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Coupled Fixed Point Theorems for Contractive Type Mappings in Dislocated Metric Spaces

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Abstract: *The purpose of this paper is to prove some new coupled fixed point theorems for contractive type mappings defined on dislocated metric spaces. Also, we give an example to support our main theorem and some corollaries of the main result.*
Keywords: *coupled fixed point, dislocated metric spaces, Contractive type mappings*

I. Introduction and preliminaries

In 2000, P. Hitzler and A. K. Seda [5] introduced the notion of dislocated metric space in which self distance of a point need not be equal to zero. They also generalized the famous Banach contraction principle in this space. The study of common fixed points of mappings in dislocated metric space satisfying certain contractive conditions has been at the center of vigorous research activity. Dislocated metric space plays very important role in topology, logical programming and in electronics engineering. C.T. Aage and J. N. Salunke [1], A. Isufati [6] established some important fixed point theorems in single and pair of mappings in dislocated metric space. K. Jha, D Panthi [7] established a common fixed point theorem in dislocated metric spaces. In 2006, Bhaskar et al introduced the notion of coupled fixed point and proved some fixed point results in this context.

Definition 1.1: Let X be a nonempty set and let $d_l: X \times X \to [0,+\infty)$ be a function, called a dislocated metric (or simply d_l-metric) if the following conditions hold for any $x, y, z \in X$:
(i)if $d_l(x,y) = 0$, then $x = y$,
(ii) $d_l(x,y) = d_l(y,x)$,
(iii) $d_l(x,y) \le d_l(x,z) + d_l(z,y) - d_l(z,z)$.
The pair (X, d_l) is then called a dislocated metric space. It is clear that if$d_l(x,y) = 0$, then from (i), $x = y$. But if $x = y$, $d_l(x,y)$ may not be 0.
Example1.2: If $X = [0,+\infty)$, then $d_l(x,y) = x + y$ defines a dislocated metric d_l on X.
Example1.3: If $X = [0,+\infty)$, then $d_l(x,y) = \max\{x,y\}$ defines a dislocated metric d_l on X.
Definition 1.4: A sequence $\{x_n\}$ in a d_l-metric space (X, d_l) is called a Cauchy sequence if given $\varepsilon > 0$, there exists $n_0 \in N$ such that for all $n, m \ge n_0$, we have $d_l(x_m, x_n) < \varepsilon$.
Definition 1.5: A sequence $\{x_n\}$ in a d_l-metric space converges with respect to d_l if there exists $x \in X$ such that $d_l(x_n, x) \to 0$ as $n \to +\infty$. In this case, x is called limit of $\{x_n\}$, and we write $x_n \to x$.
Definition 1.6: A d_l-metric space (X, d_l) is called complete if every Cauchy sequence in X converges to a point in X.
Definition 1.7: An element $(x,y) \in X \times X$ is called a coupled fixed point of the mapping $F: X \times X \to X$ if $F(x,y) = x$ and $F(y,x) = y$.
Example 1.8: Let $X = [0, \infty)$ and $F: X \times X \to X$ be defined by $F(x, y) = x + y$
For all $x, y \in X$. It is easy to see that F has a unique coupled fixed point (0, 0).

II. Main Results

Theorem 2.1 Let (X, d) be a dislocated metric space. Suppose that the mapping
$F: X \times X \to X$ Satisfies

$$\begin{aligned} d\big(F(x,y), F(u,v)\big) \le\ & ad(x,u) + bd(y,v) \\ & + hd(F(x,y),u) + kd(F(u,v),x) \\ & + ld(F(x,y),x) + md(F(u,v),u) \\ & + qd(F(y,x),y) + rd(F(v,u),v) \ldots (1) \end{aligned}$$

For all $x, y, u, v \in X$, where a, b, h, k, l, m, q and r are non-negative constants with $a + b + h + k + l + m + q + r < 1/2$. Then F has a unique coupled fixed point.
Proof: Choose $x_0, y_0 \in X$ and set

$x_1 = F(x_0, y_0), \quad y_1 = F(y_0, x_0)$

$\dots$

$x_{n+1} = F(x_n, y_n), \quad y_{n+1} = F(y_n, x_n)$

From (2.1), we have

$$\begin{aligned} d(x_n, x_{n+1}) = d\big(F(x_{n-1}, y_{n-1}), F(x_n, y_n)\big) &\le ad(x_{n-1}, x_n) + bd(y_{n-1}, y_n) \\ &+hd(F(x_{n-1}, y_{n-1}), x_n) + kd(F(x_n, y_n), x_{n-1}) \\ &+ld(F(x_{n-1}, y_{n-1}), x_{n-1}) + md(F(x_n, y_n), x_n) \\ &+qd(F(y_{n-1}, x_{n-1}), y_{n-1}) + rd(F(y_n, x_n), y_n) \end{aligned}$$

Similarly, we have

$$\begin{aligned} d(y_n, y_{n+1}) = d\big(F(y_{n-1}, x_{n-1}), F(y_n, x_n)\big) &\le ad(y_{n-1}, y_n) + bd(x_{n-1}, x_n) \\ &+ hd(F(y_{n-1}, x_{n-1}), y_n) + kd(F(y_n, x_n), y_{n-1}) + ld(F(y_{n-1}, x_{n-1}), y_{n-1}) \\ &+md(F(y_n, x_n), y_n) + qd(F(x_{n-1}, y_{n-1}), x_{n-1}) + rd(F(x_n, y_n), x_n) \end{aligned}$$

Therefore, by letting

$d_n = d(x_n, x_{n+1}) + d(y_n, y_{n+1})$, we have

$$\begin{aligned} d_n &= d(x_n, x_{n+1}) + d(y_n, y_{n+1}) \le ad(x_{n-1}, x_n) + bd(y_{n-1}, y_n) \\ &+ hd(F(x_{n-1}, y_{n-1}), x_n) + kd(F(x_n, y_n), x_{n-1}) + d(F(x_{n-1}, y_{n-1}), x_{n-1}) \\ &+md(F(x_n, y_n), x_n) + qd(F(y_{n-1}, x_{n-1}), y_{n-1}) + rd(F(y_n, x_n), y_n) \\ &+ad(y_{n-1}, y_n) + bd(x_{n-1}, x_n) + d(F(y_{n-1}, x_{n-1}), y_n) + kd(F((y_n, x_n), y_{n-1}) \\ &+ld(F(y_{n-1}, x_{n-1}), y_{n-1}) + md(F(y_n, x_n), y_n) + qd(F((x_{n-1}, y_{n-1}), x_{n-1}) + rd(F(x_n, y_n), x_n) \\ &\le ad(x_{n-1}, x_n) + bd(y_{n-1}, y_n) + hd(x_n, x_n) + kd(x_{n+1}, x_{n-1}) + ld(x_n, x_{n-1}) \\ &+md(x_{n+1}, x_n) + qd(y_n, y_{n-1}) + rd(y_{n+1}, y_n) + ad(y_{n-1}, y_n) + bd(x_{n-1}, x_n) \\ &+hd(y_n, y_n) + kd(y_{n+1}, y_{n-1}) + ld(y_n, y_{n-1}) + md(y_{n+1}, y_n) + qd(x_n, x_{n-1}) + rd(x_{n+1}, x_n) \\ &\le ad(x_{n-1}, x_n) + bd(y_{n-1}, y_n) + hd(x_{n-1}, x_n) + hd(x_n, x_{n+1}) + kd(x_n, x_{n-1}) + kd(x_{n+1}, x_n) \\ &+ld(x_n, x_{n-1}) + md(x_{n+1}, x_n) + qd(y_n, y_{n-1}) + rd(y_{n+1}, y_n) + ad(y_{n-1}, y_n) + bd(x_{n-1}, x_n) \\ &+hd(y_{n-1}, y_n) + hd(y_n, y_{n+1}) + kd(y_n, y_{n-1}) + kd(y_{n+1}, y_n) + ld(y_n, y_{n-1}) \\ &+md(y_{n+1}, y_n) + qd(x_n, x_{n-1}) + rd(x_{n+1}, x_n) \\ &\le \begin{pmatrix} a+b+h \\ +k+l+q \end{pmatrix} [d(x_{n-1}, x_n) + d(y_{n-1}, y_n)] + \begin{pmatrix} h+k \\ +m+r \end{pmatrix} [d(x_n, x_{n+1}) + d(y_n, y_{n+1})] \end{aligned}$$

$d_n \le \frac{(a+b+h+k+l+q)}{[1-(h+k+m+r)]} d_{n-1}, \;\; d_n \le \delta\, d_{n-1}, \;\; \text{where } \delta = \frac{(a+b+h+k+l+q)}{[1-(h+k+m+r)]} < 1$

In general, we have for n = 0, 1, 2 . . .

$d_n \le \delta d_{n-1} \le \delta^2 d_{n-2} \le \dots \le \delta^n d_0$, Now, for all $m > n$,

$d(x_m, x_n) \le d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{m-1}, x_m)$ and

$d(y_m, y_n) \le d(y_n, y_{n+1}) + d(y_{n+1}, y_{n+2}) + \dots + d(y_{m-1}, y_m)$

Therefore, we have $d(x_m, x_n) + d(y_m, y_n) \le \delta^n d_0 + \delta^{n-1} d_0 + \dots + \delta^{m-1} d_0$

Thus we have $|d(x_m, x_n) + d(y_m, y_n)| \le \frac{\delta^n}{1-\delta} |d_0|$

Which implies that $\{x_n\}$ and $\{y_n\}$ are Cauchy sequences in X. By X is complete, there exists$x^*, y^* \in X$, such that $\lim_{n\to\infty} x_n = x^*$ and $\lim_{n\to\infty} y_n = y^*$. We now show that (x^*, y^*) is a coupled fixed point of F. Let $\epsilon > 0$. Continuity of F at (x^*, y^*) implies that, for given $\frac{\epsilon}{2} > 0$, there exist $d > 0$, such that $d(x^*, u) + d(y^*, v) < d$ implies, $d(F(x^*, y^*), F(u, v) < \frac{\epsilon}{2}$. Since $\{x_n\} \to x$ and$\{y_n\} \to y$, for $\phi = \min\left(\frac{\epsilon}{2}, \frac{d}{2}\right) > 0$, there exist n_0, m_0, such that, for $n \ge n_0$, $m \ge m_0$, we have $d(x_n, x^*) < \phi$ and $d(x_m, x^*) < \phi$. Therefore, for $n \in N, n \ge x\{n_0, m_0\}$

$$\begin{aligned} d(F(x^*, y^*), x^*) &\le d(F(x^*, y^*), x_{n+1}) + d(x_{n+1}, x^*) \\ &= d\big(F(x^*, y^*), F(x_n, y_n)\big) + d(x_{n+1}, x^*) < \frac{\epsilon}{2} + \phi \le \epsilon, \end{aligned}$$

From which it follows that $F(x^*, y^*) = x*$. In a similar manner, we can show that $F(y^*, x^*) = y*$. Hence (x^*, y^*) is a coupled fixed point of F.

Corollary 2.1 Let (X, d) be a dislocated metric space. Suppose that the mapping $F: X \times X \to X$ Satisfies $d\big(F(x, y), F(u, v)\big) \le a[d(x, u) + d(y, v)] + h[d(F(x, y), u) + d(F(u, v), x)] + l[d(F(x, y), x) + d(F(u, v), u)] + q[d(F(y, x), y) + d(F(v, u), v)]$………… (2)

For all x, y, u, v ∈ X, where a, h, l and q are non-negative constants with a + h + l + q < 1/2.

Then F has a unique coupled fixed point.

Proof: We can prove this result by putting a = b, h = k, l = m and q = r in Theorem 2.1

Corollary 2.2: Let (X, d) be a dislocated metric space. Suppose that the mapping $F: X \times X \to X$ satisfies

$$\begin{aligned} d\big(F(x, y), F(u, v)\big) \le a\{&[d(x, u) + d(y, v)] + [d(F(x, y), u) + d(F(u, v), x)] \\ &+[d(F(x, y), x) + d(F(u, v), u)] + [d(F(y, x), y) + d(F(v, u), v)]\} \dots\dots . (3) \end{aligned}$$

For all x, y, u, v ∈ X, where a is non-negative constant, with a < 1/2. Then F has a unique coupled fixed point.

Proof: We can prove this result by putting a = b = h = k = l = m = q = r in Theorem 2.1.

Corollary 2.3: Let (X, d) be a dislocated metric space. Suppose that the mapping $F: X \times X \to X$ satisfies

$$d\big(F(x,y),F(u,v)\big) \le hd(F(x,y),u) + kd(F(u,v),x) + ld(F(x,y),x) + md(F(u,v),u) \\ +qd(F(y,x),y) + rd(F(v,u),v) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots .. (4)$$

For all x, y, u, v ∈ X, where h, k, l, m, q, and r are non-negative constants

With h + k + l + m +q + r < 1/2. Then F has a unique coupled fixed point.

Proof: We can prove this result by applying Theorem 2.1 by putting a = b = 0.

Corollary 2.4: Let (X, d) be a dislocated metric space. Suppose that the mapping $F: X \times X \to X$ Satisfies

$$d\big(F(x,y),F(u,v)\big) \le ad(x,u) + bd(y,v) + hd(F(x,y),u) + kd(F(u,v),x) \\ +ld(F(x,y),x) + md(F(u,v),u) \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots . (5)$$

For all x, y, u, v ∈ X, where a, b, h, k, l, and m are non-negative constants

With a + b + h + k + l + m < 1/2. Then F has a unique coupled fixed point.

Proof: We can prove this result by putting q = r = 0 in Theorem 2.1.

III. Example

Now, we introduce an example to support our results

Example 3.1:

On the set $X = [0,1]$, define $d: X \times X \to \mathbb{R}^+$, $d(x,y) = |x-y|$

Also, define $F: X \times X \to X$, $F(x,y) = \frac{(x+y)}{4}$

Then

(1) (X, d) is a complete dislocated metric space.

(2) For any x, y, u, v ∈ X.

$$d\big(F(x,y),F(u,v)\big) \le \frac{1}{4}[d(x,u) + d(y,v)]$$

Proof:

Here we have $F(x,y) = \frac{(x+y)}{4}$

Therefore $d\big(F(x,y),F(u,v)\big) = d(\frac{x+y}{4},\frac{(u+v)}{4}) = \frac{1}{4}[|(x+y)-(u+v)|]$

$$\le \frac{1}{4}[|(x-u)| + |(y-v)| \le \frac{1}{4}[d(x,u) + d(y,v)]$$

Holds for all x, y ∈ X. Here (0, 0) is the unique coupled fixed point of F.

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