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A Two Grid Discretization Method For Decoupling Time-Harmonic Maxwell's Equations

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Abstract: *In this work, we study a two grid finite element methods for solving coupled partial differential equations of Time-Harmonic Maxwell's equations. A brief survey of finite element methods for Maxwell's equation and related fundamentals, such as Sobolev spaces, elliptic regularity results, finite element methods for Second order problems and its algorithms were reviewed. The method is based on discretization using continuous H^1-conforming elements for decoupling systems of partial differential equations. With this method, the solution of the coupled equations on a fine grid is reduced to the solution of coupled equations on a much coarser grid together with the solution of decoupled equations on the fine grid.*

Keywords: *Time-Harmonic Maxwell's equation, finite element methods, two-grid scheme, fine and coarser grids.*

I. Introduction

Maxwell's equations consist of two pairs of coupled partial deferential equations relating to four fields, two of which model the sources of electromagnetism. These equations characterized the fundamental relations between the electric field and magnetic field as recognized by the founder [James Clark Maxwell(1831-1879)] of the Modern theory of electromagnetism. However, the modern version of the Maxwell's equations has two fundamental field vector functions $E(\underline{x},t)$ and $H(\underline{x},t)$ in the classical electromagnetic field, with space variable $\underline{x}\in \mathrm{R}^3$ and time variable $t\in \mathrm{R}$. The distribution of electric charges is given by a scalar charge density function $\rho(\underline{x},t)$ and the current is described by the current density function $J(\underline{x},t)$.

This paper is concerned with the discretization of time-harmonic Maxwell equations with finite elements. The analysis of Maxwell's equations for simplified cases can be reduced to the solution of the Helmholtz equation, which in turn can be discretized using standard H^1-conforming (continuous) elements.

II. Literature Review

Given that

$$\nabla\times\nabla\times u+\alpha u = f \quad \text{in} \quad \Omega,\ n\times u \quad =0 \quad \text{on} \quad \partial\Omega\} \tag{1}$$

Considering the weak form for the curl-curl problem (1):

Find $u\in H_0(curl;\Omega)$ such that

$$(\nabla\times u,\nabla\times v)+\alpha(u,v)=(f,v) \tag{2}$$

for all $\underline{\upsilon}\in H_0(curl;\Omega),$ where (.,.) denotes the inner product of $[L_2(\Omega)]^2$. Hence the space $H_0(curl;\Omega)$ is defined as follows:

$$H(curl;\Omega)=\{\underline{v}=\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}\in[L_2(\Omega)]^2:\nabla\times\underline{\upsilon}=\frac{\partial v_2}{\partial x_1}-\frac{\partial v_1}{\partial x_2}\in L_2(\Omega) \tag{3}$$

$$H_0(curl;\Omega)=\{\upsilon\in H(curl;\Omega):n\times v=0 \text{ on } \partial\Omega \tag{4}$$

where n is the unit outer normal.

Remark: The $n\times v=0$ on $\partial\Omega$ is equivalent to $\tau\cdot\upsilon=0$ on $\partial\Omega,$ where τ is the unit tangent vector along $\partial\Omega.$ The curl-curl problem (2) is usually solved directly using $H(\text{curl})$ conforming vector finite-elements [13,14,15,16,17]. However, this is non-elliptic when the $H_0(curl)$ formulation is used and hence the convergence analysis of both the numerical scheme and its solvers are more complicated. For any $\underline{u}\in H_0(curl;\Omega)$ due to the well-known Helmholtz decomposition[18, 16], we have the following orthogonal decomposition:

$$\underline{u} = \dot{u} + \nabla\phi \qquad (5)$$

where $\underline{\dot{u}} \in H_0(curl;\Omega) \cap H(div^\circ;\Omega)$ and $\phi \in H_0^1(\Omega).$ The space $H(div^\circ;\Omega)$ is defined as follows:

$$H(div;\Omega) = \{\underline{\upsilon} \in [L_2(\Omega)]^2 : \nabla\cdot\nu = \frac{\partial \nu_1}{\partial x_1} + \frac{\partial \nu_2}{\partial x_2} \in L_2(\Omega)\} \qquad (6)$$

$$H(div^\circ;\Omega) = \{\underline{\upsilon} \in H(div;\Omega) : \nabla\cdot\upsilon = 0\} \qquad (7)$$

It is trivial to show that $\phi \in H_0^1(\Omega)$ satisfies

$$\alpha(\nabla\phi, \nabla\eta) = (f, \nabla\eta) \qquad (8)$$

for all $\eta \in H_0^1(\Omega),$ (8) is the variational form of the Poisson problem. Many successful schemes have been developed for solving this problem. Considering $\dot{u}$ as the weak solution of the following reduced curl-curl problem [39], then Find $\dot{u} \in H_0(curl;\Omega) \cap H(div^\circ;\Omega)$ such that

$$(\nabla\times\dot{u}, \nabla\times\underline{\upsilon}) + \alpha(\dot{u}, \underline{\upsilon}) = (f, \underline{\upsilon}) \qquad (9)$$

for all $\underline{\upsilon} \in H_0(curl;\Omega) \cap H(div^\circ;\Omega).$

Unlike the non-elliptic curl-curl problem (1), the reduced problem (9) is an elliptic problem. In particular, the solution $\underline{u}$ has elliptic regularity under the assumption that $f \in [L_2(\Omega)]^2,$ which greatly simplifies the analysis.

III. A Model Maxwell's Equation

The Maxwell's equation stated as the following equations in a region of space in R^3 occupied by the electromagnetic field:

$$\nabla\times E = -\mu\frac{\partial H}{\partial t} \qquad (10)$$

$$\nabla\cdot E = \frac{\rho}{\varepsilon} \qquad (11)$$

$$\nabla\times H = \varepsilon\frac{\partial E}{\partial t} + J \qquad (12)$$

$$\nabla\cdot H = 0 \qquad (13)$$

where ε is the electric permitivity, and μ is the magnetic permeability. Equation (10) is called Faraday's law and describes how the changing of magnetic field affects the electric field. The equation (12) is referred as Ampère's law. The divergence conditions (11) and (13) are Gauss' laws of electric displacement and magnetic induction respectively.

Let the radiation frequency be $\omega,$ such that $\omega > 0,$ then we can find solutions of the Maxwell's equations of the form

$$E(\underline{x},t) = \exp^{-i\omega t}\hat{E}(\underline{x}) \qquad (14)$$

$$H(\underline{x},t) = \exp^{-i\omega t}\hat{H}(\underline{x}) \qquad (15)$$

$$J(\underline{x},t) = \exp^{-i\omega t}\hat{J}(\underline{x}) \qquad (16)$$

$$\rho(\underline{x},t) = \exp^{-i\omega t}\hat{\rho}(\underline{x}) \qquad (17)$$

Differentiating (15) yields

$$\frac{\partial H}{\partial t} = -i\omega t \exp^{-i\omega t} \hat{H}(\underline{x}) \tag{18}$$

Substituting (14) and (18) into (10), we obtain

$$\nabla \times \hat{E} = i\omega\mu\hat{H}(\underline{x}) \tag{19}$$

Also, putting (14) and (17) into (11), gives

$$\nabla \cdot \hat{E} = \frac{\hat{\rho}}{\varepsilon} \tag{20}$$

Further differentiation of (14), yields

$$\frac{\partial E}{\partial t} = -iw \exp^{-iwt} \hat{E}(\underline{x}) \tag{21}$$

Substituting (15),(16) and (21) into (12), we obtain

$$\nabla \times \hat{H} = -i\omega\varepsilon\hat{E} + \hat{J} \tag{22}$$

Combining (15) and (13), gives

$$\nabla \cdot \hat{H} = 0 \tag{23}$$

It can be observed that when the charge is consumed, the divergence conditions (20) and (23) are always satisfied, provided that the equations (19) and (22) holds. Then combining the equations (13) and (22) we have

$$\nabla \times \nabla \times \hat{E} - \omega^2 \mu\varepsilon\hat{E} = i\omega\mu\hat{J} \tag{24}$$

and

$$\nabla \times \nabla \times \hat{H} - \omega^2 \mu\varepsilon\hat{H} = \nabla \times \hat{J} \tag{25}$$

Considering equation (1) with perfectly conducting boundary condition for the curl-curl problem (24)-(25), where $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain, $\alpha \in \mathbb{R}$ is a constant, and $f \in [L_2(\Omega)]^2$. The curl-curl problem (1) appears in the semi-discretization of electric fields in the time-dependent (time-domain) Maxwell's equations when $\alpha > 0$ and the time-harmonic (frequency domain) Maxwell's equations when $\alpha \leq 0$. If the vector equation is written as the scalar-valued in the real and imaginary parts respectively, we obtain the following equivalent coupled equations and the boundary condition:

$$-\Delta u_1 + \alpha_1 u_1 - \alpha_2 u_2 = f_1 \quad \text{in } \Omega \tag{26}$$

$$-\Delta u_2 + \alpha_1 u_2 + \alpha_2 u_1 = f_2 \quad \text{in } \Omega \tag{27}$$

$$u_j = 0, \quad j = 1,2,\ldots \tag{28}$$

IV. Variational Formulation

The variational formulation can be obtained in terms of either electric field E or magnetic field H. This depends upon the choice in which one of the equations (24)-(25) is to be satisfied weakly when discretized in the distributional sense, and the other one strongly. Hence, in the E-field formulation, we select weak sense by multiplying the curl-curl coupled equation (26) by a suitable test function ψ_1 and integrate over the domain Ω to obtain

$$-\int_\Omega \Delta u_1 \psi_1 dx + \alpha_1 \int_\Omega u_1 \psi_1 dx - \alpha_2 \int_\Omega u_2 \psi_1 dx = \int_\Omega f_1 \psi_1 dx \tag{29}$$

Therefore, we get

$$-\int_\Omega \psi_1 \Delta u_1 dx = \int_\Omega \nabla \psi_1 \times \nabla f_1 dx \tag{30}$$

Thus, the boundary integral varnishes since $\psi_1 = 0$ on $\partial\Omega$. Therefore, we obtain the variational formulation of the coupled equation (26) as follows:

$$\int_\Omega (\nabla u_1 \times \nabla \psi_1) = \int_\Omega \nabla f_1 \times \nabla \psi_1 \tag{31}$$

Expressing (31) in form of inner product, we get

$$(\nabla u_1 \times \nabla \psi_1) = (f_1 \times \nabla \psi_1) \tag{32}$$

Similarly, we obtain the weak formulation of the couple equation (27) as

$$(\nabla u_2 \times \nabla \psi_2) = (f_2 \times \nabla \psi_2) \tag{33}$$

Combining (29) and (31) and expressing in terms of inner products, yields

$$(\nabla u_1 \times \nabla \psi_1) + \alpha_1(u_1 \times \psi_1) - \alpha_2(u_2 \times \psi_1) = (\nabla f_1 \times \nabla \psi_1) \tag{34}$$

In a similar manner, we get

$$(\nabla u_2 \times \nabla \psi_2) + \alpha_2(u_2 \times \psi_2) - \alpha_2(u_1 \times \psi_2) = (\nabla f_2 \times \nabla \psi_2) \tag{35}$$

Combining the equations (34) and (35), we have

$$\hat{\alpha}(\psi, u) = (\nabla \psi_1 \times \nabla u_1) + (\nabla \psi_2 \times \nabla u_2) \tag{36}$$

and

$$N(\psi, u) = \alpha_1(u_1 \times \psi_1) - \alpha_2(u_2 \times \psi_1) + \alpha_2(u_2 \times \psi_2) - \alpha_2(u_1 \times \psi_2) \tag{37}$$

where

$$(\nabla f_1 \times \nabla \psi_1) + (\nabla f_2 \times \nabla \psi_2) = (f, \psi) \tag{38}$$

$$\alpha(u, \psi) = \hat{\alpha}(u, \psi) + N(u, \psi) \tag{39}$$

Hence, the variational formulation for equations (26)-(28) is derived as

$$\alpha(u, \psi) = (f, \psi) \;\; \forall \;\; u, \psi \in \; H_0(curl; \Omega) \tag{40}$$

V. Theorem(Regularity Result)

Assume that

$$f \in L^2(\Omega) \text{ x } L^2(\Omega), \;\; V \in L^\infty(\Omega) \text{ x } L^\infty(\Omega), \;\; 0 \le V_1(x) \tag{41}$$

Then the variational problem (40) has a unique solution $\psi \in H^2(\Omega) \text{ x } H^2(\Omega)$ and

$$\|\psi\|_2 \le C \|f\|_0 \tag{42}$$

Proof. By the bilinear and linear continuous operators which implies the linearity and boundedness of a set, we have

$$|a(u,w)| \le C \|u\|_1 \|w\|_1, \;\; \forall \;\; u, w \in H_0^1(\Omega) \text{ x } H_0^1(\Omega)$$

Hence, by the Lax Milgram Lemma, the weak formulation(40) has a unique solution such that $\psi_1 \in H_0^1(\Omega) \times H_0^1(\Omega).$ Also, from the regularity theory for the second-order elliptic equations as proposed by Evans C. Lawrence, we have

$$\|\psi\|_{H^2(L^2(\Omega))} \le C(\|f\|_{L^2(\Omega)} + \|\psi\|_{L^2(\Omega)}) \tag{43}$$

where the constant C depends only on the domain $\Omega.$ This simply means that

$$\|\psi\|_2 \le C \|\psi_1\|_0 + \|f_2\|_0 \tag{44}$$

Set $u = \psi$ in (40), we have

$$\|\psi\|_1 \le C \|f\|_0 \tag{45}$$

Hence, (42) follows from (44) and (45).

In the following section, we define triangulation concept which is used in the fundamental theorems on error analysis.

VI. Definition (Triangulation)

Let τ be a quasi-uniform triangulation of the finite element space Ω with mesh size $h > 0,$ and let $S_0^h \subset H_0^1(\Omega)$ be the corresponding piecewise linear function space. It therefore means that S_0^h is a finite-dimensional subspace of the Hilbert space $H_0^1.$ Then the finite element approximation of the equation (42) is defined as

Find

$$\alpha(\psi_h, u_h) = (f, u_h) \forall \ \psi_h \in S_0^h \times S_0^h \tag{46}$$

VII. Theorem (Error Analysis Of Finite Element Discretization)

Assume that theorem (0.5) holds. Then ψ_h has the error estimate

$$\| \psi - \psi_h \|_s \lesssim h^{2-s} \| \psi \|_2 \quad s = 0,1\ldots \tag{47}$$

Proof. From the variational formulation (40) and (46), we obtain

$$\alpha(\psi - \psi_h, u_h) = 0 \ \ \forall \, u_h \in S_0^h \times S_0^h$$

then we have

$$\alpha(e_h, u_h) = 0 \ \ \forall \, u_h \in S_0^h \times S_0^h \tag{48}$$

Let $\psi^I \in S_0^h \times S_0^h$ be the interpolation error bound of ψ_1 and by (48), we have

$$\| e_h \|_1 \leq \| \psi - \psi^I \|_1 \lesssim h \| \psi \|_2 \tag{49}$$

In order to show uniqueness of the solution, the auxiliary problem in equations (28)- (30) are written as

$$-\Delta u_1(x) + V_1(x) u_1(x) - V_2(x) u_2(x) = g_1(x) \qquad \forall \, x \in \Omega \tag{50}$$

$$-\Delta u_2(x) + V_1(x) u_2(x) + V_2(x) u_1(x) = g_2(x) \qquad \forall \, x \in \Omega \tag{51}$$

$$u_j(x) = 0, \qquad j = 1,2 \qquad\qquad \forall \, x \in \partial\Omega \tag{52}$$

VIII. Theorem(Uniqueness Of Solution)

Let $g \in L^2(\Omega) \text{ x } L^2(\Omega),$ then there exists a unique solution

$$\mathbf{u} \in H^2(\Omega) \cap H_0^1(\Omega) \text{ x } H^2(\Omega) \cap H_0^1(\Omega)$$

such that

$$\alpha(\mathbf{w}, \mathbf{u}) = (\mathbf{g}, \mathbf{w}) \ \ \forall \mathbf{w} \in H_0^1(\Omega) \text{ x } H_0^1(\Omega) \tag{53}$$

and

$$\| \mathbf{u} \|_2 \leq \| \mathbf{g} \|_0 \tag{54}$$

Proof. The equation (54) is satisfied by (53). Also, let $u^I \in S_0^h \times S_0^h$ be the interpolation of u and from (53), we obtain

$$\| e_h \|_0 \lesssim h \| e_h \|_1 \tag{55}$$

Therefore, (47) follows from (48) and (55), which means that

$$\alpha(e_h, \psi) = (\psi, \psi - \psi_h) \tag{56}$$

and we obtain for s=0

$$\| \psi - \psi_h \|_0 \leq \| e_h \|_0 \tag{57}$$

Then by equations (55) and(49), we have

$$\| \psi - \psi_h \|_0 \leq \| e_h \|_1 \tag{58}$$

hence we get

$$\| \psi - \psi_h \|_0 \lesssim h^2 \| \psi \|_2 \tag{59}$$

This completes the proof of the uniqueness

IX. Definition Of Two Grid Technique

Two grid method is a discretization technique for nonlinear equations based on two grids of different sizes. The main idea is to use a coarse grid space to produce a rough approximation of the solution of nonlinear problems. This method involves a nonlinear problem being solved on the coarse grid with grid size H and a linear problem on the fine grid with grid size $h << H$.

X. A New Two-Grid Finite Element Method

The finite element discretization (46) corresponds to a coupled system of equations in the general case. Thus, to reduce the rigour another finite element space $S_0^H (\subset S_0^h \subset H_0^1(\Omega))$ defined on a coarser quasiuniform triangulation (with mesh size $H > h$) of Ω is proposed as the following algorithm:

XI. Algorithm A1

Step 1 Let $\psi_h \in S_0^H \times S_0^H$ such that

$$\alpha(\psi_H, \chi) = (f, \chi) \quad \forall\, \chi \in S_0^H \times S_0^H \tag{60}$$

Step 2 Let $\psi_h^* \in S_0^h \times S_0^h$ such that

$$\hat{\alpha}(\psi^*, u_h) = (f, u_h) - N(\psi_H, u_h) \quad \forall\, u_h \in S_0^h \times S_0^h \tag{61}$$

It can be observed from step 2 that the linear system is decoupled and consists of two separate Poisson equations. However, on the coarser space it is required that a coupled system is solved in step 1, since the smallness of the S_0^H rather than the dimension of S_0^h is evident. It therefore follows that ψ_h^* can attain the optimal accuracy in H^1-norm if the coarse mesh size H is taken to be $\sqrt{h}$.

XII. Theorem

Assume that (41) holds. Then ψ_h^* has the following error estimates:

$$\|\psi - \psi_h^*\|_1 \lesssim H^2 \tag{62}$$

Consequently,

$$\|\psi - \psi_h^*\|_1 \lesssim h + H^2 \tag{63}$$

then, ψ^* has the same accuracy as ψ_h in H^1-norm for H^2

Proof. Let $e_h = \psi - \psi_h$ and $\hat{e}_h = \psi_h - \psi_h^*$, it then follows that by (61) and (46) we have

$$\hat{\alpha}(\hat{e}_h, u_h) + N(\psi_h - \psi_H, u_h) = 0 \quad \forall\, u_h \in S_0^h \times S_0^h \tag{64}$$

Putting $u_h = \hat{e}_h$ in the last expression, yields

$$\|\hat{e}_h\|_1^2 \lesssim \hat{\alpha}(\hat{e}_h, \hat{e}_h) \lesssim \|\psi_h - \psi_H\|_0 \|\hat{e}_h\|_0$$

and then

$$\|\hat{e}_h\|_1 \lesssim \|\psi_h - \psi_H\|_0 \tag{65}$$

By taking (47),(44) and the inequality,

$$\|\psi_h - \psi_H\|_0 \le \|\psi - \psi_h\|_0 + \|\psi - \psi_H\|_0$$

we get

$$\|\psi_h - \psi_H\|_0 \le h^2 + H^2 \tag{66}$$

Therefore, (62) follows from (65) and the above inequality. Also, (63) follows from (47), (62) and the following inequality:

$$\|\psi - \psi_h^*\|_1 \le \|\psi - \psi_h\|_1 + \|\hat{e}_h\|_1$$

Algorithm A1 can be improved in a successive technique as follows:

XIII. Algorithm A2

Let $\psi_h^0 = 0$. Assume that $\psi_h^k \in S_0^h \times S_0^h$ has been obtained, then $\psi_h^{k+1} \in S_0^h \times S_0^h$ is defined as follows:

Step 3. Find $e_H \in S_0^H \times S_0^H$ such that

$$\alpha(e_H, \chi) = (\mathbf{f}, \chi) - \alpha(\psi_h^k, \chi), \forall\, \chi\, S_0^H \times S_0^H \tag{67}$$

Step 4. Find $\psi_h^{k+1} \in S_0^h \times S_0^h$ such that

$$\hat{\alpha}(\psi_h^{k+1}, u_h) = (\mathbf{f}, u_h) - N(\psi_h^k + e_H, u_h), \forall\, u_h \in S_0^h \times S_0^h \tag{68}$$

XIV. Theorem

Under the assumption (41), ψ_h^k has the following error estimate:

$$\|\psi_h - \psi_h^k\|_1 \lesssim H^{k+1}, \; k \ge 1 \tag{69}$$

Consequently,

$$\|\psi - \psi_h^k\|_1 \lesssim h + H^{k+1},\ k \geq 1 \tag{70}$$

where $\psi_h^k,\ k \geq 1,$ has the same accuracy as ψ_h in H^1-norm if $H = h^{\frac{1}{k+1}}$.

Proof. From (46) and (61), we have

$$\hat{\alpha}(\psi_h - \psi_h^{k+1}, u_h) = -N(\psi_h(\psi_h^k + e_H), u_h),\ u_h \in S_0^h \times S_0^h \tag{71}$$

which, by taking $u_h = \psi_h - \psi_h^{k+1},$ gives

$$\|\psi_h - \psi_h^{k+1}\|_1^2 \lesssim \hat{\alpha}(\psi_h - \psi_h^{k+1}, \psi_h - \psi_h^{k+1})$$
$$\lesssim \|\ \psi_h - (\psi_h^k + e_H)\|_0 \|\psi_h - \psi_h^{k+1}\|_0,$$

and then

$$\|\psi_h - \psi_h^{k+1}\|_1 \lesssim \|\ \psi_h - (\psi_h^k + e_H)\|_0 \tag{72}$$

It follows from (46) and (60) that

$$\alpha(\psi_h - (\psi_h^k + e_H), \chi) = 0, \forall\ \chi \in S_0^H \times S_0^H \tag{73}$$

This implies that

$$\|\psi_h - (\psi_h^k + e_H)\|_1^2 \lesssim \alpha(\psi_h - (\psi_1^k + e_H), \psi_h - (\psi_h^k + e_H))$$
$$= \alpha(\psi_h - (\psi_h^k + e_H), \psi_h - \psi_h^k)$$
$$\lesssim \|\ \psi_h - (\psi_h^k + e_H)\|_1 \|\psi_h - \psi_h^k\|_1,$$

and then

$$\|\psi_h - (\psi_h^k + e_H)\|_1 \lesssim \|\ \psi_h - \psi_h^k\|_1 \tag{74}$$

Let u be the solution of problem (53) with $g = \psi_h - (\psi_h^k + e_H)$ and $u^I \in S_0^H \times S_0^H$ be the interpolation of u. Then according to (54) and (73), we have

$$\|\psi_h - (\psi_h^k + e_H)\|_0^2 = \alpha(\psi_h - (\psi_h^k + e_H), u) = \alpha(\psi_h - (\psi_h^k + e_H), u - u^I)$$
$$\lesssim \|\ \psi_h - (\psi_h^k + e_H)\|_1 \|u - u^I\|_1$$
$$\lesssim H\|\psi_h - (\psi_h^k + e_H)\|_1 \|u\|_2$$
$$\lesssim H\|\psi_h(\psi_h^k + e_H)_1\|\|\psi\psi_h - (\psi_h^k + e_H)\|_0$$

which implies that

$$\|\psi_h - (\psi_h^k + e_H)\|_0 \lesssim H\|\psi_h - (\psi_h^k + e_H)\|_1 \tag{75}$$

Therefore, from (71), (75) and (73), we have

$$\|\psi_h - \psi_h^k\|_1 \lesssim H\|\psi_h - \psi_h^{k-1}\|_1 \lesssim H^{k-1}\|\psi_h - \psi_h^1\|_1,\ k \geq 1 \tag{76}$$

Observe that ψ_h^1 is the solution $\psi*_h$ obtained by Algorithm A1, thus, (68) follows from (76) and (62). Furthermore, (70) follows from (68), (47) and the following inequality:

$$\|\psi - \psi_h^k\| \leq \|\psi - \psi_h\|_1 + \|\psi_h - \psi_h^k\|_1$$

As stated in Theorem 12.0, it is sufficient to take $H = h^{\frac{1}{k+1}}$ to obtain the optimal approximation in h^1-norm. Therefore, the dimension of S_0^H can be much smaller than the dimension of S_0^h. Finally, the numerical examples and error estimates on the efficiency of the algorithms is demonstrated in [7] with boundary value problem of the Schrödinger type.

XV. Comments

Using the Maxiwell equation as an illustration, we presented in this paper a new two-grid discretization technique to decouple systems of partial differential equations. This new application of the two-grid decoupling technique can obviously be extended in many different ways, for different discretizations such as finite volume and finite difference methods for other types of partial differential equations.

References

[1]. J. Xu and A. Zhou, Local and parallel finite element algorithms based on two-grid discretizations, Math. Comp. 69 (2000), 881-909.
[2]. M. Marion and J. Xu, Error estimates on a new nonlinear Galerkin method based on two-grid finite elements, SIAM J. Numer. Anal., 32, 4(1995), 1170-1184.
[3]. J. Xu, A novel two-grid method for semilinear equations, SIAM J. Sci. Comput., 15, 1(1994), 231-237.
[4]. P. G. Ciarlet, The finite element method for elliptic problems, North-Holland, Amsterderm, 1978.
[5]. R. P. Fedorenko, A relaxation method for elliptic difference equations, USSR Computational Math Phys...1(5) 1092-1096,(1961).
[6]. M. Marion and J. Xu, Error estimates on a new nonlinear Galerkin method based on two grid finite elements, SIAM J. Numer. Anal., 32,4(1995), 1170-1184.
[7]. J. Jin, S. Shu, and J. Xu, A two-grid discretization method for decoupling systems of partial differential equations (Volume 75,Number 256, October 2006, pages 1617-1626)
[8]. P. Solin, Partial Differential Equations and the Finite element method (2006).
[9]. S. Larson and V. Thomee, partial differential equations with numerical methods (2009)
[10]. J. Xu, Two grid finite element discretization techniques for linear and nonlinear partial differential equation, SIAM, J. Numer. Anal, 33, 5(1996). 1757-1777.
[11]. C. B. Susanne and L. R. Scott, The Mathematical Theory of finite element methods, Third edition (2008)
[12]. L. C. Evans Partial differential equations, Graduate studies in Mathematics volume 19(1997)
[13]. Bossavit. Discretization of electromagnetic problems: the "generalized finite differences" In P.G. Ciarlet, W.H.A. Schilders, and E.J.W. Ter Maten, editors, Handbook of Numerical Analysis, Vol. XIII, pages 105-197, North-Holland, Amsterdam, 2005.
[14]. L. Demkowicz. Finite element methods for Maxwell's equations. In E. Stein, R. de Borst, and T. J. R. Hughes, editors, Encyclopedia of Computational Mechanics, pages 723-737. Wiley, Weinheim, 2004
[15]. R. Hiptmair. Finite elements in computational electromagnetic. Acta Numer., 11:237-339, 2002
[16]. P. Monk. Finite Element Methods for Maxwell's Equations. Oxford University Press, New York, 2003
[17]. J.C. Nédélec. A new family of mixed finite elements in R^3. Numer. Math., 50:57-81, 1986
[18]. V. Girault and P.A. Raviart. Finite Element Methods for Navier-stokes Equations. Theory and Algorithms. Springer-Verlag, Berlin, 1986.
[19]. S.C. Brenner, F. Li and L.Y. Sung. A locally divergence free interior penalty method for two dimensional curl-curl problems. SIAM J. Numer. Anal., 46:1190-1211, 2008.