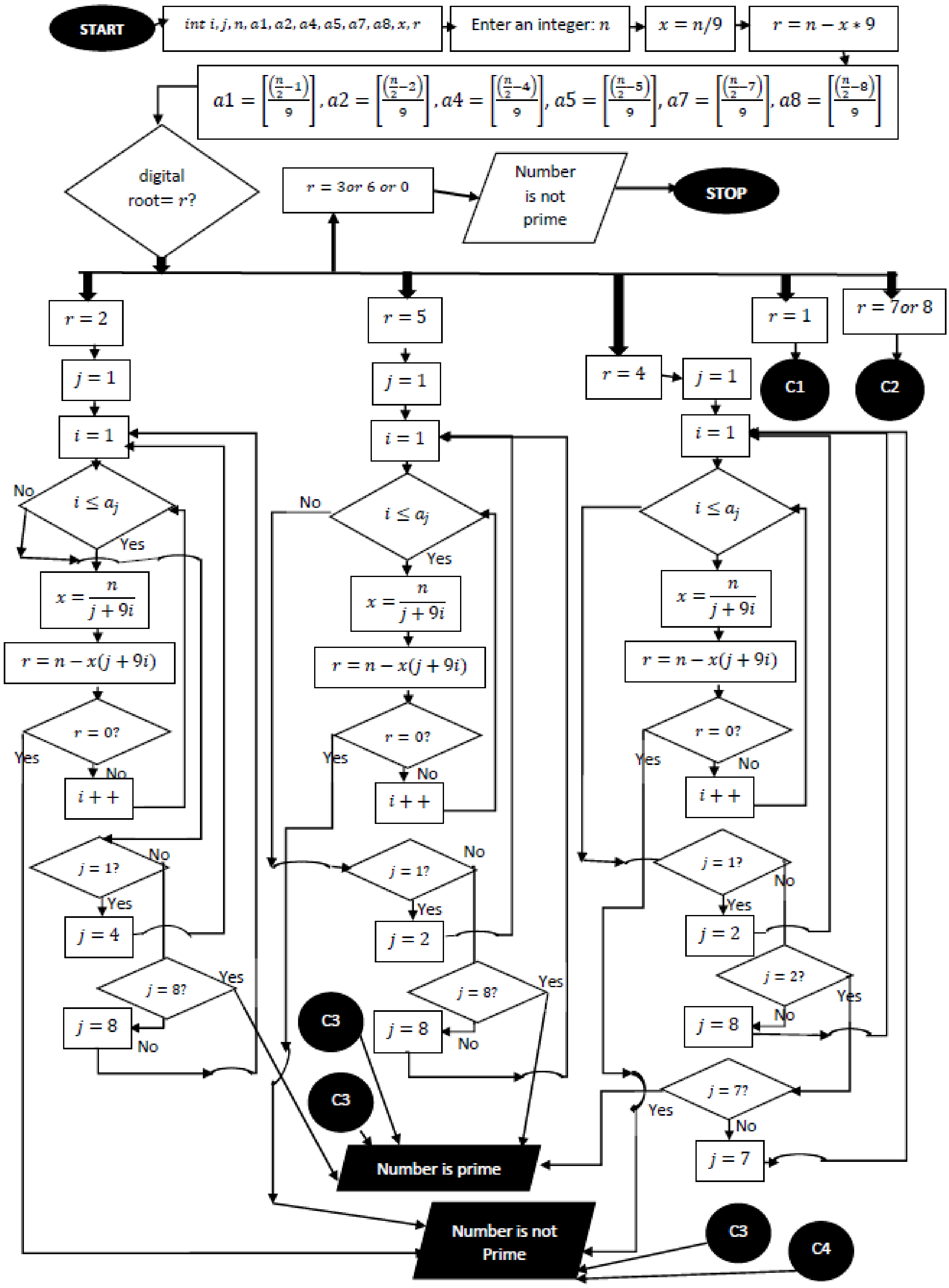
Here we can see that digital roots of $p(2)$, $p(3), p(5), p(8), p(9) \neq 9$, therefore by Theorem 4.4, the numbers equivalent to 2 *or* 3 *or* 5 *or* 8 *or* 9 can never satisfy $p(x) = x^3 - 92x^2 + 2401x - 15810$

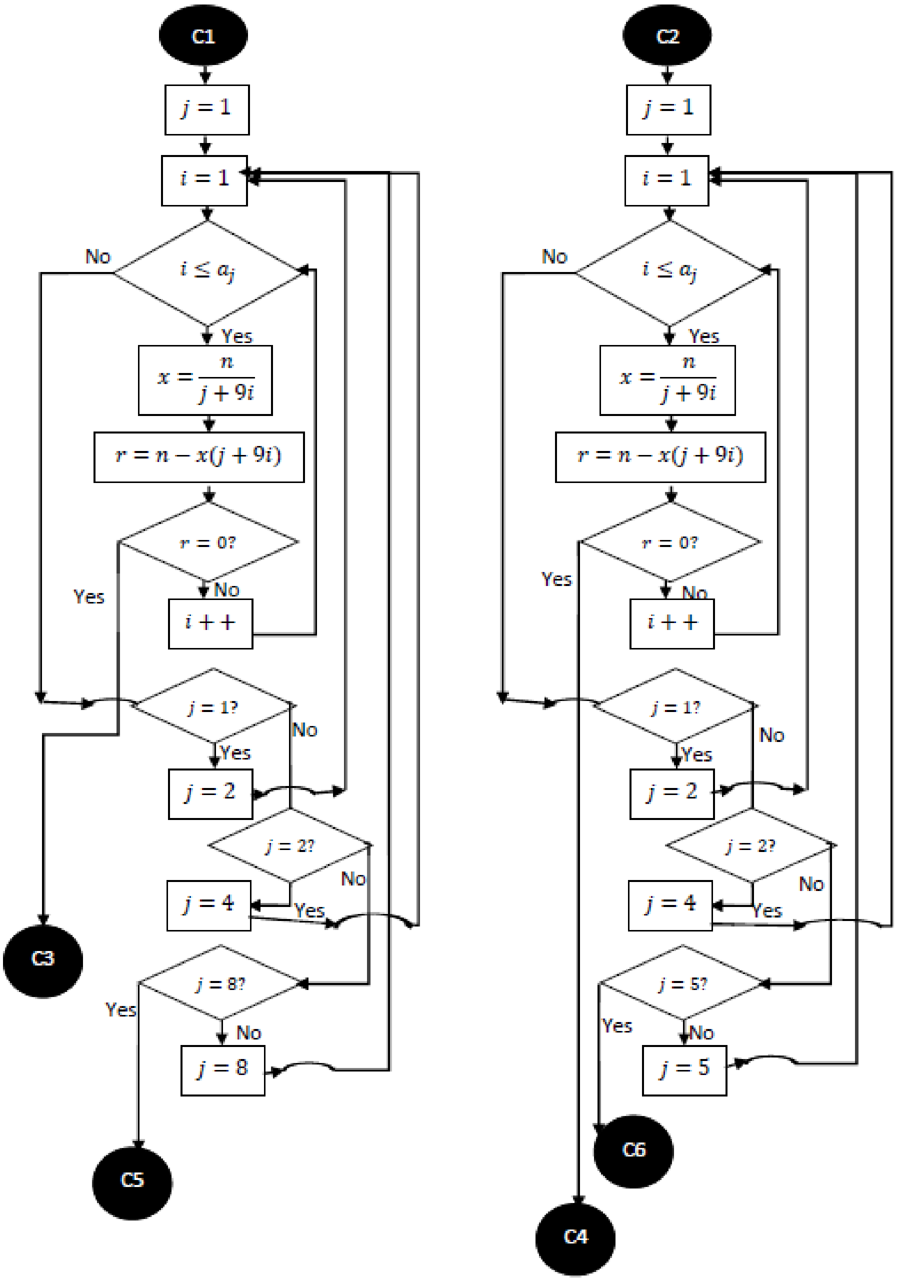
Remark: Alternative method for finding Digital root

Adding digits of a large number till a single digit is obtained may be a lengthy. So there is an alternative method for finding digital root.

Digital root of any number x shall be equal to remainder r (if r is not equal to 0) when x is divisible by 9. If $r = 0$ then take digital root value as 9.

V. Flow Chart for primality test: See in figure 1.





Further Research prospective: The idea of digital root can be extended to rational number. Properties of digital root is also valid if it take digital root as a function from set of rational number to D.
Programme based on the given algorithm can be made for primality test. Also this algorithm may be improved and can be applied in the field of cryptography.

References

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